(1) Overview of SYZ approach to construction of mirrors

- in CY case (mirror = CY)
- in Fano case (mirror = LG model)

} ignoring instanton corrections

(2) Instanton corrections – examples
Mirror symmetry for pairs

(3) More examples: blowups of toric varieties along codim. 2 subvarieties
→ mirror symmetry for general varieties [conj.[1] by Katzarkov]
[work in progress w/ Abouzaid-Katzarkov]

<u>SYZ conjecture:</u> | X Calabi-Yau & mirror $X^\vee$ carry dual special Lag. torus fibrations over a same affine manifold B.

⚠ → existence of SLag T^n fibrations is not clear unless X is near "large complex structure limit" degeneration

→ geometry of mirror is modified by "instanton corrections"

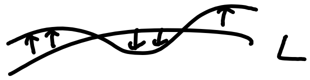
- <u>Def.</u> ‖ (X^n, ω, J) Kähler mfld is (almost) CY if $\Omega^{n,0} \simeq \mathcal{O}_X$
  Then $\exists\, \Omega \in \Omega^{n,0}$ holomorphic volume form

We don't necess. require $|\Omega|_g$ = constant

<u>Def.</u> ‖ $L^n \subset X$ is <u>special Lagrangian</u> if $\omega_{|L} = 0$ and $\operatorname{Im} \Omega_{|L} = 0$
(or more generally $\operatorname{Im}(e^{-i\varphi}\Omega)_{|L} = 0$ for some fixed φ)

Then $\Omega_{|L} = \psi \cdot \mathrm{vol}_{g_{|L}}$ where $\psi = |\Omega|_g \in C^\infty(L, \mathbb{R}_+)$

- <u>Deformations:</u>



$v \in C^\infty(NL)$ is a first-order SLag deform.[n] if $\begin{cases} L_v \omega = 0 \\ L_v \operatorname{Im} \Omega = 0 \end{cases}$

ie.: let $-L_v \omega = \alpha \in \Omega^1(L, \mathbb{R})$
$L_v \operatorname{Im} \Omega = \psi \, *_g \alpha \in \Omega^{n-1}(L, \mathbb{R})$

Then:

$$\{\text{Slag deformations}\} \underset{v \mapsto -\iota_v \omega}{\xrightarrow{\sim}} \left\{ \alpha \in \Omega^1(L,\mathbb{R}) \;/\; \begin{matrix} d\alpha = 0 \\ d^*(\psi\alpha) = 0 \end{matrix} \right\} =: \mathcal{H}^1_\psi(L)$$

"ψ-harmonic" 1-forms

$\exists !$ ψ-harmonic representative in each cohomology class, so

$$\begin{matrix} \mathcal{H}^1_\psi(L) & \simeq & H^1(L,\mathbb{R}) & \simeq & H^{n-1}(L,\mathbb{R}) \\ \alpha & & [\alpha] & & [\psi * \alpha] \end{matrix}$$

Prop (McLean, Joyce)

The moduli space of special Lagrangians is a smooth mfld $\mathcal{B}$,

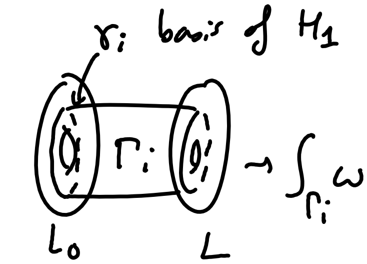
$$T_L \mathcal{B} \underset{\text{can.}}{\simeq} H^1(L,\mathbb{R}) \underset{\text{can.}}{\simeq} H^{n-1}(L,\mathbb{R}).$$

For $L \simeq T^n$, can expect locally a fibration $T^n \to X \to \mathcal{B}$

Rmk: $\mathcal{B}$ carries two natural affine structures:

- "symplectic": $T_L\mathcal{B} \simeq H^1(L,\mathbb{R})$,

  local coords. = symplectic areas swept by basis of $H_1(L)$



- "complex": $T_L\mathcal{B} \simeq H^{n-1}(L,\mathbb{R})$

  local coords $= \int_{\Delta_i} \operatorname{Im}\Omega$ swept by a basis of $H_{n-1}(L)$

Dual fibration: the dual torus of L is $\operatorname{Hom}(\pi_1(L), U(1))$

parametrizes flat $U(1)$ conn.'s on $\mathbb{C} \to L$ / gauge

Thus, given a Slag fibration $T^n \to X \xrightarrow{\pi} B$,

Def: $M = \{ (L,\nabla) \;/\; L \subset X$ Slag fiber of π, ∇ flat $U(1)$ conn. on $\mathbb{C} \to L$ / gauge $\}$

$$T_{(L,\nabla)} M = \left\{ (\underset{\substack{\downarrow \\ \text{normal vect. field}}}{v}, \underset{\substack{\downarrow \\ \text{connection 1-form}}}{\alpha}) \in C^\infty(NL) \oplus \Omega^1(L,\mathbb{R}) \;/\; -\iota_v\omega + i\alpha \in \mathcal{H}^1_\psi(L) \otimes \mathbb{C} \right\}$$

naturally a $\underline{\underline{\mathbb{C}\text{ vector space}}}$

$\Rightarrow$ This defines an a.c.s. $J^\vee$ on M

Prop: Set
- $J^\vee$ given by $T_{(L,\nabla)}M \simeq \mathcal{H}^1_\psi(L)\otimes\mathbb{C}$
- $\Omega^\vee((v_1,\alpha_1),\dots,(v_n,\alpha_n)) = \int_L (-\iota_{v_1}\omega + i\alpha_1)\wedge\dots\wedge(-\iota_{v_n}\omega + i\alpha_n)$
- $\omega^\vee((v_1,\alpha_1),(v_2,\alpha_2)) = \frac{1}{[\Omega]\cdot[L]}\int_L \alpha_2\wedge\iota_{v_1}\mathrm{Im}\,\Omega - \alpha_1\wedge\iota_{v_2}\mathrm{Im}\,\Omega$

Then $J^\vee$ is integrable, $\omega^\vee$ is a compatible Kähler form, $(M,J^\vee,\Omega^\vee,\omega^\vee)$ is almost-CY, and $\pi^\vee : M \to B$, $(L,\nabla)\mapsto L$ is a SLag torus fibration

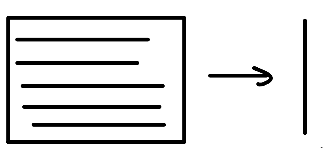
★ M is the (<u>uncorrected</u>) mirror of X.

$$\begin{tikzcd} X \arrow[dr, "\pi"'] & & M \arrow[dl, "\pi^\vee"] \\ & B & \end{tikzcd}$$

dual SLag torus fibrations;
duality $\equiv$ interchange of the two affine structures on B.

★ In real life: most SLag fibrations have singular fibers, where dualization breaks down. Instanton corrections will deal with this.

- Ex: $X = T^2 = \mathbb{C}/\mathbb{Z}+i\tau\mathbb{Z}$, $\Omega = dz$, $\int_{T^2}\omega = \lambda$
  (restr. because I don't want to mention B-fields)



T^2 $\qquad$ S^1

$L = \{\mathrm{Im}\, z = \text{const.}\}$ are SLag. S^1's

$\pi : T^2 \to B = S^1$

For sympl. affine structure, B has size $\lambda = \text{area}(T^2)$
Complex ——"—— ——"—— τ = modular param.

Dualizing, $M = T^2$ with $\begin{cases} \text{complex str. } \mathbb{C}/\mathbb{Z}+i\lambda\mathbb{Z} \\ \text{sympl. area } \tau \end{cases}$

- <u>Motivation from HMS:</u> expect $D^b\mathrm{Coh}(M) \simeq D^\pi \mathcal{F}uk(X)$

so: point $p \in M \iff \mathcal{O}_p \in D^b\mathrm{Coh}(M) \iff \mathcal{L}_p \in D^\pi \mathcal{F}uk(X)$

$\mathrm{Ext}^*(\mathcal{O}_p, \mathcal{O}_p) \simeq H^*(T^n, \mathbb{C}) \rightarrow$ Floer cohomology of $\mathcal{L}_p$ is $\simeq H^*(T^n)$

expect most likely $\mathcal{L}_p$ is a Lagr. torus + flat $U(1)$ conn.

(However: some pts of M might not correspond to honest Lagr. in X)

If only want M as a complex manifold, enough to work with Lagrangian tori in X; "special" condition needed to define $\omega^\vee$ on M.

<u>Non-CY case:</u> from now on (X, J, ω) Kähler, $D \subset X$ hypersurface $D \in |-K_X|$
effective anticanonical divisor
reduced, at most normal crossings.

$\Rightarrow \Omega = \sigma_D^{-1} \in \Omega^{n,0}(X \setminus D)$ holom. vol. form with poles along D.

Can look for a Slag torus fibration on the almost-CY manifold $X \setminus D$,

let $M = \left\{ (L, \nabla) \;/\; \begin{array}{l} L \subset X \setminus D \text{ Slag torus} \\ \nabla \text{ flat } U(1) \text{ conn.} \end{array} \right\}$ mirror of $X \setminus D$ as above.

The mirror of X (mod. instanton corrections) will be a
<u>Landau-Ginzburg model</u> (M, W), $W: M \to \mathbb{C}$ holomorphic $f^{\underline{n}}$
= <u>"superpotential"</u>

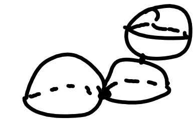
W modifies geometric interpretation of mirror symmetry, esp. B-model = singularities of W.

<u>Construction:</u> $(L, \nabla) \in M$, $\beta \in \pi_2(X, L)$

$\Rightarrow \mathcal{M}(L, \beta)$ moduli space of holom. maps $u: (D^2, \partial D^2) \to (X, L)$, $[u] = \beta$

exp. $\dim_{\mathbb{R}} \mathcal{M}(L,\beta) = n - 3 + \mu(\beta)$

Maslov index: for L Slag,
$\mu(\beta) = 2(\beta \cap D)$

$\exists$ compactification $\overline{\mathcal{M}}(L,\beta)$ formed by adding bubbled configurations 

$L \simeq T^n$ spin $\Rightarrow \mathcal{M}$ orientable.

- Assume there are no discs of $\mu(\beta) \leq 0$, and $\mu = 2$ discs are regular.

Then, for $\mu(\beta) = 2$, $\exists\ n_\beta(L) = \#$ holom. discs in class β passing through a generic point $p \in L$

(⚠ signed count, need orientations)

(or: $\quad ev: \underset{\substack{\uparrow \\ \text{1 boundary marked pt}}}{\overline{\mathcal{M}}_1}(L,\beta) \longrightarrow L, \quad n_\beta(L) = \deg\left(ev_*[\overline{\mathcal{M}}_1(L,\beta)]\right)$)

Def:
$$W(L,\nabla) = \sum_{\substack{\beta \in \pi_2(X,L) \\ \mu(\beta)=2}} n_\beta(L)\, z_\beta(L,\nabla)$$
where $z_\beta(L,\nabla) = \underbrace{\exp\left(-\int_\beta \omega\right)}_{\mathbb{R}_+} \underbrace{\mathrm{hol}_{\partial\beta}(\nabla)}_{U(1)} \in \mathbb{C}^*$

Note: • z_β are local holom. coordinates on M!

Indeed $d \log z_\beta\ (v,\alpha) = \int_{\partial\beta} -\iota_v\omega + i\alpha$

($\rightarrow \log z_\beta$ = complexified version of affine coordinate on B)

Hence W is a holom. function on M as long as things go well.... but.... 2 major issues:

1. convergence of the sum is unknown in general (except specific cases e.g. toric Fano)
   so W might only be def^d as a formal sum $\in$ Novikov ring, not as an actual complex number

2. in general, $n_\beta(L)$ might be ill-defined – may depend on construction of virtual fund. chain for $\overline{\mathcal{M}}(L,\beta)$, and on additional data. This makes W multivalued / discontinuous.

This will be remedied by instanton corrections.

Example: $\mathbb{CP}^2$ (or any other toric Fano) [see: Hori, Cho-Oh, FO^3]

$X = \mathbb{CP}^2$, $D = \{x_0 x_1 x_2 = 0\}$, $X \setminus D \simeq (\mathbb{C}^*)^2$, $\Omega = \dfrac{dx \wedge dy}{xy}$, ω toric
(in general: X toric Fano, D toric divisor)

Then product tori $L = S^1(r_1) \times S^1(r_2) \subset (\mathbb{C}^*)^2 \subset \mathbb{CP}^2$ are special Lagrangian
Base B = orbit space for T^2-action on $(\mathbb{C}^*)^2$

$\rightarrow$ sympl. affine structure: $B = \mathrm{int}(\Delta)$: interior of moment polytope
fibration = moment map

$\rightarrow$ complex affine structure: $B = \mathbb{R}^2$, fibration = Log map
(tropical geometry!)

- Dualizing, $\Big\Vert\ M \cong \left\{ (z_1, z_2) \,/\, \left(-\frac{1}{2\pi} \log |z_i|\right) \in \mathrm{int}\ \Delta \right\} \subset (\mathbb{C}^*)^2$

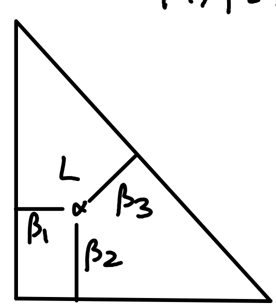
- There are no $\mu \le 0$ discs in (X, L) (would have to be $\subset (\mathbb{C}^*)^2$)
- $\mu = 2$ discs hit $D = D_0 \cup D_1 \cup D_2$ exactly once transversely
  Exactly one family for each component of D.

$L = S^1(r_1) \times S^1(r_2)$ bounds
- $D^2(r_1) \times \{pt\} \subset \mathbb{C}^2 \subset \mathbb{CP}^2$
- $\{pt\} \times D^2(r_2)$
- 3^{rd} family through line at infinity

in classes β_1, β_2, and $\beta_3 = [\mathbb{CP}^1] - \beta_1 - \beta_2$

Pictorially:

Δ



$\rightarrow$ variables $z_{\beta_1} = z_1$ $\quad (|z_1| = e^{-2\pi \mu_1}\ !)$

$z_{\beta_2} = z_2$

$z_{\beta_3} = \dfrac{e^{-\mathrm{Area}(\mathbb{CP}^1)}}{z_1 z_2}$

Moreover $n_{\beta_1} = n_{\beta_2} = n_{\beta_3} = 1$ (one disc through each point of L)

$$\Rightarrow \quad W = z_1 + z_2 + \frac{e^{-\mathrm{Area}(\mathbb{CP}^1)}}{z_1 z_2} \qquad \text{(classical).}$$

| For general toric Fano, W = Laurent polynomial with one monomial per facet of Δ, with exponent $\Leftrightarrow$ normal vector to the facet.