MONODROMY INVARIANTS IN SYMPLECTIC TOPOLOGY

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This text is a set of lecture notes for a series of four talks given at I.P.A.M., Los Angeles, on March 18-20, 2003. The first lecture provides a quick overview of symplectic topology and its main tools: symplectic manifolds, almost-complex structures, pseudo-holomorphic curves, Gromov-Witten invariants and Floer homology. The second and third lectures focus on symplectic Lefschetz pencils: existence (following Donaldson), monodromy, and applications to symplectic topology, in particular the connection to Gromov-Witten invariants of symplectic 4-manifolds (following Smith) and to Fukaya categories (following Seidel). In the last lecture, we offer an alternative description of symplectic 4-manifolds by viewing them as branched covers of the complex projective plane; the corresponding monodromy invariants and their potential applications are discussed.

1. Introduction to symplectic topology

In this lecture, we recall basic notions about symplectic manifolds, and briefly review some of the main tools used to study them. Most of the topics discussed here can be found in greater detail in standard graduate books such as [McS].

1.1. **Symplectic manifolds.**

Definition 1.1. *A* symplectic structure *on a smooth manifold M is a closed non-degenerate 2-form ω, i.e. an element $\omega \in \Omega^2(M)$ such that $d\omega = 0$ and $\forall v \in TM - \{0\}$, $\iota_v\omega \neq 0$.*

For example, $\mathbb{R}^{2n}$ carries a standard symplectic structure, given by the 2-form $\omega_0 = \sum dx_i \wedge dy_i$. Similarly, every orientable surface is symplectic, taking for ω any non-vanishing volume form.

Since ω induces a non-degenerate antisymmetric bilinear pairing on the tangent spaces to M, it is clear that every symplectic manifold is even-dimensional and orientable (if $\dim M = 2n$, then $\frac{1}{n!}\omega^n$ defines a volume form on M).

Two important features of symplectic structures that set them apart from most other geometric structures are the existence of a large number of symplectic automorphisms, and the absence of local geometric invariants.

The first point is illustrated by the following construction. Consider a smooth function $H : M \to \mathbb{R}$ (a *Hamiltonian*), and define X_H to be the vector field on M such that $\omega(X_H, \cdot) = dH$. Let $\phi_t : M \to M$ be the family of diifeomorphisms generated by the flow of X_H, i.e., $\phi_0 = \mathrm{Id}$ and $\frac{d}{dt}\phi_t(x) = X_H(\phi_t(x))$. Then ϕ_t is a *symplectomorphism*, i.e. $\phi_t^*\omega = \omega$. Indeed, we have $\phi_0^*\omega = \omega$, and

$$\frac{d}{dt}\phi_t^*\omega = \phi_t^*(L_{X_H}\omega) = \phi_t^*(d\iota_{X_H}\omega + \iota_{X_H}d\omega) = \phi_t^*(d(dH) + 0) = 0.$$

Therefore, the group of symplectomorphisms $\mathrm{Symp}(M, \omega)$ is infinite-dimensional, and its Lie algebra contains all Hamiltonian vector fields. This is in contrast with the case of Riemannian metrics, where isometry groups are much smaller.

The lack of local geometric invariants of symplectic structures is illustrated by two classical results of fundamental importance, which show that the study of symplectic manifolds largely reduces to *topology* (i.e., to discrete invariants): Darboux's theorem, and Moser's stability theorem. The first one shows that all symplectic forms are locally equivalent, in sharp contrast to the case of Riemannian metrics where curvature provides a local invariant, and the second one shows that exact deformations of symplectic structures are trivial.

Theorem 1.2 (Darboux)**.** *Every point in a symplectic manifold (M^{2n},ω) admits a neighborhood that is symplectomorphic to a neighborhood of the origin in $(\mathbb{R}^{2n},\omega_0)$.*

Proof. We first use local coordinates to map a neighborhood of a given point in M diffeomorphically onto a neighborhood V of the origin in $\mathbb{R}^{2n}$. Composing this diffeomorphism f with a suitable linear transformation of $\mathbb{R}^{2n}$, we can ensure that the symplectic form $\omega_1 = (f^{-1})^*\omega$ coincides with ω_0 at the origin. This implies that, restricting to a smaller neighborhood if necessary, the closed 2-forms $\omega_t = t\,\omega_1 + (1-t)\omega_0$ are non-degenerate over V for all $t\in[0,1]$.

Using the Poincaré lemma, consider a family of 1-forms α_t on V such that $\frac{d}{dt}\omega_t = -d\alpha_t$. Subtracting a constant 1-form from α_t if necessary, we can assume that α_t vanishes at the origin for all t. Using the non-degeneracy of ω_t we can find vector fields X_t such that $\iota_{X_t}\omega_t = \alpha_t$. Let $(\phi_t)_{t\in[0,1]}$ be the flow generated by X_t, i.e. the family of diffeomorphisms defined by $\phi_0 = \mathrm{Id}$, $\frac{d}{dt}\phi_t(x) = X_t(\phi_t(x))$; we may need to restrict to a smaller neighborhood $V'\subset V$ of the origin in order to make the flow ϕ_t well-defined for all t. We then have

$$\frac{d}{dt}\phi_t^*\omega_t = \phi_t^*(L_{X_t}\omega_t) + \phi_t^*\Big(\frac{d\omega_t}{dt}\Big) = \phi_t^*(d(\iota_{X_t}\omega_t) - d\alpha_t) = 0,$$

and therefore $\phi_1^*\omega_1 = \omega_0$. Therefore, $\phi_1^{-1}\circ f$ induces a symplectomorphism from a neighborhood of x in (M,ω) to a neighborhood of the origin in $(\mathbb{R}^{2n},\omega_0)$. □

Theorem 1.3 (Moser)**.** *Let $(\omega_t)_{t\in[0,1]}$ be a continuous family of symplectic forms on a compact manifold M. Assume that the cohomology class $[\omega_t]\in H^2(M,\mathbb{R})$ does not depend on t. Then (M,ω_0) is symplectomorphic to (M,ω_1).*

Proof. We use the same argument as above: since $[\omega_t]$ is constant there exist 1-forms α_t such that $\frac{d}{dt}\omega_t = -d\alpha_t$. Define vector fields X_t such that $\iota_{X_t}\omega_t = \alpha_t$ and the corresponding flow ϕ_t. By the same calculation as above, we conclude that $\phi_1^*\omega_1 = \omega_0$. □

1.2. **Submanifolds in symplectic manifolds.**

Definition 1.4. *A submanifold $W\subset(M^{2n},\omega)$ is called* symplectic *if $\omega_{|W}$ is non-degenerate at every point of W (it is then a symplectic form on W);* isotropic *if $\omega_{|W}=0$; and* Lagrangian *if it is isotropic of maximal dimension* $\dim W = n = \frac{1}{2}\dim M$.

An important example is the following: given any smooth manifold N, the cotangent bundle T^*N admits a canonical symplectic structure that can be expressed locally as $\omega = \sum dp_i\wedge dq_i$ (where (q_i) are local coordinates on N and (p_i) are the dual coordinates on the cotangent spaces). Then the zero section is a Lagrangian submanifold of T^*N.

Since the symplectic form induces a non-degenerate pairing between tangent and normal spaces to a Lagrangian submanifold, the normal bundle to a Lagrangian submanifold is always isomorphic to its cotangent bundle. The fact that this isomorphism extends beyond the infinitesimal level is a classical result of Weinstein:

Theorem 1.5 (Weinstein). *For any Lagrangian submanifold $L \subset (M^{2n}, \omega)$, there exists a neighborhood of L which is symplectomorphic to a neighborhood of the zero section in the cotangent bundle $(T^*L, \sum dp_i \wedge dq_i)$.*

There is also a neighborhood theorem for symplectic submanifolds; in that case, the local model for a neighborhood of the submanifold $W \subset M$ is a neighborhood of the zero section in the symplectic vector bundle NW over W (since $Sp(2n)$ retracts onto $U(n)$, the classification of symplectic vector bundles is the same as that of complex vector bundles).

1.3. **Almost-complex structures.**

Definition 1.6. *An* almost-complex structure *on a manifold M is an endomorphism J of the tangent bundle TM such that $J^2 = -\mathrm{Id}$. An almost-complex structure J is said to be* tamed *by a symplectic form ω if for every non-zero tangent vector u we have $\omega(u, Ju) > 0$; it is* compatible *with ω if it is ω-tame and $\omega(u, Jv) = -\omega(Ju, v)$; equivalently, J is ω-compatible if and only if $g(u, v) = \omega(u, Jv)$ is a Riemannian metric.*

Proposition 1.7. *Every symplectic manifold (M, ω) admits a compatible almost-complex structure. Moreover, the space of ω-compatible (resp. ω-tame) almost-complex structures is contractible.*

This result follows from the fact that the space of compatible (or tame) complex structures on a symplectic vector space is non-empty and contractible (this can be seen by constructing explicit retractions); it is then enough to observe that a compatible (resp. tame) almost-complex structure on a symplectic manifold is simply a section of the bundle $End(TM)$ that defines a compatible (resp. tame) complex structure on each tangent space.

An almost-complex structure induces a splitting of the complexified tangent and cotangent bundles: $TM \otimes \mathbb{C} = TM^{1,0} \oplus TM^{0,1}$, where $TM^{1,0}$ and $TM^{0,1}$ are respectively the $+i$ and $-i$ eigenspaces of J (i.e., $TM^{1,0} = \{v - iJv,\ v \in TM\}$, and similarly for $TM^{0,1}$; for example, on $\mathbb{C}^n$ equipped with its standard complex structure, the $(1,0)$ tangent space is generated by $\partial/\partial z_i$ and the $(0,1)$ tangent space by $\partial/\partial \bar{z}_i$. Similarly, J induces a complex structure on the cotangent bundle, and $T^*M \otimes \mathbb{C} = T^*M^{1,0} \oplus T^*M^{0,1}$ (by definition $(1,0)$-forms are those which pair trivially with $(0,1)$-vectors, and vice versa). This splitting of the cotangent bundle induces a splitting of differential forms into “types”: $\bigwedge^r T^*M \otimes \mathbb{C} = \bigoplus_{p+q=r} \bigwedge^p T^*M^{1,0} \otimes \bigwedge^q T^*M^{0,1}$. Moreover, given a function $f : M \to \mathbb{C}$ we can write $df = \partial f + \bar\partial f$, where $\partial f = \frac{1}{2}(df - i\, df \circ J)$ and $\bar\partial f = \frac{1}{2}(df + i\, df \circ J)$ are the $(1,0)$ and $(0,1)$ parts of df respectively. Similarly, given a complex vector bundle E over M equipped with a connection, the covariant derivative ∇ can be split into operators $\partial^\nabla : \Gamma(E) \to \Omega^{1,0}(E)$ and $\bar\partial^\nabla : \Gamma(E) \to \Omega^{0,1}(E)$.

Although the tangent space to a symplectic manifold (M, ω) equipped with a compatible almost-complex structure J can be pointwise identified with $(\mathbb{C}^n, \omega_0, i)$, there is an important difference between a symplectic manifold equipped with a compatible almost-complex structure and a complex Kähler manifold: the possible lack of *integrability* of the almost-complex structure, namely the fact that the Lie bracket of two $(1,0)$ vector fields is not necessarily of type $(1,0)$.

Definition 1.8. *The* Nijenhuis tensor *of an almost-complex manifold* (M, J) *is the quantity defined by* $N_J(X, Y) = \frac{1}{4}([X, Y] + J[X, JY] + J[JX, Y] - [JX, JY])$. *The almost-complex structure* J *is said to be* integrable *if* $N_J = 0$.

It can be checked that N_J is a tensor (i.e., only depends on the values of the vector fields X and Y at a given point), and that $N_J(X, Y) = 2\,\mathrm{Re}([X^{1,0}, Y^{1,0}]^{(0,1)})$. The non-vanishing of N_J is therefore an obstruction to the integrability of a local frame of $(1, 0)$ tangent vectors, i.e. to the existence of local holomorphic coordinates. The Nijenhuis tensor is also related to the fact that the exterior differential of a $(1, 0)$-form may have a non-zero component of type $(0, 2)$, so that the ∂ and $\bar\partial$ operators on differential forms do not have square zero ($\bar\partial^2$ can be expressed in terms of ∂ and the Nijenhuis tensor).

Theorem 1.9 (Newlander-Nirenberg)**.** *Given an almost-complex manifold* (M, J)*, the following properties are equivalent: (i)* $N_J = 0$*; (ii)* $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$*; (iii)* $\bar\partial^2 = 0$*; (iv)* (M, J) *is a complex manifold, i.e. admits complex analytic coordinate charts.*

1.4. **Pseudo-holomorphic curves and Gromov-Witten invariants.**

Pseudo-holomorphic curves, first introduced by Gromov in 1985 [Gr], have since then become the most important tool in modern symplectic topology. In the same way as the study of complex curves in complex manifolds plays a central role in algebraic geometry, the study of pseudo-holomorphic curves has revolutionized our understanding of symplectic manifolds.

The equation for holomorphic maps between two almost-complex manifolds becomes overdetermined as soon as the complex dimension of the domain exceeds 1, so we cannot expect the presence of any almost-complex submanifolds of complex dimension ≥ 2 in a symplectic manifold equipped with a generic almost-complex structure. On the other hand, J-holomorphic curves, i.e. maps from a Riemann surface (Σ, j) to the manifold (M, J) such that $J \circ df = df \circ j$, are governed by an elliptic PDE, and their study makes sense even in non-Kähler symplectic manifolds. The questions that we would like to answer are of the following type:

Given a compact symplectic manifold (M, ω) *equipped with a generic compatible almost-complex structure* J *and a homology class* $\beta \in H_2(M, \mathbb{Z})$*, what is the number of pseudo-holomorphic curves of given genus* g*, representing the homology class* β *and passing through* r *given points in* M *(or through* r *given submanifolds in* M*) ?*

The answer to this question is given by *Gromov-Witten invariants*, which count such curves (in a sense that is not always obvious, as the result can e.g. be negative, and need not even be an integer). One starts by introducing a *moduli space* $\mathcal{M}_{g,r}(\beta)$ of genus g pseudo-holomorphic curves with r marked points representing a given homology class $\beta \in H_2(M, \mathbb{Z})$. This moduli space comes equipped with r *evaluation maps* $ev_i : \mathcal{M}_{g,r}(\beta) \to M$, associating to a curve the image of its i-th marked point. Considering r submanifolds of M representing homology classes Poincaré dual to cohomology classes $\alpha_1, \ldots, \alpha_r \in H^*(M, \mathbb{R})$, and assuming the dimensions to be the right ones, we then want to define a number

$$\mathrm{GW}_{g,\beta}(M; \alpha_1, \ldots, \alpha_r) = \int_{\mathcal{M}_{g,r}(\beta)} ev_1^*\alpha_1 \wedge \cdots \wedge ev_r^*\alpha_r. \tag{1}$$

We can also choose to impose restrictions on the complex structure of the domain (or on the positions of the marked points in the domain) by introducing into the integral a cohomology class pulled back from the moduli space of genus g Riemann surfaces with r marked points.

Giving a precise meaning to the right-hand side of (1) is a very delicate task, which has been a central preoccupation of symplectic geometers for more than ten years following Gromov's seminal work. We only give a very simplified description of the main issues; the reader is referred to the book [McS2] for a detailed discussion of the so-called "weakly monotone" case, and to more recent work (Fukaya-Ono, Li-Tian [LT], Ruan, Siebert, Hofer-Salamon, ...) for the general case, which requires more sophisticated techniques.

To start with, one must study deformations of pseudo-holomorphic curves, by linearizing the equation $\bar\partial_J f = 0$ near a solution. The linearized Cauchy-Riemann operator $D_{\bar\partial}$, whose kernel describes infinitesimal deformations of a given curve $(f : \Sigma \to M) \in \mathcal{M}_{g,r}(\beta)$, is a Fredholm operator of (real) index

$$2d := \operatorname{ind} D_{\bar\partial} = (1-g)(\dim_{\mathbb{R}} M - 6) + 2r + 2\,c_1(TM)\cdot\beta.$$

When the considered curve is *regular*, i.e. when the linearized operator $D_{\bar\partial}$ is surjective, the deformation theory is unobstructed, and if the curve has no automorphisms (which is the case of a generic curve provided that $r \geq 3$ when $g = 0$ and $r \geq 1$ when $g = 1$), we expect the moduli space $\mathcal{M}_{g,r}(\beta)$ to be locally a smooth manifold of real dimension $2d$.

The main result underlying the theory of pseudo-holomorphic curves is Gromov's compactness theorem (see [Gr], [McS2], ...):

Theorem 1.10 (Gromov)**.** *Let $f_n : (\Sigma_n, j_n) \to (M, \omega, J)$ be a sequence of pseudo-holomorphic curves in a compact symplectic manifold, representing a fixed homology class. Then a subsequence of $\{f_n\}$ converges (in the "Gromov-Hausdorff topology") to a limiting map f_∞, possibly singular.*

The limiting curve f_∞ can have a very complicated structure, and in particular its domain may be a nodal Riemann surface with more than one component, due to the phenomenon of *bubbling*. For example, the sequence of degree 2 holomorphic curves $f_n : \mathbb{CP}^1 \to \mathbb{CP}^2$ defined by $f_n(u\!:\!v) = (u^2 : uv : \frac{1}{n}v^2)$ converges to a singular curve with two degree 1 components: for $(u\!:\!v) \neq (0\!:\!1)$, we have $\lim f_n(u\!:\!v) = (u\!:\!v\!:\!0)$, so that the sequence apparently converges to a line in $\mathbb{CP}^2$. However the derivatives of f_n become unbounded near $(0\!:\!1)$, and composing f_n with the coordinate change $\phi_n(u\!:\!v) = (\frac{1}{n}u : v)$ we obtain $f_n \circ \phi_n(u\!:\!v) = (\frac{1}{n^2}u^2 : \frac{1}{n}uv : \frac{1}{n}v^2) = (\frac{1}{n}u^2 : uv : v^2)$, which converges to $(0\!:\!u\!:\!v)$ everywhere except at $(1\!:\!0)$, giving the other component (the "bubble") in the limiting curve.

The presence of a symplectic structure on the target manifold M is crucial for compactness, as it provides an a priori estimate of the *energy* of a pseudo-holomorphic curve, and hence makes it possible to control the bubbling phenomenon: for a J-holomorphic map $f : (\Sigma, j) \to (M, \omega, J)$, we have

$$\int_\Sigma \|df\|^2 = \int_\Sigma f^*\omega = [\omega]\cdot f_*[\Sigma].$$

The definition of Gromov-Witten invariants requires a compactification of the moduli space of pseudo-holomorphic curves, in order to be able to define a fundamental cycle for $\mathcal{M}_{g,r}(\beta)$ against which cohomology classes can be evaluated. It follows from Gromov's compactness theorem that this compactification can be achieved by considering the moduli space $\overline{\mathcal{M}}_{g,r}(\beta)$ of *stable maps*, i.e. J-holomorphic maps $f : \bigsqcup(\Sigma_\alpha, j_\alpha) \to M$ with domain a tree of Riemann surfaces, and with a discrete set of automorphisms (which imposes conditions on the genus 0 or 1 "ghost components"). If we assume that none of the stable maps in the compactified moduli space has multiply covered components (i.e. components that factor through non-trivial coverings of Riemann surfaces), then the moduli space of stable maps can be used to define a fundamental class $[\overline{\mathcal{M}}_{g,r}(\beta)]$. However, because the presence of curves with non-trivial automorphisms makes the moduli space an orbifold rather than a manifold, the fundamental class is in general a *rational* homology class in $H_{2d}(\overline{\mathcal{M}}_{g,r}(\beta), \mathbb{Q})$.

Technically, the hardest case is when some of the stable maps in the moduli space have components which are multiply covered. The stable map compactification can then fail to provide a suitable fundamental cycle, because the actual dimension of the moduli space of pseudo-holomorphic curves may exceed that predicted by the index calculation, even for a generic choice of J, due to the possibility of arbitrarily moving the branch points of the multiple components. It is then necessary to break the symmetry and restore transversality by perturbing the holomorphic curve equation $\bar\partial_J f = 0$ into another equation $\bar\partial_J f = \nu(f)$ whose space of solutions has the correct dimension ([LT], ...), making it possible to define a *virtual fundamental cycle* $[\overline{\mathcal{M}}_{g,r}(\beta)]^{vir}$. For the same reasons as above, this fundamental cycle is in general a *rational* homology class in $H_{2d}(\overline{\mathcal{M}}_{g,r}(\beta), \mathbb{Q})$.

1.5. **Floer homology for Hamiltonians.** Floer homology has been introduced in symplectic geometry as an attempt to prove the *Arnold conjecture* for fixed points of Hamiltonian diffeomorphisms: for every non-degenerate Hamiltonian diffeomorphism $\phi : (M, \omega) \to (M, \omega)$ (i.e., the flow of a time-dependent family of Hamiltonian vector fields), the number of fixed points of ϕ is bounded from below by the sum of the Betti numbers of M. Briefly speaking, the idea is to obtain invariants from the Morse theory of a functional defined on an infinite-dimensional space, and hence to deduce existence results for critical points of the functional.

To define the Floer homology of the diffeomorphism of (M, ω) generated by a 1-periodic family of Hamiltonians $H : S^1 \times M \to \mathbb{R}$, one introduces the *action functional* on the space of contractible loops in M: given a loop $\gamma : S^1 \to M$ bounding a disk $u : D^2 \to M$, we define

$$\mathcal{A}_H(\gamma) = -\int_{D^2} u^*\omega - \int_{S^1} H(t, \gamma(t))\, dt.$$

The critical points of $\mathcal{A}_H$ correspond to the 1-periodic closed orbits of the Hamiltonian flow, and the gradient trajectories correspond to maps $f : \mathbb{R} \times S^1 \to M$ satisfying an equation of the form

$$\frac{\partial f}{\partial s} + J(f)\frac{\partial f}{\partial t} - \nabla H(t, f) = 0. \tag{2}$$

When H is independent of t, and if we start with a constant loop γ, a gradient trajectory of $\mathcal{A}_H$ is simply a gradient trajectory of H; however, when $H = 0$, the equation for gradient trajectories reduces to that of pseudo-holomorphic curves.

The *Floer complex* $CF_*(M, H)$ is a free graded module with one generator for each contractible 1-periodic orbit $\gamma : S^1 \to M$ of the Hamiltonian flow, or more precisely for each pair $(\gamma, [u])$, where $[u] \in \pi_2(M, \gamma)$ is a relative homotopy class (taking this more refined description, $CF_*(M, H)$ is naturally a module over the

Novikov ring). The grading is defined by the *Conley-Zehnder index* of the critical point $(\gamma, [u])$ of $\mathcal{A}_H$. Given two periodic orbits γ^- and γ^+, we can consider the moduli space $\mathcal{M}(\gamma^-, \gamma^+; A)$ of gradient trajectories joining γ^- to γ^+ (i.e., of solutions to the gradient flow equation (2) that converge to γ^- for $s \to -\infty$ and to γ^+ for $s \to +\infty$) realizing a given relative homology class A. The expected dimension of this moduli space, which always carries an $\mathbb{R}$-action by translation in the s direction, is the difference betwen the Conley-Zehnder indices of $(\gamma^-, [u])$ and $(\gamma^+, [u]\#A)$; hence, assuming regularity and compactness of the moduli spaces $\mathcal{M}(\gamma^-, \gamma^+; A)$ we can define an operator ∂ on $CF_*(M, H)$ by the formula

$$\partial(\gamma^-, [u]) = \sum_{\gamma^+, A} \#(\mathcal{M}(\gamma^-, \gamma^+; A)/\mathbb{R})\ (\gamma^+, [u]\#A),$$

where the sum runs over pairs (γ^+, A) such that the expected dimension of the moduli space is equal to 1. After a suitable discussion of regularity and compactness (bubbling) issues, which are essentially identical to those encountered in the definition of Gromov-Witten invariants, a proper meaning can be given to this definition of the operator ∂, and the following result can be obtained:

Theorem 1.11. *For regular (H, J), the Floer differential satisfies $\partial^2 = 0$, and hence we can define the Floer homology $HF_*(M, \omega, H, J) = \operatorname{Ker} \partial / \operatorname{Im} \partial$. Moreover, the Floer homologies obtained for different (H, J) are naturally isomorphic.*

Moreover, by considering the limit as $H \to 0$ one can relate Floer homology to either the quantum or classical homology of M (with coefficients in the Novikov ring), which yields an inequality between the number of critical points of $\mathcal{A}_H$ and the total rank of $H_*(M)$, and hence the Arnold conjecture.

The technically easier *monotone* case of these results is due to Floer; the general case has been treated subsequently using the same methods as for the definition of general Gromov-Witten invariants (the reader is referred to e.g. [McS2] and [Sa] for detailed expositions on Hamiltonian Floer homology).

1.6. **Floer homology for Lagrangians.** Floer homology has also been successfully introduced for the study of Lagrangian submanifolds, a construction which has taken a whole new importance after Kontsevich's formulation of the homological mirror symmetry conjecture.

Consider two compact orientable (relatively spin) Lagrangian submanifolds L_0 and L_1 in a symplectic manifold (M, ω) equipped with a compatible almost-complex structure J. Lagrangian Floer homology corresponds to the Morse theory of a functional on (a covering of) the space of arcs joining L_0 to L_1, whose critical points are constant paths.

The *Floer complex* $CF^*(L_0, L_1)$ is a free module with one generator for each intersection point $p \in L_0 \cap L_1$, and grading given by the *Maslov index.* To be more precise, as in the case of Hamiltonian Floer homology, in the general case one needs to consider pairs $(p, [u])$ where $[u]$ is the equivalence class of a map $u : [0, 1] \times [0, 1] \to M$ such that $u(\cdot, 0) \in L_0$, $u(\cdot, 1) \in L_1$, $u(1, \cdot) = p$, and $u(0, \cdot)$ is a fixed arc joining L_0 to L_1 (two maps u, u' are equivalent if they have the same symplectic area and the same Maslov index).

Given two points $p_\pm \in L_0 \cap L_1$, we can define a moduli space $\mathcal{M}(p_-, p_+; A)$ of pseudo-holomorphic maps $f : \mathbb{R} \times [0, 1] \to M$ such that $f(\cdot, 0) \in L_0$, $f(\cdot, 1) \in L_1$, and $\lim_{t \to \pm\infty} f(t, \tau) = p_\pm\ \forall \tau \in [0, 1]$, realizing a given relative homology class A;

the expected dimension of this moduli space is the difference of Maslov indices. Assuming regularity and compactness of $\mathcal{M}(p_-, p_+; A)$, we can define an operator ∂ on $CF^*(L_0, L_1)$ by the formula

$$\partial(p_-, [u]) = \sum_{p_+, A} \#(\mathcal{M}(p_-, p_+; A)/\mathbb{R})\ (p_+, [u]\#A),$$

where the sum runs over pairs (p_+, A) for which the expected dimension of the moduli space is 1.

In all good cases we expect to have $\partial^2 = 0$, which allows us to define the Floer homology $HF^*(L_0, L_1) = \operatorname{Ker}\partial/\operatorname{Im}\partial$. However, a serious technical difficulty arises from the fact that, in the case of the moduli spaces $\mathcal{M}(p_-, p_+; A)$ of pseudo-holomorphic strips, bubbling can occur on the boundary of the domain, so that limit curves can contain both S^2 and D^2 bubble components. Because disc bubbling is a phenomenon that arises in real codimension 1, the fundamental chain associated to the compactified moduli space is not always a cycle; in that case, to define Floer homology we need to add other curves to the moduli space $\mathcal{M}(p_-, p_+; A)$ in order to obtain a cycle. This gives rise to a series of obstructions that may prevent Floer homology from being well-defined. The detailed analysis of the moduli space and of the obstructions to the definition of Lagrangian Floer homology has been carried out in the recent work of Fukaya, Oh, Ohta and Ono [FO^3], refining the earlier work of Floer and Oh on technically easier special cases.

When Floer homology is well-defined, it has important consequences on the intersection properties of Lagrangian submanifolds. Indeed, for every *Hamiltonian* diffeomorphism ϕ we have $HF^*(L_0, L_1) = HF^*(L_0, \phi(L_1))$; and if L_0 and L_1 intersect transversely, then the total rank of $HF^*(L_0, L_1)$ gives a lower bound on the number of intersection points of L_0 and L_1. Moreover, $HF^*(L, L)$ is related to the usual cohomology $H^*(L)$ via a spectral sequence which degenerates in the case where $i_* : H_*(L, \mathbb{Q}) \to H_*(M, \mathbb{Q})$ is injective. Therefore, given a compact orientable relatively spin Lagrangian submanifold $L \subset (M, \omega)$ such that $i_* : H_*(L, \mathbb{Q}) \to H_*(M, \mathbb{Q})$ is injective, for any Hamiltonian diffeomorphism $\phi : (M, \omega) \to (M, \omega)$ such that $\phi(L)$ is transverse to L, we have

$$\#(L \cap \phi(L)) \geq \sum_k \operatorname{rank} H_k(L, \mathbb{Q}). \tag{3}$$

For example, consider a symplectic manifold of the form $(M \times M, \omega \oplus -\omega)$, and let L_0 be the diagonal, and L_1 be the graph of a Hamiltonian diffeomorphism $\psi : (M, \omega) \to (M, \omega)$. Then L_0 and L_1 are Lagrangian submanifolds, and (3) yields the Arnold conjecture for the fixed points of ψ.

Besides a differential, Floer complexes for Lagrangians are also equipped with a product structure, i.e. a morphism $CF^*(L_0, L_1) \otimes CF^*(L_1, L_2) \to CF^*(L_0, L_2)$ (assuming that the obstruction classes vanish for the given Lagrangians L_0, L_1, L_2). This product structure is defined as follows: consider three points $p_1 \in L_0 \cap L_1$, $p_2 \in L_1 \cap L_2$, $p_3 \in L_0 \cap L_2$, together with the equivalence classes $[u_i]$ needed to specify generators of CF^*. Consider the moduli space $\mathcal{M}(p_1, p_2, p_3; A)$ of all pseudo-holomorphic maps f from a disc with three marked points q_1, q_2, q_3 on its boundary to M, such that $f(q_i) = p_i$ and the three portions of boundary delimited by the marked points are mapped to L_0, L_1, L_2 respectively, and realizing a given relative homology class A. We compactify this moduli space and complete it if necessary in order to obtain a well-defined fundamental cycle. Assuming that the relative homology class A is compatible with the given equivalence classes $[u_i]$, the virtual dimension of this moduli space is the difference between the Maslov index of $(p_3, [u_3])$ and the sum of those of $(p_1, [u_1])$ and $(p_2, [u_2])$. The product of $(p_1, [u_1])$ and $(p_2, [u_2])$ is then defined as

$$(p_1, [u_1]) \cdot (p_2, [u_2]) = \sum_{p_3, A} \#\mathcal{M}(p_1, p_2, p_3; A)\, (p_3, [u_3]), \tag{4}$$

where the sum runs over all pairs (p_3, A) for which the expected dimension of the moduli space is zero, and for each such pair $[u_3]$ is the equivalence class determined by $[u_1]$, $[u_2]$ and A.

While the product structure on CF^* defined by (4) satisfies the Leibniz rule with respect to the differential ∂ (and hence descends to a product structure on Floer homology), it differs from usual products by the fact that it is only associative *up to homotopy.* In fact, Floer complexes come equipped with a full set of *higher-order products*

$$\mu^n : CF^*(L_0, L_1) \otimes \cdots \otimes CF^*(L_{n-1}, L_n) \to CF^*(L_0, L_n) \quad \text{for all } n \geq 1,$$

with each μ^n shifting degree by $2 - n$. The first two maps μ^1 and μ^2 are respectively the Floer differential ∂ and the product described above. The definition of μ^n is similar to those of ∂ and of the product structure: given generators $(p_i, [u_i]) \in CF^*(L_{i-1}, L_i)$ for $1 \leq i \leq n$ and $(p_{n+1}, [u_{n+1}]) \in CF^*(L_0, L_n)$ such that $\deg(p_{n+1}, [u_{n+1}]) = \sum_{i=1}^{n} \deg(p_i, [u_i]) + 2 - n$, the coefficient of $(p_{n+1}, [u_{n+1}])$ in $\mu^n((p_1, [u_1]), \ldots, (p_n, [u_n]))$ is obtained by counting (in a suitable sense) pseudo-holomorphic maps f from a disc with $n+1$ marked points $q_1, \ldots, q_{n+1}$ on its boundary to M, such that $f(q_i) = p_i$ and the portions of boundary delimited by the marked points are mapped to $L_0, \ldots, L_n$ respectively, and representing a relative homology class compatible with the given data $[u_i]$.

Assume that there are no non-trivial pseudo-holomorphic discs with boundary contained in one of the considered Lagrangian submanifolds (otherwise one needs to introduce a μ^0 term as well, which is perfectly legitimate but makes the structure much more different from that of a usual category): then the maps $(\mu^n)_{n \geq 1}$ define an *A_∞-structure* on Floer complexes, i.e. they satisfy an infinite sequence of algebraic relations:

$$\begin{cases} \mu^1(\mu^1(a)) = 0, \\ \mu^1(\mu^2(a, b)) = \mu^2(\mu^1(a), b) + (-1)^{\deg a} \mu^2(a, \mu^1(b)), \\ \mu^1(\mu^3(a, b, c)) = \mu^2(\mu^2(a, b), c) - \mu^2(a, \mu^2(b, c)) \\ \qquad\qquad\qquad \pm \mu^3(\mu^1(a), b, c) \pm \mu^3(a, \mu^1(b), c) \pm \mu^3(a, b, \mu^1(c)), \\ \ldots \end{cases}$$

This leads to the concept of "Fukaya category" of a symplectic manifold. Conjecturally, for every symplectic manifold (M, ω) one should be able to define an A_∞-category $\mathcal{F}(M)$ whose objects are Lagrangian submanifolds (compact, orientable, relatively spin, "twisted" by a flat unitary vector bundle); the space of morphisms between two objects L_0 and L_1 is the Floer complex $CF^*(L_0, L_1)$ equipped with its differential $\partial = \mu^1$, with (non-associative) composition given by the product μ^2, and higher order compositions μ^n.

The importance of Fukaya categories in modern symplectic topology is largely due to the *homological mirror symmetry* conjecture, formulated by Kontsevich.

Very roughly, this conjecture states that the phenomenon of mirror symmetry, i.e. a conjectural correspondence between symplectic manifolds and complex manifolds ("mirror pairs") arising from a duality among string theories, should be visible at the level of Fukaya categories of symplectic manifolds and categories of coherent sheaves on complex manifolds: given a mirror pair consisting of a symplectic manifold M and a complex manifold X, the derived categories $D\mathcal{F}(M)$ and $D^bCoh(X)$ should be equivalent (in a more precise form of the conjecture, one should actually consider families of manifolds and deformations of categories). However, due to the very incomplete nature of our understanding of Fukaya categories in comparison to the much better understood derived categories of coherent sheaves, this conjecture has so far only been verified on very specific examples.

1.7. **The topology of symplectic 4-manifolds.** An important question in symplectic topology is to determine which smooth manifolds admit symplectic structures. In the case of open manifolds, Gromov's h-principle implies that the existence of an almost-complex structure is sufficient. In contrast, the case of compact manifolds is much less understood, except in dimension 4.

Whereas the existence of a class $\alpha \in H^2(M,\mathbb{R})$ such that $\alpha^{\cup n} \neq 0$ and of an almost-complex structure already provide elementary obstructions to the existence of a symplectic structure on a given compact manifold, in the case of 4-manifolds a much stronger obstruction arises from *Seiberg-Witten invariants*.

The Seiberg-Witten equations are defined on a compact 4-manifold M equipped with a Riemannian metric and a spin^c structure $\mathfrak{s}$, characterized by a pair of rank 2 complex vector bundles $S^{\pm}$ (positive and negative spinors, interchanged by the *Clifford action* of the tangent bundle), with the same determinant line bundle L. The choice of a Hermitian connection A on L determines (together with the Levi-Civita connection on the tangent bundle) a spin^c connection on $S^{\pm}$, which in turn yields a Dirac operator $D_A : \Gamma(S^{\pm}) \to \Gamma(S^{\mp})$ (obtained by contraction of the connection operator with the Clifford action); moreover, the bundle of self-dual 2-forms $\Lambda^2_+ T^*M$ is canonically isomorphic (via the Clifford action) to that of traceless antihermitian endomorphisms of S^+. With this understood, the Seiberg-Witten equations are the following equations for a pair (A,ψ) consisting of a $U(1)$ connection on the determinant line bundle L and a section of the bundle of positive spinors [Wi]:

$$\begin{cases} D_A\psi = 0 \\ F_A^+ = q(\psi) + \mu \end{cases} \tag{5}$$

where $q(\psi)$ is the imaginary self-dual 2-form corresponding to the traceless part of $\psi^* \otimes \psi \in \mathrm{End}(S^+)$ and μ is a constant parameter. The space of solutions to (5) is invariant under $U(1)$-valued gauge transformations $(A,\psi) \mapsto (A + 2g^{-1}dg, g\psi)$, and the quotient moduli space $\mathcal{M}(L)$ is compact, orientable, smooth for generic choice of the parameter μ whenever $b_2^+(M) \neq 0$, and of expected dimension $d = \frac{1}{4}(c_1(L)^2 - 2\chi(M) - 3\sigma(M))$; the Seiberg-Witten invariants of X are then defined by counting points of the moduli space (with signs): when $d = 0$ (the most important case) we set $SW(L) = \#\mathcal{M}(L)$. Whenever $b_2^+(M) \geq 2$ this invariant depends only on the manifold M and the given spin^c-structure; when $b_2^+(M) = 1$ there are two different chambers depending on the choice of the metric and the parameter μ, giving rise to possibly different values of the invariant.

The most important results concerning the Seiberg-Witten invariants of symplectic 4-manifolds have been obtained by Taubes ([Ta1], [Ta2], ...). They are summarized in the following statement:

Theorem 1.12 (Taubes). *Let (M^4, ω) be a compact symplectic 4-manifold with $b_2^+ \geq 2$. Then:*
(i) $SW(K_M^{\pm 1}) = \pm 1$;
(ii) $c_1(K_M) \cdot [\omega] \geq 0$, and $SW(L) = 0$ whenever $|c_1(L) \cdot [\omega]| > c_1(K_M) \cdot [\omega]$.
(iii) $SW(K_M^{-1} + 2e) = Gr_T(e)$, where Gr_T is a specific version of Gromov-Witten invariants (counting possibly disconnected pseudo-holomorphic curves, with special weights attributed to multiply covered square zero tori);
(iv) the homology class $c_1(K_M)$ admits a (possibly disconnected) pseudo-holomorphic representative, every component of which satisfies $g = 1 + [C_i] \cdot [C_i]$. Hence, if M is minimal i.e. contains no (-1)-spheres, then $c_1(K_M)^2 = 2\chi(M) + 3\sigma(M) \geq 0$.

These criteria prevent many 4-manifolds from admitting symplectic structures, e.g. those which decompose as the connected sum of two manifolds with $b_2^+ \geq 1$ (by a result of Witten, their SW invariants vanish): for example, $\mathbb{CP}^2 \# \mathbb{CP}^2 \# \mathbb{CP}^2$ does not admit any symplectic structure even though it satisfies the cohomological condition for the existence of an almost-complex structure (there exists a class $c \in H^2(M, \mathbb{Z})$ such that $c^2 = 2\chi + 3\sigma = 19$).

When $b_2^+(M) = 1$, some of the statements remain valid (being careful about the choice of chamber for SW invariants). Using Gromov's characterization of the Fubini-Study symplectic structure of $\mathbb{CP}^2$ in terms of the existence of pseudo-holomorphic lines, Taubes has shown that the symplectic structure of $\mathbb{CP}^2$ is unique up to scaling. This result has been extended by Lalonde and McDuff to the case of rational ruled surfaces, where ω is determined by its cohomology class.

In parallel to the above constraints on symplectic 4-manifolds, surgery techniques have led to many interesting new examples of compact symplectic manifolds.

One of the most efficient techniques in this respect is the *symplectic sum* construction, investigated by Gompf [Go1]: if two symplectic manifolds (M_1^{2n}, ω_1) and (M_2^{2n}, ω_2) contain compact symplectic hypersurfaces W_1^{2n-2}, W_2^{2n-2} that are mutually symplectomorphic and whose normal bundles have opposite Chern classes, then we can cut M_1 and M_2 open along the submanifolds W_1 and W_2, and glue them to each other along their common boundary, performing a fiberwise connected sum in the normal bundles to W_1 and W_2, to obtain a new symplectic manifold $M = M_1{}_{W_1}\#_{W_2} M_2$. This construction has in particular allowed Gompf to show that every finitely presented group can be realized as the fundamental group of a compact symplectic 4-manifold.

A more specifically 4-dimensional construction is the *link surgery* technique developed by Fintushel and Stern [FS1]. By modifying a given 4-manifold in the neighborhood of an embedded torus with trivial normal bundle, it allows one to construct large families of mutually homeomorphic 4-manifolds whose diffeomorphism types can be distinguished using Seiberg-Witten invariants; these manifolds are symplectic in some cases. More can be found about 4-dimensional surgery techniques in the book [GS]; we will also discuss some symplectic surgery constructions in more detail in the sections below.

By comparing the available examples and the topological constraints arising from Seiberg-Witten invariants, one can get a fairly good understanding of the topology of compact symplectic 4-manifolds. Nonetheless, a large number of questions remain open, in particular concerning symplectic manifolds of *general type*, i.e. with $b_2^+ \geq 2$ and $c_1(K_M)^2 > 0$. For example, Seiberg-Witten invariants fail to provide any useful information on complex surfaces of general type, whose diffeomorphism types are hence not well understood; similarly, it is unknown to this date whether the Bogomolov-Miyaoka-Yau inequality $c_1^2 \leq 3c_2$, satisfied by all complex surfaces of general type, also holds in the symplectic case.

2. Symplectic Lefschetz fibrations

2.1. **Symplectic fibrations.** The first construction of a symplectic non-Kähler manifold is due to Thurston [Th], who showed the existence of a symplectic structure on a T^2-bundle over T^2 with $b_1 = 3$ (whereas Kähler manifolds always have even b_1 since their first cohomology splits into $H^{1,0} \oplus H^{0,1}$). The existence of a symplectic structure on this manifold follows from a general result about symplectic fibrations (see also §6 of [McS]):

Theorem 2.1 (Thurston)**.** *Let $f : M \to B$ be a compact locally trivial fibration with symplectic fiber (F, ω_F) and symplectic base (B, ω_B). Assume that the structure group of the fibration reduces to the symplectomorphisms of F, and that there exists a cohomology class $c \in H^2(M, \mathbb{R})$ whose restriction to the fiber is equal to $[\omega_F]$. Then, for all sufficiently large $K > 0$, M admits a symplectic form in the cohomology class $c + K\, f^*[\omega_B]$, for which all fibers of f are symplectic submanifolds.*

When the fibers are 2-dimensional, many of the assumptions of this theorem are always satisfied: the fiber always admits a symplectic structure (provided it is orientable), and the structure group of the fibration always reduces to symplectomorphisms; moreover, the cohomological condition is equivalent to the requirement that the fibers of f represent a non-zero class in $H_2(M, \mathbb{R})$.

Proof. Let $\eta \in \Omega^2(M)$ be a closed 2-form representing the cohomology class c. Cover the base B by balls U_i over which the fibration is trivial: we have a diffeomorphism $\phi_i : f^{-1}(U_i) \to U_i \times F$, and the assumption on the structure group of the fibration means that, over $U_i \cap U_j$, the trivializations ϕ_i and ϕ_j differ by symplectomorphisms of the fibers. The diffeomorphism ϕ_i determines a projection $p_i : f^{-1}(U_i) \to F$ such that $\phi_i(x) = (f(x), p_i(x))$ for every $x \in f^{-1}(U_i)$.

After restriction to $f^{-1}(U_i) \simeq U_i \times F$, the closed 2-forms η and $p_i^*\omega_F$ belong to the same cohomology class, so that we can write $p_i^*\omega_F = \eta + d\alpha_i$ for some 1-form α_i over $f^{-1}(U_i)$. Let $\{\rho_i\}$ be a partition of unity by smooth functions $\rho_i : B \to [0,1]$ supported over the balls U_i, and let $\tilde\eta = \eta + \sum_i d((\rho_i \circ f)\, \alpha_i)$. The 2-form $\tilde\eta$ is well-defined since the support of $\rho_i \circ f$ is contained in $f^{-1}(U_i)$, and it is obviously closed and cohomologous to η. Moreover, over $F_p = f^{-1}(p)$ for any $p \in B$, we have $\tilde\eta_{|F_p} = \eta_{|F_p} + \sum_i \rho_i(p) d\alpha_{i|F_p} = \sum_i \rho_i(p)\, (\eta + d\alpha_i)_{|F_p} = \sum_i \rho_i(p)\, (p_i^*\omega_F)_{|F_p}$. However since the local trivializations ϕ_i differ by symplectomorphisms of the fiber, the 2-forms $p_i^*\omega_F$ are all equal to each other. Therefore, the 2-form $\tilde\eta_{|F_p}$ can be identified with ω_F (recall that $\sum \rho_i \equiv 1$).

So far we have constructed a closed 2-form $\tilde\eta \in \Omega^2(M)$, with $[\tilde\eta] = c$, whose restriction to any fiber of f is symplectic (and in fact coincides with ω_F). At any point $x \in M$, the tangent space T_xM splits into a vertical subspace $V_x = \operatorname{Ker} df_x$ and a horizontal subspace $H_x = \{v \in T_xM,\ \tilde\eta(v, v') = 0\ \forall v' \in V_x\}$. Since the restriction of $\tilde\eta$ to the vertical subspace is non-degenerate, we have $T_xM = H_x \oplus V_x$, and so $f^*\omega_B$ is non-degenerate over H_x. Therefore, for sufficiently large $K > 0$ the 2-form $\tilde{\eta} + K\, f^*\omega_B$ is non-degenerate over H_x, and since its restriction to V_x coincides with $\tilde{\eta}$, it is also non-degenerate over T_xM. Using the compactness of M we can find a constant $K > 0$ for which this property holds at every point; the form $\tilde{\eta} + K\, f^*\omega_B$ then defines a symplectic structure on M. $\square$

Thurston's example of a non-Kähler symplectic manifold is obtained in the following way: start with the trivial bundle with fiber $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ (with coordinates $x, y \in \mathbb{R}/\mathbb{Z}$) over $\mathbb{R}^2$ (with coordinates z, t), and quotient it by the action of $\mathbb{Z}^2$ generated by $((x, y), (z, t)) \mapsto ((x, y), (z + 1, t))$ and $((x, y), (z, t)) \mapsto ((x+y, y), (z, t+1))$. This action of $\mathbb{Z}^2$ maps fibers to fibers, and hence the quotient M carries a structure of T^2-fibration over $\mathbb{R}^2/\mathbb{Z}^2 = T^2$. The fiber class is homologically non-trivial since it has intersection number 1 with the section $\{x = y = 0\}$, and the monodromy of the fibration is given by symplectomorphisms of the fiber ($(x, y) \mapsto (x, y)$ and $(x, y) \mapsto (x + y, y)$). Therefore, by Theorem 2.1 the compact 4-manifold M admits a symplectic structure. Let $\gamma_1, \dots, \gamma_4$ be the closed loops in M corresponding to the four coordinate axes: by translating γ_2 along the t axis, one can deform it into a loop homologous to $\gamma_1 + \gamma_2$ in the fiber, so that $[\gamma_1] = 0$ in $H_1(M, \mathbb{Z})$. On the other hand, it is not hard to see that $[\gamma_2], [\gamma_3], [\gamma_4]$ are linearly independent and generate $H_1(M, \mathbb{Z}) \simeq \mathbb{Z}^3$. Hence $b_1(M) = 3$ and M does not admit any Kähler structure (on the other hand, M actually carries an integrable complex structure, but it is not compatible with any symplectic form).

2.2. **Symplectic Lefschetz fibrations.**

Definition 2.2. *A map f from a compact oriented manifold M^{2n} to the sphere S^2 (or more generally a compact oriented Riemann surface) is a* Lefschetz fibration *if the critical points of f are isolated, and for every critical point $p \in M$ the map f is modelled on a complex Morse function, i.e. there exist neighborhoods $U \ni p$ and $V \ni f(p)$ and orientation-preserving local diffeomorphisms $\phi : U \to \mathbb{C}^n$ and $\psi : V \to \mathbb{C}$ such that $\psi \circ f \circ \phi^{-1}$ is the map $(z_1, \dots, z_n) \mapsto \sum z_i^2$.*

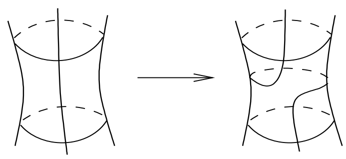
To simplify the description, we can additionally require that the critical values of f are all distinct (so that each fiber contains at most one singular point).

The local model near a singular fiber is easiest to understand in the 4-dimensional case. The fiber F_λ of the map $(z_1, z_2) \to z_1^2 + z_2^2$ above $\lambda \in \mathbb{C}$ is given by the equation $(z_1 + iz_2)(z_1 - iz_2) = \lambda$: the fiber F_λ is smooth (topologically a cylinder) for all $\lambda \neq 0$, while the fiber above the origin presents a transverse double point, and is obtained from the nearby fibers by collapsing an embedded simple closed loop called the *vanishing cycle*. For example, for $\lambda > 0$ the vanishing cycle is the loop $\{(x_1, x_2) \in \mathbb{R}^2,\ x_1^2 + x_2^2 = \lambda\} = F_\lambda \cap \mathbb{R}^2 \subset F_\lambda$. In arbitrary dimension, the fiber over a critical point of f presents an *ordinary double point*, and the nearby fibers are smoothings of this singularity; this can be seen in the local model, where the singular fiber $F_0 = \{\sum z_i^2 = 0\} \subset \mathbb{C}^n$ is obtained from a nearby smooth fiber $F_\lambda = \{\sum z_i^2 = \lambda\}$ by collapsing the vanishing cycle $S_\lambda = F_\lambda \cap (e^{i\theta/2}\mathbb{R})^n \subset F_\lambda$ (where $\theta = \arg(\lambda)$). In fact, S_λ is obtained from the unit sphere in $\mathbb{R}^n \subset \mathbb{C}^n$ by multiplication by $\lambda^{1/2}$, and F_λ is diffeomorphic to the cotangent bundle T^*S_λ.

Fix a base point $q_0 \in S^2 - \mathrm{crit}(f)$, and consider a closed loop $\gamma : [0, 1] \to S^2 - \mathrm{crit}(f)$ (starting and ending at q_0). By fixing a horizontal distribution we can perform parallel transport in the fibers of f along γ, which induces a diffeomorphism from $F_{q_0} = f^{-1}(q_0)$ to itself. The isotopy class of this diffeomorphism, which is well-defined independently of the chosen horizontal distribution, is called the *monodromy* of f along γ. Hence, we obtain a monodromy homomorphism characteristic of the Lefschetz fibration f,

$$\psi : \pi_1(S^2 - \mathrm{crit}(f), q_0) \to \pi_0 \mathrm{Diff}^+(F_{q_0}).$$

By considering the local model near a critical point of f, one can show that the monodromy of f around one of its singular fibers is a *positive Dehn twist* along the vanishing cycle – a diffeomorphism supported in a neighborhood of the vanishing cycle, and inducing the antipodal map on the vanishing cycle itself (recall from above that in the local model we have $S_\lambda = F_\lambda \cap (e^{i\theta/2}\mathbb{R})^n$). The effect of this Dehn twist is most easily seen when $\dim M = 4$: in the fiber F_{q_0}, a tubular neighborhood of the vanishing cycle can be identified with a cylinder $\mathbb{R} \times S^1$, and the opposite ends of the cylinder twist relatively to each other as one moves around the singular fiber, as shown in the figure below.



In higher dimension, a neighborhood of the vanishing cycle is diffeomorphic to the neighborhood $\{(x, u) \in T^*S^{n-1},\ |u| \leq \pi\}$ of the zero section in the cotangent bundle of S^{n-1}, and the Dehn twist is the diffeomorphism supported in this neighborhood obtained by composing the time 1 geodesic flow on T^*S^{n-1} with the differential of the antipodal involution of S^{n-1}.

We are particularly interested in the case where the total space M carries a symplectic structure compatible with the fibration:

Definition 2.3. *A* symplectic Lefschetz fibration $f : (M, \omega) \to S^2$ *is a Lefschetz fibration whose total space carries a symplectic structure with the following properties:* (i) *the fibers of* f *are symplectic submanifolds;* (ii) *near a critical point of* f*, there exist neighborhoods* $U \ni p$ *and* $V \ni f(p)$ *and orientation-preserving local diffeomorphisms* $\phi : U \to \mathbb{C}^n$ *and* $\psi : V \to \mathbb{C}$*, such that the symplectic form* $\phi_*\omega$ *evaluates positively on all complex lines in* $\mathbb{C}^n$ $(\phi_*\omega(v, iv) > 0\ \forall v \neq 0)$*, and such that* $\psi \circ f \circ \phi^{-1}$ *is the map* $(z_1, \dots, z_n) \mapsto \sum z_i^2$.

By considering at each point $p \in M$ where $df \neq 0$ the subspace of T_pM symplectically orthogonal to the fiber through p, one obtains a specific horizontal distribution for which parallel transport preserves the symplectic structures of the fibers of f. Hence, the monodromy of a *symplectic Lefschetz fibration* can be defined with values into the *symplectic mapping class group*, i.e. the group $\pi_0\mathrm{Symp}(F)$ of isotopy classes of symplectomorphisms of the fiber. Moreover, the vanishing cycles are always *Lagrangian* spheres in the fibers (this is e.g. clearly true in the local model in $\mathbb{C}^n$ equipped with its standard symplectic structure; more generally, it follows easily from the fact that parallel transport preserves the symplectic structure and collapses the vanishing cycle into the critical point of a singular fiber).

In the case of Lefschetz fibrations over a disc, the monodromy homomorphism is sufficient to reconstruct the total space of the fibration up to diffeomorphism (resp. symplectic deformation). When considering fibrations over S^2, the monodromy data determines the fibration over a large disc D containing all critical values, after which we only need to add a trivial fibration over a small disc $D' = S^2 - D$, to be glued in a manner compatible with the fibration structure over the common boundary $\partial D = \partial D' = S^1$. This gluing involves the choice of a map from S^1 to the group of diffeomorphisms (resp. symplectomorphisms) of the fiber, i.e. an element of $\pi_1 \mathrm{Diff}(F)$ or $\pi_1 \mathrm{Symp}(F)$.

In particular, in the 4-dimensional case, the total space of a (symplectic) Lefschetz fibration with fibers of genus $g \geq 2$ is completely determined by its monodromy, up to diffeomorphism (resp. symplectic deformation).

In view of the above discussion, it is not very surprising that a result similar to Theorem 2.1 holds for Lefschetz fibrations [GS, Go2]:

Theorem 2.4 (Gompf)**.** *Let $f : M^{2n} \to S^2$ be a Lefschetz fibration with symplectic fiber (F, ω_F). Assume that the structure group of the fibration reduces to the symplectomorphisms of F, and in particular that the vanishing cycles are embedded Lagrangian spheres in (F, ω_F). Assume moreover that there exists a cohomology class $c \in H^2(M, \mathbb{R})$ whose restriction to the fiber is equal to $[\omega_F]$; in the case $n = 2$, the cohomology class c is also required to evaluate positively on every component of every reducible singular fiber. Then, for all sufficiently large $K > 0$, M admits a symplectic form in the cohomology class $c + K\, f^*[\omega_B]$, for which all fibers of f are symplectic submanifolds.*

The proof of Theorem 2.4 is essentially a refinement of the argument given for Theorem 2.1. The main difference is that one first uses the local models near the critical points of f in order to build an exact perturbation of η with the desired behavior near these points; in the subsequent steps, one then works with differential forms with support contained in the complement of small balls centered at the critical points [Go2].

An important motivation for the study of symplectic Lefschetz fibrations is the fact that, up to blow-ups, every compact symplectic manifold carries such a structure, as shown by Donaldson [Do2, Do3]:

Theorem 2.5 (Donaldson)**.** *For any compact symplectic manifold (M^{2n}, ω), there exists a smooth codimension 4 symplectic submanifold $B \subset M$ such that the blow-up of M along B carries a structure of symplectic Lefschetz fibration over S^2.*

In particular, this result (together with the converse statement of Gompf) provides a very elegant topological description of symplectic 4-manifolds. In order to describe Donaldson's construction of symplectic Lefschetz pencils, we need a digression into *approximately holomorphic geometry.*

2.3. **Approximately holomorphic geometry.** The idea introduced by Donaldson in the mid-90's is the following: in presence of an almost-complex sructure, the lack of integrability usually prevents the existence of holomorphic sections of vector bundles or pseudo-holomorphic maps to other manifolds, but one can work in a similar manner with approximately holomorphic objects.

Let (M^{2n}, ω) be a compact symplectic manifold of dimension $2n$. We will assume throughout this paragraph that $\frac{1}{2\pi}[\omega] \in H^2(M, \mathbb{Z})$; this integrality condition does not restrict the topological type of M, since any symplectic form can be perturbed into another symplectic form ω' whose cohomology class is rational (we can then achieve integrality by multiplication by a constant factor). Morever, it is easy to check that the submanifolds of M that we will construct are not only ω'-symplectic but also ω-symplectic, hence making the general case of Theorem 2.5 follow from the integral case.

Let J be an almost-complex structure compatible with ω, and let $g(.,.) = \omega(., J.)$ be the corresponding Riemannian metric. We consider a complex line bundle L over M such that $c_1(L) = \frac{1}{2\pi}[\omega]$, endowed with a Hermitian metric and a Hermitian connection ∇^L with curvature 2-form $F(\nabla^L) = -i\omega$. The almost-complex structure induces a splitting of the connection : $\nabla^L = \partial^L + \bar\partial^L$, where $\partial^L s(v) = \frac{1}{2}(\nabla^L s(v) - i\nabla^L s(Jv))$ and $\bar\partial^L s(v) = \frac{1}{2}(\nabla^L s(v) + i\nabla^L s(Jv))$.

If the almost-complex structure J is integrable, i.e. if M is a Kähler complex manifold, then L is an ample holomorphic line bundle, and for large enough values of k the line bundles $L^{\otimes k}$ admit many holomorphic sections. Therefore, the manifold M can be embedded into a projective space (Kodaira); generic hyperplane sections are smooth hypersurfaces in M (Bertini), and more generally the linear system formed by the sections of $L^{\otimes k}$ allows one to construct various structures (Lefschetz pencils, ...).

When the manifold M is only symplectic, the lack of integrability of J prevents the existence of holomorphic sections. Nonetheless, it is possible to find an *approximately holomorphic* local model: a neighborhood of a point $x \in M$, equipped with the symplectic form ω and the almost-complex structure J, can be identified with a neighborhood of the origin in $\mathbb{C}^n$ equipped with the standard symplectic form ω_0 and an almost-complex structure of the form $i + O(|z|)$. In this local model, the line bundle $L^{\otimes k}$ endowed with the connection $\nabla = (\nabla^L)^{\otimes k}$ of curvature $-ik\omega$ can be identified with the trivial line bundle $\underline{\mathbb{C}}$ endowed with the connection $d + \frac{k}{4}\sum(z_j\, d\bar z_j - \bar z_j\, dz_j)$. The section of $L^{\otimes k}$ given in this trivialization by $s_{k,x}(z) = \exp(-\frac{1}{4}k|z|^2)$ is then approximately holomorphic [Do1].

More precisely, a sequence of sections s_k of $L^{\otimes k}$ is said to be approximately holomorphic if, with respect to the rescaled metrics $g_k = kg$, and after normalization of the sections to ensure that $\|s_k\|_{C^r, g_k} \sim C$, an inequality of the form $\|\bar\partial s_k\|_{C^{r-1}, g_k} < C' k^{-1/2}$ holds, where C and C' are constants independent of k. The change of metric, which dilates all distances by a factor of $\sqrt{k}$, is required in order to be able to obtain uniform estimates, due to the larger and larger curvature of the line bundle $L^{\otimes k}$. The intuitive idea is that, for large k, the sections of the line bundle $L^{\otimes k}$ with curvature $-ik\omega$ probe the geometry of M at small scale ($\sim 1/\sqrt{k}$), which makes the almost-complex structure J almost integrable and allows one to achieve better and better approximations of the holomorphicity condition $\bar\partial s = 0$.

It is worth noting that, since the above requirement is an open condition, it is not possible to define a "space of approximately holomorphic sections" of $L^{\otimes k}$ in any simple manner (cf. the work of Borthwick and Uribe [BU] or Shiffman and Zelditch for other approaches to this problem).

Once many approximately holomorphic sections have been made available, the aim is to find among them some sections whose geometric behavior is as generic as possible. Donaldson has obtained the following result [Do1]:

Theorem 2.6 (Donaldson)**.** *For $k \gg 0$, $L^{\otimes k}$ admits approximately holomorphic sections s_k whose zero sets W_k are smooth symplectic hypersurfaces.*

This result starts from the observation that, if the section s_k vanishes transversely and if $|\bar\partial s_k(x)| \ll |\partial s_k(x)|$ at every point of $W_k = s_k^{-1}(0)$, then the submanifold W_k is symplectic, and even approximately J-holomorphic (i.e. $J(TW_k)$ is close to TW_k). The crucial point is therefore to obtain a lower bound for ∂s_k at every point of W_k, in order to make up for the lack of holomorphicity.

Sections s_k of $L^{\otimes k}$ are said to be *uniformly transverse to* 0 if there exists a constant $\eta > 0$ (independent of k) such that the inequality $|\partial s_k(x)|_{g_k} > \eta$ holds at any point of M where $|s_k(x)| < \eta$. In order to prove Theorem 2.6, it is sufficient to achieve this uniform estimate on the tranversality of some approximately holomorphic sections s_k. The idea of the construction of such sections consists of two main steps. The first one is an effective local transversality result for complex-valued functions. Donaldson's argument makes use of a result of Yomdin on the complexity of real semi-algebraic sets; however a somewhat simpler argument can be used instead [Au5]. The second step is a remarkable globalization process, which makes it possible to achieve uniform transversality over larger and larger open subsets by means of successive perturbations of the sections s_k, until transversality holds over the entire manifold M [Do1].

The symplectic submanifolds constructed by Donaldson present several remarkable properties which make them closer to complex submanifolds than to arbitrary symplectic submanifolds. For instance, they satisfy the Lefschetz hyperplane theorem: up to half the dimension of the submanifold, the homology and homotopy groups of W_k are identical to those of M [Do1]. More importantly, these submanifolds are, in a sense, asymptotically unique: for given large enough k, the submanifolds W_k are, up to symplectic isotopy, independent of all the choices made in the construction (including that of the almost-complex structure J) [Au1].

It is worth mentioning that analogues of Donaldson's construction for contact manifolds have been obtained by Ibort, Martinez-Torres and Presas ([IMP], ...); see also recent work of Giroux and Mohsen [GM].

2.4. **Symplectic Lefschetz pencils.** We now move on to Donaldson's construction of symplectic Lefschetz pencils [Do2, Do3]. In comparison with Theorem 2.6, the general setup is the same, the main difference being that we consider no longer one, but two sections of $L^{\otimes k}$. A pair of suitably chosen approximately holomorphic sections (s^0_k, s^1_k) of $L^{\otimes k}$ defines a family of symplectic hypersurfaces

$$\Sigma_{k,\alpha} = \{x \in M,\ s^0_k(x) - \alpha s^1_k(x) = 0\},\quad \alpha \in \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}.$$

The submanifolds $\Sigma_{k,\alpha}$ are all smooth except for finitely many of them which present an isolated singularity; they intersect transversely along the *base points* of the pencil, which form a smooth symplectic submanifold $Z_k = \{s^0_k = s^1_k = 0\}$ of codimension 4.

The two sections s^0_k and s^1_k determine a projective map $f_k = (s^0_k : s^1_k) : M - Z_k \to \mathbb{CP}^1$, whose critical points correspond to the singularities of the fibers $\Sigma_{k,\alpha}$. In the case of a symplectic Lefschetz pencil, the function f_k is a complex Morse function, i.e. near any of its critical points it is given by the local model $f_k(z) = z_1^2 + \cdots + z_n^2$ in approximately holomorphic coordinates. After blowing up M along Z_k, the Lefschetz pencil structure on M gives rise to a well-defined map $\hat{f}_k : \hat{M} \to \mathbb{CP}^1$; this map is a symplectic Lefschetz fibration. Hence, Theorem 2.5 may be reformulated more precisely as follows:

Theorem 2.7 (Donaldson)**.** *For large enough k, the given manifold (M^{2n}, ω) admits symplectic Lefschetz pencil structures determined by pairs of suitably chosen approximately holomorphic sections s^0_k, s^1_k of $L^{\otimes k}$. Moreover, for large enough k these Lefschetz pencil structures are uniquely determined up to isotopy.*

As in the case of submanifolds, Donaldson's argument relies on successive perturbations of given approximately holomorphic sections s_k^0 and s_k^1 in order to achieve uniform transversality properties, not only for the sections (s_k^0, s_k^1) themselves but also for the derivative ∂f_k [Do3].

The precise meaning of the uniqueness statement is the following: assume we are given two sequences of Lefschetz pencil structures on (M, ω), determined by pairs of approximately holomorphic sections of $L^{\otimes k}$ satisfying uniform transversality estimates, but possibly with respect to two different ω-compatible almost-complex structures on M. Then, beyond a certain (non-explicit) value of k, it becomes possible to find one-parameter families of Lefschetz pencil structures interpolating between the given ones. In particular, this implies that for large k the monodromy invariants associated to these Lefschetz pencils only depend on (M, ω, k) and not on the choices made in the construction.

The monodromy invariants associated to a symplectic Lefschetz pencil are essentially those of the symplectic Lefschetz fibration obtained after blow-up along the base points, with only a small refinement. After the blow-up operation, each fiber of $\hat{f}_k : \hat{M} \to \mathbb{CP}^1$ contains a copy of the base locus Z_k embedded as a smooth symplectic hypersurface. This hypersurface lies away from all vanishing cycles, and is preserved by the monodromy. Hence, the monodromy homomorphism can be defined to take values in the group of isotopy classes of symplectomorphisms of the fiber Σ_k whose restriction to the submanifold Z_k is the identity.

We can even do better by carefully examining the local model near the base points and observing that parallel transport (with respect to the natural symplectic connection) along a loop $\gamma \subset \mathbb{CP}^1 - \mathrm{crit}(\hat{f}_k)$ acts on a neighborhood of Z_k by complex multiplication by $e^{-2i\pi A(\gamma)}$ in the fibers of the normal bundle, where $A(\gamma)$ is the proportion of the area of $\mathbb{CP}^1$ enclosed by γ. If we remove a regular value of $\hat{f}_k$ (e.g. the point at infinity in $\mathbb{CP}^1$), the area enclosed by a given oriented loop becomes a well-defined element of $\mathbb{R}$ (rather than $\mathbb{R}/\mathbb{Z}$), so that we can unambiguously represent every homotopy class by a loop for which the enclosed area is equal to zero. This makes it possible to define a monodromy homomorphism

$$\psi_k : \pi_1(\mathbb{C} - \mathrm{crit}(\hat{f}_k)) \to \mathrm{Map}^{\omega}(\Sigma_k, Z_k) \tag{6}$$

with values in the *relative* symplectic mapping class group

$$\mathrm{Map}^{\omega}(\Sigma_k, Z_k) = \pi_0(\{\phi \in \mathrm{Symp}(\Sigma_k, \omega_{|\Sigma_k}),\ \phi_{|V(Z_k)} = \mathrm{Id}\}),$$

where Σ_k is a generic fiber of $\hat{f}_k$ (above the chosen base point in $\mathbb{C} - \mathrm{crit}(\hat{f}_k)$), and $V(Z_k)$ is a neighborhood of Z_k inside Σ_k. As before, the monodromy around a singular fiber is a positive Dehn twist along the vanishing cycle, which is an embedded Lagrangian sphere $S^{n-1} \subset \Sigma_k$. However, the product of all the monodromies around the individual singular fibers (i.e., the "monodromy at infinity") is not trivial, but rather equal to an element $\delta_{Z_k} \in \mathrm{Map}^{\omega}(\Sigma_k, Z_k)$, the positive Dehn twist along the unit sphere bundle in the normal bundle of Z_k in Σ_k (i.e. δ_{Z_k} restricts to each fiber of the normal bundle as a Dehn twist around the origin).

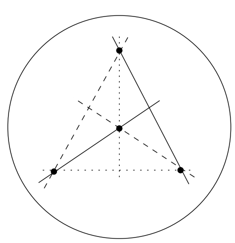
Things are easiest when M is a 4-manifold: the fibers are then Riemann surfaces, and Z_k is a finite set of points. The group $\mathrm{Map}^{\omega}(\Sigma_k, Z_k)$ therefore identifies with the mapping class group $\mathrm{Map}_{g,N}$ of a Riemann surface of genus $g = g(\Sigma_k)$ with $N = \mathrm{card}\, Z_k$ boundary components. The monodromy around a singular fiber (a Riemann surface with an ordinary double point) is a Dehn twist along an embedded loop, and the element δ_{Z_k} is the product of the Dehn twists along N small loops encircling the punctures.

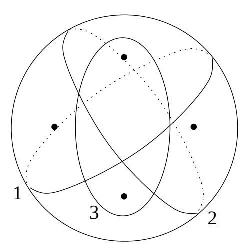
Finally, we mention that Gompf's result (Theorem 2.4) admits an adaptation to the case of Lefschetz pencils [Go2]: in the presence of base points, the cohomology class $[\omega]$ is determined in advance by the topology of the pencil, so that (using Moser's theorem) the constructed symplectic form on blown-down manifold M becomes canonical *up to symplectomorphism* (rather than just deformation equivalence).

3. Lefschetz fibrations: examples and applications

3.1. **Examples and classification questions.** In this section, we give examples of Lefschetz fibrations, and mention some results and questions related to their classification. We restrict ourselves to the case where the total space is of dimension 4, because it is by far the best understood. In fact, the structure of the symplectic mapping class group $\mathrm{Map}^\omega(\Sigma) = \pi_0\mathrm{Symp}(\Sigma)$ of a symplectic 4-manifold is almost never known, except in the simplest cases (always rational or ruled surfaces), which makes it very difficult to say much about higher-dimensional symplectic Lefschetz fibrations (see §4 for an alternative approach to this problem).

As a first example, we consider a pencil of degree 2 curves in $\mathbb{CP}^2$, a case in which the monodromy homomorphism (6) can be directly explicited in a remarkably simple manner. Consider the two sections $s_0 = x_0(x_1 - x_2)$ and $s_1 = x_1(x_2 - x_0)$ of the line bundle $O(2)$ over $\mathbb{CP}^2$: their zero sets are singular conics, in fact the unions of two lines each containing two of the four intersection points $(1:0:0)$, $(0:1:0)$, $(0:0:1)$, $(1:1:1)$. Moreover, the zero set of the linear combination $s_0 + s_1 = x_2(x_1 - x_0)$ is also singular; on the other hand, it is fairly easy to check that all other linear combinations $s_0 + \alpha s_1$ (for $\alpha \in \mathbb{CP}^1 - \{0, 1, \infty\}$) vanish along smooth conics. By blowing up the four base points of the pencil generated by s_0 and s_1, we obtain a genus 0 Lefschetz fibration with four exceptional sections. The three singular fibers are nodal configurations consisting of two tranversely intersecting spheres, with each component containing two of the four base points; each of the three different manners in which four points can be split into two groups of two is realized at one of the singular fibers. The following diagram represents the three singular conics of the pencil inside $\mathbb{CP}^2$ (left), and the corresponding vanishing cycles inside a smooth fiber (right):





As seen above, the monodromy of this Lefschetz pencil can be expressed by a morphism $\psi : \pi_1(\mathbb{C} - \{p_1, p_2, p_3\}) \to \mathrm{Map}_{0,4}$. After choosing a suitable ordered basis of the free group $\pi_1(\mathbb{C} - \{p_1, p_2, p_3\})$, we can make sure that ψ maps the generators to the Dehn twists τ_1, τ_2, τ_3 along the three loops shown on the diagram. On the other hand, as seen in §2.4 the monodromy at infinity is given by the boundary twist $\prod \delta_i$, the product of the four Dehn twists along small loops encircling the four base points in the fiber. The monodromy at infinity can be decomposed into the product of the monodromies around each of the three singular fibers (τ_1, τ_2, τ_3). Hence, the monodromy of a pencil of conics in $\mathbb{CP}^2$ can be expressed by the relation $\prod \delta_i = \tau_1 \cdot \tau_2 \cdot \tau_3$ in the mapping class group $\mathrm{Map}_{0,4}$ (*lantern relation*).

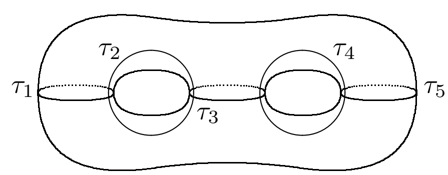
In our subsequent examples, we forget the base points, and simply consider Lefschetz fibrations and their monodromy in Map_g (rather than Lefschetz pencils with monodromy in $\mathrm{Map}_{g,N}$).

The classification of genus 1 Lefschetz fibrations over S^2 is a classical result of Moishezon and Livne [Mo1], who have shown that, up to isotopy, any such fibration is holomorphic. The mapping class group Map_1 is isomorphic to $SL(2, \mathbb{Z})$ (the isomorphism is given by considering the action of a diffeomorphism of T^2 on $H_1(T^2, \mathbb{Z}) = \mathbb{Z}^2$). It is generated by the Dehn twists

$$\tau_a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \tau_b = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

along the two generating loops $a = S^1 \times \{pt\}$ and $b = \{pt\} \times S^1$, with relations $\tau_a \tau_b \tau_a = \tau_b \tau_a \tau_b$ and $(\tau_a \tau_b)^6 = 1$. If we consider a *relatively minimal* genus 1 Lefschetz fibration (i.e., if there are no homotopically trivial vanishing cycles, an assumption that always becomes true after blowing down spherical components of reducible singular fibers), then the number of singular fibers is a multiple of 12, and for a suitable choice of an ordered system of generating loops $\gamma_1, \dots, \gamma_{m=12k}$ in the base of the fibration, the vanishing cycles can always be assumed to be $a, b, a, b, \dots$ In other terms, the monodromy morphism maps the defining relation $\gamma_1 \cdot \ldots \cdot \gamma_m = 1$ of the group $\pi_1(\mathbb{CP}^1 - \mathrm{crit})$ to the positive relation $(\tau_a \cdot \tau_b)^{6k} = 1$ among Dehn twists in Map_1.

The situation becomes more interesting for genus 2 Lefschetz fibrations. The mapping class group of a genus 2 Riemann surface is generated by five Dehn twists τ_i $(1 \le i \le 5)$ represented on the following diagram:



It is well-known (cf. e.g. [Bi], Theorem 4.8) that Map_2 admits a presentation with generators $\tau_1, \dots, \tau_5$, and the following relations: $\tau_i \tau_j = \tau_j \tau_i$ if $|i - j| \ge 2$; $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$; $(\tau_1 \tau_2 \tau_3 \tau_4 \tau_5)^6 = 1$; $I = \tau_1 \tau_2 \tau_3 \tau_4 \tau_5 \tau_5 \tau_4 \tau_3 \tau_2 \tau_1$ (the hyperelliptic involution) is central, and $I^2 = 1$.

We now give two very classical examples of holomorphic genus 2 Lefschetz fibrations. The first one is the fibration $f_0 : X_0 \to \mathbb{CP}^1$, obtained from a pencil of curves of bi-degree $(2, 3)$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$ by blowing up the 12 base points: this fibration has 20 singular fibers, and its monodromy is described by the positive relation $(\tau_1 \tau_2 \tau_3 \tau_4 \tau_5 \tau_5 \tau_4 \tau_3 \tau_2 \tau_1)^2 = 1$ in Map_2, expressing the identity element as the product of the 20 Dehn twists along the vanishing cycles. The second standard example of holomorphic genus 2 fibration is a fibration $f_1 : X_1 \to \mathbb{CP}^1$ with total space a K3 surface blown up in two points, and with 30 singular fibers; its monodromy is described by the positive relation $(\tau_1\tau_2\tau_3\tau_4\tau_5)^6 = 1$ involving the Dehn twists along the 30 vanishing cycles.

If we have two Lefschetz fibrations $f : X \to \mathbb{CP}^1$ and $f' : X' \to \mathbb{CP}^1$ with fibers of the same genus, we can perform a *fiber sum* operation: choose smooth fibers $F = f^{-1}(p) \subset X$ and $F' = f'^{-1}(p') \subset X'$, and an orientation-preserving diffeomorphism $\phi : F \to F'$. The complement U of a neighborhood of F in X is a symplectic 4-manifold with boundary $F \times S^1$, and similarly the complement U' of a neighborhood of F' in X' has boundary $F' \times S^1$; by restriction to U and U', the fibrations f and f' induce Lefschetz fibration structures over the disc D^2. Using the diffeomorphism $\phi \times \mathrm{Id}$ to identify $F \times S^1$ with $F' \times S^1$ in a manner compatible with the fibrations, we can glue U and U' along their boundary in order to obtain a compact symplectic manifold $X\#_\phi X'$ equipped with a Lefschetz fibration $f\#_\phi f' : X\#_\phi X' \to \mathbb{CP}^1$. In many cases there is a particularly natural choice of gluing diffeomorphism ϕ, leading to a preferred fiber sum; fiber sums obtained using different gluing diffeomorphisms are then said to be *twisted*.

The fiber sum construction leads to many interesting examples of Lefschetz fibrations; however, when the building blocks are the holomorphic genus 2 fibrations f_0 and f_1, the result is actually independent of the chosen gluing diffeomorphisms, and is again a holomorphic Lefschetz fibration. By fiber summing m copies of f_0 and n copies of f_1 we obtain a holomorphic Lefschetz fibration with $20m + 30n$ singular fibers and monodromy described by the relation $(\tau_1\tau_2\tau_3\tau_4\tau_5\tau_5\tau_4\tau_3\tau_2\tau_1)^{2m}(\tau_1\tau_2\tau_3\tau_4\tau_5)^{6n} = 1$ in Map_2. It is worth observing that the fiber sum of 3 copies of f_0 is actually the same Lefschetz fibration as the fiber sum of 2 copies of f_1.

It is a recent result of Siebert and Tian [ST] that all genus 2 symplectic Lefschetz fibrations without reducible singular fibers (i.e. without homologically trivial vanishing cycles) and satisfying a technical assumption (transitivity of monodromy) are holomorphic. More precisely, there exists a surjective group homomorphism $\mathrm{Map}_2 \to S_6$, mapping each generator τ_i ($1 \le i \le 5$) to the transposition $(i, i+1)$; the monodromy of a genus 2 Lefschetz fibration is said to be *transitive* if the image of its monodromy morphism (a subgroup of Map_2) is mapped surjectively onto the symmetric group S_6.

Theorem 3.1 (Siebert-Tian). *Every genus 2 Lefschetz fibration without reducible fibers and with transitive monodromy is holomorphic. It can be realized as a fiber sum of copies of f_0 and f_1, and hence its topology is completely determined by the number of singular fibers.*

In contrast to this spectacular result, which follows from a positive solution to the symplectic isotopy problem for certain smooth symplectic curves in rational ruled surfaces [ST], if we allow some of the singular fibers to be reducible, then genus 2 Lefschetz fibrations need not be holomorphic. Examples of this phenomenon were found by Ozbagci and Stipsicz [OS]: let $f : X \to S^2$ be the Lefschetz fibration obtained by blowing up the 4 base points of a pencil of algebraic curves representing the homology class $[S^2 \times pt] + 2[pt \times T^2]$ in $S^2 \times T^2$. The Lefschetz fibration f has 8 singular fibers, two of which are reducible, and its total space (a blow-up of $S^2 \times T^2$) has first Betti number $b_1 = 2$. For a suitable choice of the identification diffeomorphism ϕ, the symplectic 4-manifold $X\#_\phi X$ obtained by twisted fiber sum of two copies of the Lefschetz fibration f has first Betti number $b_1 = 1$, and hence $f\#_\phi f$ is not a holomorphic Lefschetz fibration.

However, this non-isotopy phenomenon disappears if we perform a stabilization by fiber sums [Au6]:

Theorem 3.2. *Let $f : X \to S^2$ be any genus 2 Lefschetz fibration. Then the fiber sum of f with sufficiently many copies of the holomorphic fibration f_0 described above is isomorphic to a holomorphic fibration. Moreover, this fiber sum $f\#n\,f_0$ ($n \gg 0$) is entirely determined by its total number of singular fibers and by the number of reducible fibers of each type* (*two genus* 1 *components, or genus* 0 *and* 2 *components*)*.*

Very little is known about the structure of Lefschetz fibrations with fiber genus 3 or more; no analogues of Theorems 3.1 and 3.2 are available. In fact, by imitating the construction of Ozbagci and Stipsicz it is easy to construct genus 3 Lefschetz fibrations without reducible fibers and with $b_1 = 1$ (which implies that these fibrations are not holomorphic). More precisely, these examples can be constructed from the holomorphic genus 3 fibration obtained by blowing up the 8 base points of a generic pencil of algebraic curves representing the class $2[S^2 \times pt] + 2[pt \times T^2]$ in $S^2 \times T^2$. This Lefschetz fibration has 16 singular fibers, all irreducible, and its total space has $b_1 = 2$; suitably twisted fiber sums of two copies of this fibration yield genus 3 symplectic Lefschetz fibration structures (without reducible fibers) on symplectic 4-manifolds with $b_1 = 1$. Various other examples of non-holomorphic genus 3 Lefschetz fibrations have been constructed by Amoros et al. [ABKP], Smith [Sm1], Fintushel and Stern, among others.

Two natural questions arise at this point. The first one, suggested by the above examples of non-holomorphic fibrations obtained as twisted fiber sums, is whether every symplectic Lefschetz fibration can be decomposed into a fiber sum of holomorphic fibrations. The answer to this question is known to be negative, as implied by the following result [Sm2] (see also [ABKP], ...):

Theorem 3.3 (Amoros et al., Stipsicz, Smith)**.** *If a relatively minimal Lefschetz fibration over S^2 with fibers of genus ≥ 2 contains a section of square -1, then it cannot be decomposed as any non-trivial fiber sum.*

In particular, since Lefschetz fibrations obtained by blowing up the base points of a Lefschetz pencil always admit sections of square -1, by applying Donaldson's construction (Theorem 2.7) to any non-Kähler symplectic 4-manifold we obtain indecomposable non-holomorphic Lefschetz fibrations.

The second question we may ask is whether a "stable isotopy" result similar to Theorem 3.2 remains true for Lefschetz fibrations of higher genus. In the very specific case of Lefschetz fibrations with monodromy contained in the *hyperelliptic* subgroup of the mapping class group, such a statement can be obtained as a corollary of a recent result of Kharlamov and Kulikov [KK] about braid monodromy factorizations: after repeated (untwisted) fiber sums with copies of a same fixed holomorphic fibration with $8g + 4$ singular fibers, any hyperelliptic genus g Lefschetz fibration eventually becomes holomorphic. Moreover, the fibration obtained in this manner is completely determined by its number of singular fibers of each type (irreducible, reducible with components of given genus), and when the fiber genus is odd by a certain $\mathbb{Z}_2$-valued invariant. (The proof of this result uses the fact that the hyperelliptic mapping class group is an extension by $\mathbb{Z}_2$ of the braid group of $2g+2$ points on a sphere, which is itself a quotient of B_{2g+2}; this makes it possible to transform the monodromy of a hyperelliptic Lefschetz fibration into a factorization in B_{2g+2}, with different types of factors for the various types of singular fibers and extra contributions belonging to the kernel of the morphism $B_{2g+2} \to B_{2g+2}(S^2)$, and hence reduce the problem to that studied by Kharlamov and Kulikov). However, it is not clear whether the result should be expected to remain true in the non-hyperelliptic case.

3.2. **Lefschetz fibrations and pseudo-holomorphic curves.** Taubes' work on the Seiberg-Witten invariants of symplectic 4-manifolds [Ta1, Ta2] has tremendously improved our understanding of the topology of symplectic 4-manifolds. The most immediate applications are of two types: to show that certain smooth 4-manifolds admit no symplectic structure (because their Seiberg-Witten invariants violate Taubes' structure theorem), or to obtain existence results for pseudo-holomorphic curves in a given symplectic 4-manifold (using the non-vanishing of Seiberg-Witten invariants and the identity $SW = Gr_T$). Results of the second type, and in particular the fact that any compact symplectic 4-manifold with $b_2^+ \geq 2$ contains an embedded (possibly disconnected) pseudo-holomorphic curve representing its canonical class [Ta1] (cf. Theorem 1.12 (*iv*)), are purely symplectic in nature, and it may be possible to reprove them by methods that do not involve Seiberg-Witten theory at all. Spectacular progress has been made in this direction by Donaldson and Smith, who have obtained the following result [DS]:

Theorem 3.4 (Taubes, Donaldson-Smith)**.** *Let (X,ω) be a compact symplectic 4-manifold with $b_2^+(X) > 1 + b_1(X)$, and assume that the cohomology class $[\omega] \in H^2(X,\mathbb{R})$ is rational. Then there exists a smooth (not necessarily connected) embedded symplectic surface in X which represents the homology class Poincaré dual to $c_1(K_X)$.*

Following the work of Donaldson and Smith [DS, Sm3], we now give an outline of the proof of this result. The main idea is to represent a blow-up $\hat{X}$ of the manifold X as a symplectic Lefschetz fibration, and to look for symplectic submanifolds embedded in "standard" position in $\hat{X}$ with respect to the Lefschetz fibration $f : \hat{X} \to S^2$. Namely, $\Sigma \subset \hat{X}$ is said to be a *standard surface* if it avoids the critical points of f and if it is positively transverse to the fibers of f everywhere except at isolated non-degenerate tangency points (i.e., the restriction of f to Σ defines a branched covering with simple branch points).

Let $r > 0$ be the intersection number of $[\Sigma]$ with the fiber of f (in the case of the canonical class, $r = 2g-2$, where g is the fiber genus), and consider a fibration $F : X_r(f) \to S^2$ with generic fiber the r-th symmetric product of the fiber of f. More precisely, choosing an almost-complex structure in order to make each fiber of f a (possibly nodal) Riemann surface, $X_r(f)$ is defined as the *relative Hilbert scheme* of degree r divisors in the fibers of f; each fiber of F corresponding to a smooth fiber of f is naturally identified with its r-fold symmetric product, but in the case of nodal fibers the Hilbert scheme is a partial desingularization of the symmetric product.

A standard surface Σ intersects every fiber of f in r points (counting with multiplicities), which defines a point in the r-th symmetric product (or Hilbert scheme) of the fiber. Hence, to a standard surface $\Sigma \subset \hat{X}$ we can associate a section s_Σ of the fibration $F : X_r(f) \to S^2$. The points where Σ becomes non-degenerately tangent to the fibers of f correspond to positive transverse intersections between the section s_Σ and the diagonal divisor $\Delta \subset X_r(f)$ consisting of all r-tuples in which two or more points coincide. Conversely, any section of $X_r(f)$ that intersects Δ transversely and positively determines a standard surface in $\hat{X}$.

There is a well-defined map from the space of homotopy classes of sections of $X_r(f)$ to the homology group $H_2(\hat{X},\mathbb{Z})$, which to a section of $X_r(f)$ associates the homology class represented by the corresponding surface in $\hat{X}$. This map is injective, so to a given homology class $\alpha \in H_2(X',\mathbb{Z})$ we can associate at most a single homotopy class $\bar{\alpha}$ of sections of $X_r(f)$ ($r = \alpha \cdot [f^{-1}(pt)]$).

We can equip the manifold $X_r(f)$ with symplectic and almost-complex structures whose restrictions to each fiber of F coincide with the standard Kähler and complex structures on the symmetric product of a Riemann surface. Then it is possible to define an invariant $Gr_{DS}(X,f;\alpha) \in \mathbb{Z}$ counting pseudo-holomorphic sections of $X_r(f)$ in the homotopy class corresponding to a given homology class $\alpha \in H_2(X',\mathbb{Z})$. More precisely, if the expected dimension of the moduli space of pseudo-holomorphic sections is positive then one obtains an integer-valued invariant by adding incidence conditions requiring the standard surface Σ to pass through certain points in X', or equivalently requiring the section s_Σ to pass through certain divisors in the corresponding fibers of F. The fact that we are interested in a moduli space of pseudo-holomorphic sections makes it possible to control bubbling, which can only occur inside the fibers of F.

In the case of the canonical class $\alpha = c_1(K_{\hat{X}})$, there exists a specific almost-complex structure on $X_{2g-2}(f)$ for which the moduli space of pseudo-holomorphic sections in the relevant homotopy class $\bar{\alpha}$ is a projective space of complex dimension $\frac{1}{2}(b_2^+(X) - 1 - b_1(X)) - 1$ (whereas the generically expected dimension is 0). By computing the Euler class of the obstruction bundle over this moduli space, one obtains that $Gr_{DS}(X,f;c_1(K_{\hat{X}})) = \pm 1$ [DS].

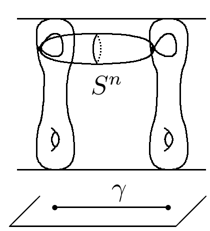
The non-vanishing of this invariant gives us an existence result for pseudo-holomorphic sections of $X_r(f)$. However, in order to obtain a standard symplectic surface from such a section s, we need to ensure the positivity of the intersections between s and the diagonal divisor Δ. This means that we need to consider on $X_r(f)$ a different almost-complex structure, for which the strata composing Δ are pseudo-holomorphic submanifolds of $X_r(f)$ (as well as the divisors associated with the exceptional sections of f arising from the blow-ups of the base points of the chosen pencil on X). For a generic choice of almost-complex structure compatible with the diagonal strata, the moduli spaces of pseudo-holomorphic sections of F have the expected dimension, and the non-vanishing of the Gr_{DS} invariant leads to the existence of a section transverse to all diagonal strata in which it is not contained.

From such a section of F we can obtain a (possibly disconnected) standard symplectic surface $\hat{\Sigma} \subset \hat{X}$, representing the homology class $c_1(K_{\hat{X}}) = \pi^* c_1(K_X) + \sum [E_i]$, where E_i are the exceptional spheres of the blow-up $\pi : \hat{X} \to X$. Moreover, because this surface has intersection number -1 with each sphere E_i, the local positivity of intersection implies that E_i is actually contained in $\hat{\Sigma}$, and therefore we have $\hat{\Sigma} = \Sigma \cup \bigcup E_i$, for some symplectic surface $\Sigma \subset \hat{X} - \bigcup E_i$; the image of Σ in X is a smooth symplectic submanifold representing the homology class Poincaré dual to the canonical bundle [DS].

While it has not yet been shown that the invariant Gr_{DS} is actually independent of the choice of a high degree Lefschetz pencil structure on (X^4, ω) and coincides with the invariant Gr_T defined by Taubes, it is worth mentioning a Serre duality-type result obtained by Smith for the Gr_{DS} invariant: under the same assumption as above on the symplectic 4-manifold X ($b_2^+(X) > 1 + b_1(X)$), for a Lefschetz pencil of sufficiently high degree we have $Gr_{DS}(X, f; \alpha) = \pm Gr_{DS}(X, f; K_X - \alpha)$ [Sm3]. This is to be compared to the duality formula for Gromov-Taubes invariants, $Gr_T(\alpha) = \pm Gr_T(K_X - \alpha)$, which follows immediately from a duality among Seiberg-Witten invariants ($SW(-L) = \pm SW(L)$) and Taubes' result (Theorem 1.12).

3.3. **Fukaya-Seidel categories for Lefschetz pencils.** One of the most exciting applications of Lefschetz pencils, closely related to the homological mirror symmetry conjecture, is Seidel's construction of "directed Fukaya categories" associated to a Lefschetz pencil. Besides the many technical difficulties arising in their definition, Fukaya categories of symplectic manifolds are intrinsically very hard to compute, because relatively little is known about embedded Lagrangian submanifolds in symplectic manifolds of dimension ≥ 4, especially in comparison to the much better understood theory of coherent sheaves over complex varieties, which play the role of their mirror counterparts.

Consider an arc γ joining a regular value p_0 to a critical value p_1 in the base of a symplectic Lefschetz fibration $f : X \to S^2$. Using the horizontal distribution given by the symplectic orthogonal to the fibers, we can transport the vanishing cycle at p_1 along the arc γ to obtain a Lagrangian disc $D_\gamma \subset X$ fibered above γ, whose boundary is an embedded Lagrangian sphere S_γ in the fiber $\Sigma_0 = f^{-1}(p_0)$. The Lagrangian disc D_γ is sometimes called the *Lefschetz thimble* over γ, and its boundary S_γ is the vanishing cycle already considered in §2. If we now consider an arc γ joining two critical values p_1, p_2 of f and passing through p_0, then the above construction applied to each half of γ yields two Lefschetz thimbles D_1 and D_2, whose boundaries are Lagrangian spheres $S_1, S_2 \subset \Sigma_0$. If S_1 and S_2 coincide exactly, then $D_1 \cup D_2$ is an embedded Lagrangian sphere in X, fibering above the arc γ (see the picture below); more generally, if S_1 and S_2 are Hamiltonian isotopic to each other, then perturbing slightly the symplectic structure we can reduce to the previous case and obtain again a Lagrangian sphere in X. The arc γ is called a *matching path* in the Lefschetz fibration f.



Matching paths are an important source of Lagrangian spheres, and more generally (extending suitably the notion of matching path) of embedded Lagrangian submanifolds. Conversely, a folklore theorem asserts that any given embedded Lagrangian sphere (or more generally, compact Lagrangian submanifold) in a compact symplectic manifold is isotopic to one that fibers above a matching path in a Donaldson-type symplectic Lefschetz pencil of sufficiently high degree.

The intersection theory of Lagrangian spheres that fiber above matching paths is much nicer than that of arbitrary Lagrangian spheres, because if two Lagrangian spheres $S, S' \subset X$ fiber above matching paths γ, γ', then all intersections of S with S' lie in the fibers above the intersection points of γ with γ'; hence, the Floer homology of S and S' can be computed by studying intersection theory for Lagrangian spheres in the fibers of f, rather than in its total space.

These considerations have led Seidel to the following construction of a Fukaya-type A_∞-category associated to a symplectic Lefschetz pencil f on a compact symplectic manifold (X, ω) [Se1]. Let f be a symplectic Lefschetz pencil determined by two sections s_0, s_1 of a sufficiently positive line bundle $L^{\otimes k}$ as in Theorem 2.7. Assume that $\Sigma_\infty = s_1^{-1}(0)$ is a smooth fiber of the pencil, and consider the symplectic manifold with boundary X^0 obtained from X by removing a suitable neighborhood of Σ_∞. The map f induces a symplectic Lefschetz fibration structure $f^0 : X^0 \to D^2$ over a disc, whose fibers are symplectic submanifolds with boundary obtained from the fibers of f by removing a neighborhood of their intersection points with the symplectic hypersurface Σ_∞ (the base points of the pencil). Choose a reference point $p_0 \in \partial D^2$, and consider the fiber $\Sigma_0 = (f^0)^{-1}(p_0) \subset X^0$.

Let $\gamma_1, \dots, \gamma_r$ be a collection of arcs in D^2 joining the reference point p_0 to the various critical values of f^0, intersecting each other only at p_0, and ordered in the clockwise direction around p_0. As discussed above, each arc γ_i gives rise to a Lefschetz thimble $D_i \subset X^0$, whose boundary is a Lagrangian sphere $L_i \subset \Sigma_0$. To avoid having to discuss the orientation of moduli spaces, we give the following definition using $\mathbb{Z}_2$ as the coefficient ring [Se1]:

Definition 3.5 (Seidel). *The directed Fukaya category $FS(f; \{\gamma_i\})$ is the following A_∞-category: the objects of $FS(f; \{\gamma_i\})$ are the Lagrangian vanishing cycles $L_1, \dots, L_r$; the morphisms between the objects are given by*

$$\mathrm{Hom}(L_i, L_j) = \begin{cases} CF^*(L_i, L_j; \mathbb{Z}_2) = \mathbb{Z}_2^{|L_i \cap L_j|} & \text{if } i < j \\ \mathbb{Z}_2 \, id & \text{if } i = j \\ 0 & \text{if } i > j; \end{cases}$$

and the differential μ^1, composition μ^2 and higher order compositions μ^n are given by Lagrangian Floer homology inside Σ_0: more precisely,

$$\mu^n : \mathrm{Hom}(L_{i_0}, L_{i_1}) \otimes \dots \otimes \mathrm{Hom}(L_{i_{n-1}}, L_{i_n}) \to \mathrm{Hom}(L_{i_0}, L_{i_n})[2-n]$$

is trivial when the inequality $i_0 < i_1 < \dots < i_n$ fails to hold (i.e. it is always zero in this case, except for μ^2 where composition with an identity morphism is given by the obvious formula). When $i_0 < \dots < i_n$, μ^n is defined by counting pseudo-holomorphic maps from the disc to Σ_0, mapping n cyclically ordered marked points on the boundary to the given intersection points between vanishing cycles, and the portions of boundary between them to $L_{i_0}, \dots, L_{i_n}$.

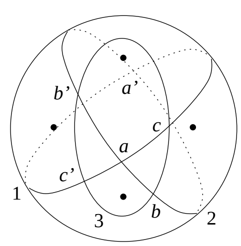
One of the most attractive features of this definition is that it only involves Floer homology for Lagrangians inside the hypersurface Σ_0; in particular, when X is a symplectic 4-manifold, the definition becomes essentially combinatorial, since in the case of a Riemann surface the pseudo-holomorphic discs appearing in the definition of Floer homology and product structures are essentially topological objects.

From a technical point of view, a key property that greatly facilitates the definition of Floer homology for the vanishing cycles L_i is *exactness*. Namely, the symplectic structure on the manifold X^0 is *exact*, i.e. it can be expressed as $\omega = d\theta$

for some 1-form θ (up to a scaling factor, θ is the 1-form describing the connection on $L^{\otimes k}$ in the trivialization of $L^{\otimes k}$ over $X - \Sigma_\infty$ induced by the section $s_1/|s_1|$). With this understood, the submanifolds L_i are all *exact Lagrangian*, i.e. the restriction $\theta_{|L_i}$ is not only closed ($d\theta_{|L_i} = \omega_{|L_i} = 0$) but also exact, $\theta_{|L_i} = d\phi_i$. Exactness has two particularly nice consequences. First, Σ^0 contains no closed pseudo-holomorphic curves (because the cohomology class of $\omega = d\theta$ vanishes). Secondly, there are no non-trivial pseudo-holomorphic discs in Σ_0 with boundary contained in one of the Lagrangian submanifolds L_i. Indeed, for any such disc D, we have $\text{Area}(D) = \int_D \omega = \int_{\partial D} \theta = \int_{\partial D} d\phi_i = 0$. Therefore, bubbling never occurs (neither in the interior nor on the boundary of the domain) in the moduli spaces used to define the Floer homology groups $HF(L_i, L_j)$. Moreover, the exactness of L_i provides a priori estimates on the area of all pseudo-holomorphic discs contributing to the definition of the products μ^n ($n \geq 1$); this implies the finiteness of the number of discs to be considered and solves elegantly the convergence problems that normally make it necessary to define Floer homology over Novikov rings.

It is natural question to ask oneself how much the category $FS(f, \{\gamma_i\})$ depends on the given data, and in particular on the chosen ordered collection of arcs $\{\gamma_i\}$. An answer is provided by a result of Seidel showing that, if the ordered collection $\{\gamma_i\}$ is replaced by another one $\{\gamma_i'\}$, then the categories $FS(f, \{\gamma_i\})$ and $FS(f, \{\gamma_i'\})$ differ by a sequence of *mutations* (operations that modify the ordering of the objects of the category while twisting some of them along others) [Se1]. Hence, the category naturally associated to the Lefschetz pencil f is not the finite directed category defined above, but rather a *derived* category, obtained from $FS(f, \{\gamma_i\})$ by considering formal direct sums and twisted complexes of Lagrangian vanishing cycles (with additional features such as idempotent splittings, formal inverses of quasi-isomorphisms, ...). It is a classical result that, if two categories differ by mutations, then their derived categories are equivalent; hence the derived category $D(FS(f))$ only depends on the Lefschetz pencil f rather than on the choice of an ordered system of arcs [Se1].

As an example, let us consider the case of a pencil of conics on $\mathbb{CP}^2$, an example for which monodromy was described explicitly at the beginning of §3.1. The fiber Σ_0 is a sphere with four punctures, and for a suitable system of arcs the three vanishing cycles $L_1, L_2, L_3 \subset \Sigma_0$ are as represented at the beginning of §3.1:



Any two of these three vanishing cycles intersect transversely in two points, so $\text{Hom}(L_1, L_2) = \mathbb{Z}_2\, a \oplus \mathbb{Z}_2\, a'$, $\text{Hom}(L_2, L_3) = \mathbb{Z}_2\, b \oplus \mathbb{Z}_2\, b'$, and $\text{Hom}(L_1, L_3) = \mathbb{Z}_2\, c \oplus \mathbb{Z}_2\, c'$ are all two-dimensional. There are no embedded 2-sided polygons in the punctured sphere Σ_0 with boundary in $L_i \cup L_j$ for any pair (i, j), since each of the four regions delimited by L_i and L_j contains one of the punctures, so $\mu^1 \equiv 0$. However, there are four triangles with boundary in $L_1 \cup L_2 \cup L_3$ (with vertices abc, $ab'c'$, $a'b'c$, $a'bc'$ respectively), and in each case the cyclic ordering of the boundary is compatible with the ordering of the vanishing cycles. Therefore, the composition of morphisms is given by the formulas $\mu^2(a,b)=\mu^2(a',b')=c$, $\mu^2(a,b')=\mu^2(a',b)=c'$. Finally, the higher compositions μ^n, $n\geq 3$ are all trivial in this category, because the ordering condition $i_0<\cdots<i_n$ never holds [Se2].

We finish this section by some vague considerations about the relationship between the derived Fukaya-Seidel category $D(FS(f))$ of the Lefschetz pencil f and the derived Fukaya category $D\mathcal{F}(X)$ of the symplectic manifold X, following ideas of Seidel. At first glance, the category $FS(f;\{\gamma_i\})$ appears to be more closely related to the Fukaya category of the fiber Σ_0 than to that of the total space of the Lefschetz pencil. However, the objects L_i actually correspond not only to Lagrangian spheres in Σ_0 (the vanishing cycles), but also to Lagrangian discs in X^0 (the Lefschetz thimbles D_i); and the Floer intersection theory in Σ_0 giving rise to $\mathrm{Hom}(L_i,L_j)$ and to the product structures can also be thought of in terms of intersection theory for the Lagrangian discs D_i in X^0. When passing to the derived category $D(FS(f))$, we hugely increase the number of objects, by considering not only the thimbles D_i but also arbitrary complexes obtained from them; this means that the objects of $D(FS(f))$ include arbitrary (not necessarily closed) Lagrangian submanifolds in X^0, with boundary in Σ_0. Since Fukaya categories are only concerned with closed Lagrangian submanifolds, it is necessary to consider a subcategory of $D(FS(f))$ whose objects correspond only to the closed Lagrangian submanifolds in X^0 (i.e., combinations of D_i for which the boundaries cancel); it is expected that this can be done in purely categorical terms by considering those objects of $D(FS(f))$ on which the Serre functor acts simply by a shift. The resulting subcategory should be closely related to the derived Fukaya category of the open manifold X^0. This leaves us with the problem of relating $\mathcal{F}(X^0)$ with $\mathcal{F}(X)$. These two categories have the same objects and morphisms (Lagrangians in X can be made disjoint from Σ_∞), but the differentials and product structures differ. More precisely, the definition of μ^n in $\mathcal{F}(X^0)$ only involves counting pseudo-holomorphic discs contained in X^0, i.e. disjoint from the hypersurface Σ_∞. In order to account for the missing contributions, one should introduce a formal parameter q and count the pseudo-holomorphic discs with boundary in $\bigcup L_i$ that intersect Σ_∞ in m points (with multiplicities) with a coefficient q^m. The introduction of this parameter q leads to a *deformation of A_∞-structures*, i.e. an A_∞-category in which the differentials and products μ^n are defined over a ring of formal power series in the variable q; the limit $q=0$ corresponds to the (derived) Fukaya category $D\mathcal{F}(X^0)$, while non-zero values of q are expected to yield $D\mathcal{F}(X)$.

The above considerations provide a strategy that should make it possible (at least in some examples) to calculate Fukaya categories by induction on dimension; an important consequence is that it becomes possible to verify the homological mirror symmetry conjecture (or parts thereof) on a wider class of examples (e.g. some K3 surfaces, cf. recent work of Seidel).

4. Symplectic branched covers of $\mathbb{CP}^2$

4.1. Symplectic branched covers.

Definition 4.1. *A smooth map $f : X^4 \to (Y^4,\omega_Y)$ from a compact oriented smooth 4-manifold to a compact symplectic 4-manifold is a* symplectic branched covering *if, given any point $p\in X$, there exist neighborhoods $U\ni p$ and $V\ni f(p)$ and orientation-preserving local diffeomorphisms $\phi:U\to\mathbb{C}^2$ and $\psi:V\to\mathbb{C}^2$, such*

that $\psi_*\omega_Y(v, iv) > 0$ $\forall v \neq 0$ (i.e. the standard complex structure is $\psi_*\omega_Y$-tame), and such that $\psi \circ f \circ \phi^{-1}$ is one of the following model maps:
(i) $(u, v) \mapsto (u, v)$ (local diffeomorphism),
(ii) $(u, v) \mapsto (u^2, v)$ (simple branching),
(iii) $(u, v) \mapsto (u^3 - uv, v)$ (cusp).

The three local models appearing in this definition are exactly those describing a generic holomorphic map between complex surfaces, except that the local coordinate systems we consider are not holomorphic.

By computing the Jacobian of f in the given local coordinates, we can see that the *ramification curve* $R \subset X$ is a smooth submanifold (it is given by $\{u = 0\}$ in the second local model and $\{v = 3u^2\}$ in the third one). However, the image $D = f(R) \subset X$ (the *branch curve*, or *discriminant curve*) may be singular. More precisely, in the simple branching model D is given by $\{z_1 = 0\}$, while in the cusp model we have $f(u, 3u^2) = (-2u^3, 3u^2)$, and hence D is locally identified with the singular curve $\{27z_1^2 = 4z_2^3\} \subset \mathbb{C}^2$. This means that, at the cusp points, D fails to be immersed. Besides the cusps, the branch curve D also generically presents *transverse double points* (or *nodes*), which do not appear in the local models because they correspond to simple branching in two distinct points p_1, p_2 of the same fibre of f. There is no constraint on the orientation of the local intersection between the the two branches of D at a node (positive or negative, i.e. complex or anti-complex), because the local models near p_1 and p_2 hold in different coordinate systems on Y.

Generically, the only singularities of the branch curve $D \subset Y$ are transverse double points ("nodes") of either orientation and complex cusps. Moreover, because the local models identify D with a complex curve, the tameness condition on the coordinate systems implies that D is a (singular) symplectic submanifold of Y.

The following result states that a symplectic branched cover of a symplectic 4-manifold carries a natural symplectic structure [Au2]:

Proposition 4.2. *If $f : X^4 \to (Y^4, \omega_Y)$ is a symplectic branched cover, then X carries a symplectic form ω_X such that $[\omega_X] = f^*[\omega_Y]$, canonically determined up to symplectomorphism.*

Proof. The 2-form $f^*\omega_Y$ is closed, but it is only non-degenerate outside of R. At any point p of R, the 2-plane $K_p = \operatorname{Ker} df_p \subset T_pX$ carries a natural orientation induced by the complex orientation in the local coordinates of Definition 4.1. Using the local models, we can construct an *exact* 2-form α such that, at any point $p \in R$, the restriction of α to K_p is non-degenerate and positive.

More precisely, given $p \in R$ we consider a small ball centered at p and local coordinates (u, v) such that f is given by one of the models of Definition 4.1, and we set $\alpha_p = d(\chi_1(|u|)\chi_2(|v|)\, x\, dy)$, where $x = \operatorname{Re}(u)$, $y = \operatorname{Im}(u)$, and χ_1 and χ_2 are suitably chosen smooth cut-off functions. We then define α to be the sum of these α_p when p ranges over a finite subset of R for which the supports of the α_p cover the entire ramification curve R. Since $f^*\omega_Y \wedge \alpha$ is positive at every point of R, it is easy to check that the 2-form $\omega_X = f^*\omega_Y + \epsilon\, \alpha$ is symplectic for a small enough value of the constant $\epsilon > 0$.

The fact that ω_X is canonical up to symplectomorphism follows immediately from Moser's stability theorem and from the observation that the space of exact perturbations α such that $\alpha_{|K_p} > 0$ $\forall p \in R$ is a convex subset of $\Omega^2(X)$ and hence connected. ∎

4.2. **Symplectic manifolds and maps to $\mathbb{CP}^2$.** Approximately holomorphic techniques make it possible to show that every compact symplectic 4-manifold can be realized as a branched cover of $\mathbb{CP}^2$. The general setup is similar to Donaldson's construction of symplectic Lefschetz pencils: we consider a compact symplectic manifold (X,ω), and perturbing the symplectic structure if necessary we may assume that $\frac{1}{2\pi}[\omega] \in H^2(X,\mathbb{Z})$. Introducing an almost-complex structure J and a line bundle L with $c_1(L) = \frac{1}{2\pi}[\omega]$, we consider triples of approximately holomorphic sections (s_k^0, s_k^1, s_k^2) of $L^{\otimes k}$: for $k \gg 0$, it is again possible to achieve a generic behavior for the projective map $f_k = (s_k^0 : s_k^1 : s_k^2) : X \to \mathbb{CP}^2$ associated with the linear system. If the manifold X is four-dimensional, then the linear system generically has no base points, and for a suitable choice of sections the map f_k is a branched covering [Au2].

Theorem 4.3. *For large enough k, three suitably chosen approximately holomorphic sections of $L^{\otimes k}$ over (X^4,ω) determine a symplectic branched covering $f_k : X^4 \to \mathbb{CP}^2$, described in approximately holomorphic local coordinates by the local models of Definition 4.1. Moreover, for $k \gg 0$ these branched covering structures are uniquely determined up to isotopy.*

Because the local models hold in approximately holomorphic (and hence ω-tame) coordinates, the ramification curve R_k of f_k is a symplectic submanifold in X (connected, since the Lefschetz hyperplane theorem applies). Moreover, if we normalize the Fubini-Study symplectic form on $\mathbb{CP}^2$ in such a way that $\frac{1}{2\pi}[\omega_{FS}]$ is the generator of $H^2(\mathbb{CP}^2,\mathbb{Z})$, then we have $[f_k^*\omega_{FS}] = 2\pi c_1(L^{\otimes k} = k[\omega]$, and it is fairly easy to check that the symplectic form on X obtained by applying Proposition 4.2 to the branched covering f_k coincides up to symplectomorphism with $k\omega$ [Au2]. In fact, the exact 2-form $\alpha = k\omega - f_k^*\omega_{FS}$ is positive over $\operatorname{Ker} df_k$ at every point of R_k, and $f_k^*\omega_{FS} + t\alpha$ is a symplectic form for all $t \in (0,1]$.

The uniqueness statement in Theorem 4.3, which should be interpreted exactly in the same way as that obtained by Donaldson for Lefschetz pencils, implies that for $k \gg 0$ it is possible to define invariants of the symplectic manifold (X,ω) in terms of the monodromy of the branched covering f_k and the topology of its branch curve $D_k \subset \mathbb{CP}^2$. However, the branch curve D_k is only determined up to creation or cancellation of (admissible) pairs of nodes of opposite orientations.

A similar construction can be attempted when $\dim X > 4$; in this case, the set of base points $Z_k = \{s_k^0 = s_k^1 = s_k^2 = 0\}$ is no longer empty. The set of base points is generically a smooth codimension 6 symplectic submanifold. With this understood, Theorem 4.3 admits the following higher-dimensional analogue [Au3]:

Theorem 4.4. *For large enough k, three suitably chosen approximately holomorphic sections of $L^{\otimes k}$ over (X^{2n},ω) determine a map $f_k : X - Z_k \to \mathbb{CP}^2$ with generic local models, canonically determined up to isotopy.*

The model maps describing the local behavior of f_k in approximately holomorphic local coordinates are now the following:

(0) $(z_1,\dots,z_n) \mapsto (z_1 : z_2 : z_3)$ near a base point,
(*i*) $(z_1,\dots,z_n) \mapsto (z_1, z_2)$,
(*ii*) $(z_1,\dots,z_n) \mapsto (z_1^2 + \dots + z_{n-1}^2, z_n)$,
(*iii*) $(z_1,\dots,z_n) \mapsto (z_1^3 - z_1 z_n + z_2^2 + \dots + z_{n-1}^2, z_n)$.

The set of critical points $R_k \subset X$ is again a (connected) smooth symplectic curve, and its image $D_k = f_k(R_k) \subset \mathbb{CP}^2$ is again a singular symplectic curve whose only singularities generically are transverse double points of either orientation and complex cusps. The fibers of f_k are codimension 4 symplectic submanifolds, intersecting along Z_k; the fiber above a point of $\mathbb{CP}^2 - D_k$ is smooth, while the fiber above a smooth point of D_k presents an ordinary double point, the fiber above a node presents two ordinary double points, and the fiber above a cusp presents an A_2 singularity.

As in the four-dimensional case, the uniqueness statement implies that, up to possible creations or cancellations of pairs of double points with opposite orientations in the curve D_k, the topology of the fibration f_k can be used to define invariants of the manifold (X^{2n}, ω).

The proof of Theorems 4.3 and 4.4 relies on a careful examination of the various possible local behaviors for the map f_k and on transversality arguments showing the existence of sections of $L^{\otimes k}$ with generic behavior. Hence, the argument relies on the enumeration of the various special cases, generic or not, that may occur; each one corresponds to the vanishing of a certain quantity that can be expressed in terms of the sections s_k^0, s_k^1, s_k^2 and their derivatives.

In order to simplify these arguments, and to make it possible to extend these results to linear systems generated by more than three sections or even more general situations, it is helpful to develop an approximately holomorphic version of singularity theory. The core ingredient of this approach is a uniform transversality result for jets of approximately holomorphic sections [Au4].

Given approximately holomorphic sections s_k of very positive bundles E_k (e.g. $E_k = \mathbb{C}^m \otimes L^{\otimes k}$) over the symplectic manifold X, one can consider the r-jets $j^r s_k = (s_k, \partial s_k, (\partial\partial s_k)_{\mathrm{sym}}, \dots, (\partial^r s_k)_{\mathrm{sym}})$, which are sections of the *jet bundles* $\mathcal{J}^r E_k = \bigoplus_{j=0}^r (T^*X^{(1,0)})^{\otimes j}_{\mathrm{sym}} \otimes E_k$. Jet bundles can naturally be stratified by approximately holomorphic submanifolds corresponding to the various possible local behaviors at order r for the sections s_k. The generically expected behavior corresponds to the case where the jet $j^r s_k$ is transerse to the submanifolds in the stratification. The result is the following [Au4]:

Theorem 4.5. *Given stratifications $\mathcal{S}_k$ of the jet bundles $\mathcal{J}^r E_k$ by a finite number of approximately holomorphic submanifolds (Whitney-regular, uniformly transverse to fibers, and with curvature bounded independently of k), for large enough k the vector bundles E_k admit approximately holomorphic sections s_k whose r-jets are uniformly transverse to the stratifications $\mathcal{S}_k$. Moreover these sections may be chosen arbitrarily close to given sections.*

A one-parameter version of this result also holds, which makes it possible to obtain results of asymptotic uniqueness up to isotopy for generic sections [Au4].

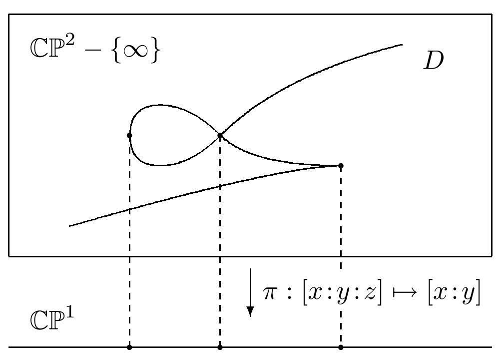
Applied to suitably chosen stratifications, Theorem 4.5 provides the main ingredient for the construction of m-tuples of approximately holomorphic sections of $L^{\otimes k}$ (and hence projective maps f_k with values in $\mathbb{CP}^{m-1}$) with generic behavior. Once uniform transversality of jets has been obtained, the only remaining task is to achieve some control over the antiholomorphic derivative $\bar\partial f_k$ near the critical points of f_k (typically its vanishing in some directions), in order to ensure that $\bar\partial f_k \ll \partial f_k$ everywhere; for low values of m such as those considered above, this task is comparatively easy.

4.3. **Monodromy invariants for branched covers of $\mathbb{CP}^2$.** The topological data characterizing a symplectic branched covering $f : X^4 \to \mathbb{CP}^2$ are on one hand the topology of the branch curve $D \subset \mathbb{CP}^2$ (up to isotopy and cancellation of pairs of nodes), and on the other hand a monodromy morphism $\theta : \pi_1(\mathbb{CP}^2 - D) \to S_N$ describing the manner in which the $N = \deg f$ sheets of the covering are arranged above $\mathbb{CP}^2 - D$.

Some simple properties of the monodromy morphism θ can be readily seen by considering the local models of Definition 4.1. For example, the image of a small loop γ bounding a disc that intersects D transversely in a single smooth point (such a loop is called a *geometric generator* of $\pi_1(\mathbb{CP}^2 - D)$) by θ is necessarily a transposition. The smoothness of X above a singular point of D implies some compatibility properties on these transpositions (geometric generators corresponding to the two branches of D at a node must map to disjoint commuting transpositions, while to a cusp must correspond a pair of adjacent transpositions). Finally, the connectedness of X implies the surjectivity of θ (because the subgroup $\mathrm{Im}(\theta)$ is generated by transpositions and acts transitively on the fiber of the covering).

It must be mentioned that the amount of information present in the monodromy morphism θ is fairly small: a classical conjecture in algebraic geometry (Chisini's conjecture, solved by Kulikov [Ku]) asserts that, given an algebraic singular plane curve D with cusps and nodes, a symmetric group-valued monodromy morphism θ compatible with D (in the above sense), if it exists, is unique except in a small finite list of cases. Whether Chisini's conjecture also holds for symplectic branch curves is an open question, but in any case the number of possibilities for θ is always finite.

The study of a singular complex curve $D \subset \mathbb{CP}^2$ can be carried out using the braid monodromy techniques developed in complex algebraic geometry by Moishezon and Teicher [Mo2, Te1]: the idea is to choose a linear projection $\pi : \mathbb{CP}^2 - \{\mathrm{pt}\} \to \mathbb{CP}^1$, for example $\pi(x\!:\!y\!:\!z) = (x\!:\!y)$, in such a way that the curve D lies in general position with respect to the fibers of π, i.e. D is positively transverse to the fibers of π everywhere except at isolated non-degenerate smooth complex tangencies. The restriction $\pi_{|D}$ is then a singular branched covering of degree $d = \deg D$, with *special points* corresponding to the singularities of D (nodes and cusps) and to the tangency points. Moreover, we can assume that all special points lie in distinct fibers of π. A plane curve satisfying these topological requirements is called a *braided* (or *Hurwitz*) curve.



Except for those which contain special points of D, the fibers of π are lines intersecting the curve D in d distinct points. If one chooses a reference point $q_0 \in \mathbb{CP}^1$ (and the corresponding fiber $\ell \simeq \mathbb{C} \subset \mathbb{CP}^2$ of π), and if one restricts to an affine subset in order to be able to trivialize the fibration π, the topology of the branched covering $\pi_{|D}$ can be described by a *braid monodromy* morphism

$$\rho : \pi_1(\mathbb{C} - \{\text{pts}\}, q_0) \to B_d, \tag{7}$$

where B_d is the braid group on d strings. The braid $\rho(\gamma)$ corresponds to the motion of the d points of $\ell \cap D$ inside the fibers of π when moving along the loop γ.

Recall that the braid group B_d is the fundamental group of the configuration space of d distinct points in $\mathbb{R}^2$; it is also the group of isotopy classes of compactly supported orientation-preserving diffeomorphisms of $\mathbb{R}^2$ leaving invariant a set of d given distinct points. It is generated by the standard *half-twists* $X_1, \dots, X_{d-1}$ (braids which exchange two consecutive points by rotating them counterclockwise by 180 degrees around each other), with relations $X_i X_j = X_j X_i$ for $|i-j| \geq 2$ and $X_i X_{i+1} X_i = X_{i+1} X_i X_{i+1}$ (the reader is referred to Birman's book [Bi] for more details).

Another equivalent way to consider the monodromy of a braided curve is to choose an ordered system of generating loops in the free group $\pi_1(\mathbb{C} - \{\text{pts}\}, q_0)$. The morphism ρ can then be described by a *factorization* in the braid group B_d, i.e. a decomposition of the monodromy at infinity into the product of the individual monodromies around the various special points of D. By observing that the total space of π is the line bundle $O(1)$ over $\mathbb{CP}^1$, it is easy to see that the monodromy at infinity is given by the central element $\Delta^2 = (X_1 \dots X_{d-1})^d$ of B_d (called "full twist" because it represents a rotation of a large disc by 360 degrees). The individual monodromies around the special points are conjugated to powers of half-twists, the exponent being 1 in the case of tangency points, 2 in the case of positive nodes (or -2 for negative nodes), and 3 in the case of cusps.

The braid monodromy ρ and the corresponding factorization depend on trivialization choices, which affect them by *simultaneous conjugation* by an element of B_d (change of trivialization of the fiber ℓ of π), or by *Hurwitz operations* (change of generators of the group $\pi_1(\mathbb{C} - \{\text{pts}\}, q_0)$). There is a one-to-one correspondence between braid monodromy morphisms $\rho : \pi_1(\mathbb{C} - \{\text{pts}\}) \to B_d$ (mapping generators to suitable powers of half-twists) up to these two algebraic operations and singular (not necessarily complex) braided curves of degree d in $\mathbb{CP}^2$ up to isotopy among such curves (see e.g. [KK] for a detailed exposition). Moreover, it is easy to check that every braided curve in $\mathbb{CP}^2$ can be deformed into a braided symplectic curve, canonically up to isotopy among symplectic braided curves (this deformation is performed by collapsing the curve D into a neighborhood of a complex line in a way that preserves the fibers of π). However, the curve D is isotopic to a complex curve only for certain specific choices of the morphism ρ.

Unlike the case of complex curves, it is not clear *a priori* that the symplectic branch curve D_k of one of the covering maps given by Theorem 4.3 can be made compatible with the linear projection π; making the curve D_k braided relies on an improvement of Theorem 4.3 in order to control more precisely the behavior of D_k near its special points (tangencies, nodes, cusps) [AK]. Moreover, one must take into account the possible occurrence of creations or cancellations of admissible pairs of nodes in the branch curve D_k, which affect the braid monodromy morphism $\rho_k : \pi_1(\mathbb{C} - \{\text{pts}\}) \to B_d$ by insertion or deletion of pairs of factors. The uniqueness statement in Theorem 4.3 then leads to the following result, obtained in collaboration with Katzarkov [AK]:

Theorem 4.6 (A.-Katzarkov)**.** *For given large enough k, the monodromy morphisms (ρ_k, θ_k) associated to the approximately holomorphic branched covering maps $f_k : X \to \mathbb{CP}^2$ defined by triples of sections of $L^{\otimes k}$ are, up to conjugation, Hurwitz operations, and insertions/deletions, invariants of the symplectic manifold (X, ω). Moreover, these invariants are* complete*, in the sense that the data (ρ_k, θ_k) are sufficient to reconstruct the manifold (X, ω) up to symplectomorphism.*

It is interesting to mention that the symplectic Lefschetz pencils constructed by Donaldson (Theorem 2.7) can be recovered very easily from the branched covering maps f_k, simply by considering the $\mathbb{CP}^1$-valued maps $\pi \circ f_k$. In other words, the fibers $\Sigma_{k,\alpha}$ of the pencil are the preimages by f_k of the fibers of π, and the singular fibers of the pencil correspond to the fibers of π through the tangency points of D_k.

In fact, the monodromy morphisms ψ_k of the Lefschetz pencils $\pi \circ f_k$ (see §2.4) can be recovered very explicitly from θ_k and ρ_k. By restriction to the line $\bar{\ell} = \ell \cup \{\infty\}$, the S_N-valued morphism θ_k describes the topology of a fiber Σ_k of the pencil as an N-fold covering of $\mathbb{CP}^1$ with d branch points; the set of base points Z_k is the preimage of the point at infinity in $\bar{\ell}$. This makes it possible to define a *lifting homomorphism* from a subgroup $B^0_d(\theta_k) \subset B_d$ (*liftable braids*) to the mapping class group $\mathrm{Map}(\Sigma_k, Z_k) = \mathrm{Map}_{g,N}$. The various monodromies are then related by the following formula [AK]:

$$\psi_k = (\theta_k)_* \circ \rho_k. \tag{8}$$

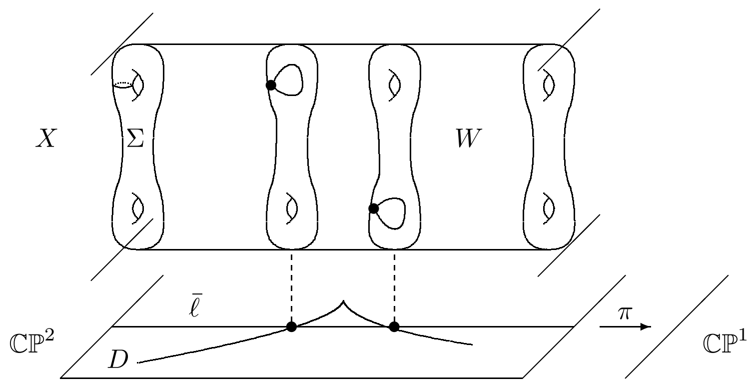
The lifting homomorphism $(\theta_k)_*$ maps liftable half-twists to Dehn twists, so that the tangencies between the branch curve D_k and the fibers of π determine explicitly the vanishing cycles of the Lefschetz pencil $\pi \circ f_k$. On the other hand, the monodromy around a node or cusp of D_k lies in the kernel of $(\theta_k)_*$.

The lifting homomorphism θ_* can be defined more precisely as follows: the space $\tilde{\mathcal{X}}_d$ of configurations of d distinct points in $\mathbb{R}^2$ together with branching data (a transposition in S_N attached to each point) is a finite covering of the space $\mathcal{X}_d$ of configurations of d distinct points. The morphism θ determines a lift $\tilde{*}$ of the base point in $\mathcal{X}_d$, and the liftable braid subgroup of $B_d = \pi_1(\mathcal{X}_d, *)$ is the stabilizer of θ for the action of B_d by deck transformations of the covering $\tilde{\mathcal{X}}_d \to \mathcal{X}_d$, i.e. $B^0_d(\theta) = \pi_1(\tilde{\mathcal{X}}_d, \tilde{*})$. Moreover, $\tilde{\mathcal{X}}_d$ is naturally equipped with a universal fibration $\mathcal{Y}_d \to \tilde{\mathcal{X}}_d$ by genus g Riemann surfaces with N marked points: the lifting homomorphism $\theta_* : B^0_d(\theta) \to \mathrm{Map}_{g,N}$ is by definition the monodromy of this fibration.

The relation (8) is very useful for explicit calculations of the monodromy of Lefschetz pencils, which is accessible to direct methods only in a few very specific cases. By comparison, the various available techniques for braid monodromy calculations [Mo3, Te1, ADKY] are much more powerful, and make it possible to carry out calculations in a much larger number of cases (see §4.5 below). In particular, in view of Theorem 2.7 we are mostly interested in the monodromy of high degree Lefschetz pencils, where the fiber genus and the number of singular fibers are very high, making them inaccessible to direct calculation even for the simplest complex algebraic surfaces.

4.4. **Monodromy invariants in higher dimension.** When $\dim X > 4$, the topology of a projective map $f_k : X - Z_k \to \mathbb{CP}^2$ (as given by Theorem 4.4) and of the discriminant curve $D_k \subset \mathbb{CP}^2$ can be described using the same methods as for a branched covering; the only difference is that the morphism θ_k describing the monodromy of the fibration above the complement of D_k now takes values in the symplectic mapping class group $\mathrm{Map}^\omega(\Sigma_k, Z_k)$ of the generic fiber of f_k. Theorem 4.6 then remains true in this context [Au3]. However, we still face the same difficulty as in the case of Lefschetz pencils of arbitrary dimension, namely the fact that the mapping class group of a symplectic manifold of dimension 4 or more is essentially unknown. Therefore, even though the monodromy morphisms (ρ_k, θ_k) provide an appealing description of a symplectic 6-manifold fibered above $\mathbb{CP}^2$, they cannot be used to accurately describe a symplectic manifold of dimension 8 or more. The definition of invariants using maps to higher-dimensional projective spaces (in order to decrease the dimension of the fibers), although possible in principle, does not seem to be a satisfactory solution to this problem, because the structure of the discriminant hypersurface in $\mathbb{CP}^m$ becomes very complicated for $m \geq 3$.

However, one can work around this difficulty by means of a dimensional reduction process. Indeed, the restriction of f_k to the line $\bar\ell \subset \mathbb{CP}^2$ defines a Lefschetz pencil structure on the symplectic hypersurface $W_k = f_k^{-1}(\bar\ell) \subset X$, with generic fiber Σ_k, base locus Z_k, and monodromy θ_k.



This structure can be enriched by adding an extra section of $L^{\otimes k}$ in order to obtain a map from W_k to $\mathbb{CP}^2$ with generic local models and braided discriminant curve; this map can in turn be characterized using monodromy invariants. This process can be repeated on successive hyperplane sections until we reduce to the 4-dimensional case. Hence, given a symplectic manifold (X^{2n}, ω) and a large integer k, we obtain $n-1$ singular curves $D_{k,(n)}, D_{k,(n-1)}, \dots, D_{k,(2)} \subset \mathbb{CP}^2$, described by $n-1$ braid monodromy morphisms, and a homomorphism $\theta_{k,(2)}$ from $\pi_1(\mathbb{CP}^2 - D_{k,(2)})$ to a symmetric group.

These invariants are sufficient to successively reconstruct the various submanifolds of X involved in the reduction process. Indeed, given a symplectic manifold $\Sigma_{k,(r-1)}$ of dimension $2r-2$ equipped with a Lefschetz pencil structure with generic fiber $\Sigma_{k,(r-2)}$ and monodromy $\theta_{k,(r)}$, and given the braid monodromy of a plane curve $D_{k,(r)} \subset \mathbb{CP}^2$, we obtain a symplectic manifold $\Sigma_{k,(r)}$ of dimension $2r$ equipped with a $\mathbb{CP}^2$-valued map with generic fiber $\Sigma_{k,(r-2)}$, monodromy $\theta_{k,(r)}$ and discriminant cuve $D_{k,(r)}$. Composing with the projection π from $\mathbb{CP}^2$ to $\mathbb{CP}^1$ we obtain on $\Sigma_{k,(r)}$ a Lefschetz pencil structure with generic fiber $\Sigma_{k,(r-1)}$, whose monodromy $\theta_{k,(r+1)}$ can be obtained from the braid monodromy of $D_{k,(r)}$ using the lifting homomorphism induced by $\theta_{k,(r)}$, using a formula similar to (8).

Therefore, the data consisting of $n-1$ braid monodromies and a symmetric group-valued homomorphism, satisfying certain compatibility conditions (each braid monodromy must be contained in an appropriate liftable braid subgroup, and the monodromies around nodes and cusps must lie in the kernel of the lifting homomorphism), completely determines a symplectic manifold (X, ω) up to symplectomorphism. Hence, we have the following combinatorial description of compact symplectic manifolds [Au3]:

Theorem 4.7. *For given $k \gg 0$, the monodromy morphisms $\rho_{k,(n)}, \dots, \rho_{k,(2)}$ with values in braid groups and $\theta_{k,(2)}$ with values in a symmetric group that can be associated to linear systems of suitably chosen sections of $L^{\otimes k}$ are, up to conjugations, Hurwitz operations and insertions/deletions, invariants of the symplectic manifold (X^{2n}, ω). Moreover, these invariants contain enough information to reconstruct the manifold (X^{2n}, ω) up to symplectomorphism.*

For higher-dimensional symplectic manifolds, this strategy of approach appears to be more promising than the direct study of $\mathbb{CP}^m$-valued maps for $m \geq 3$, because it makes it unnecessary to handle the very complicated singularities present in higher-dimensional discriminant loci. However, it must be mentioned that, whatever the approach, explicit calculations are to this date possible only on very specific low degree examples.

4.5. **Calculation techniques.** In principle, Theorems 4.6 and 4.7 reduce the classification of compact symplectic manifolds to purely combinatorial questions concerning braid groups and symmetric groups, and symplectic topology seems to largely reduce to the study of certain singular plane curves, or equivalently certain words in braid groups.

The explicit calculation of these monodromy invariants is hard in the general case, but is made possible for a large number of complex surfaces by the use of "degeneration" techniques and of approximately holomorphic perturbations. Hence, the invariants defined by Theorem 4.6 are explicitly computable for various examples such as $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$ [Mo3], a few complete intersections (Del Pezzo or K3 surfaces) [Ro], the Hirzebruch surface $\mathbb{F}_1$, and all double covers of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along connected smooth algebraic curves (among which an infinite family of surfaces of general type) [ADKY].

The degeneration technique, developped by Moishezon and Teicher [Mo3, Te1], starts with a projective embedding of the complex surface X, and deforms the image of this embedding to a singular configuration X_0 consisting of a union of planes intersecting along lines. The discriminant curve of a projection of X_0 to $\mathbb{CP}^2$ is therefore a union of lines; the manner in which the smoothing of X_0 affects this curve can be studied explicitly, by considering a certain number of standard local models near the various points of X_0 where three or more planes intersect. This method makes it possible to handle many examples in low degree, but in the case $k \gg 0$ that we are interested in (very positive linear systems over a fixed manifold), the calculations can only be carried out explicitly for very simple surfaces.

In order to proceed beyond this point, it becomes more efficient to move outside of the algebraic framework and to consider generic approximately holomorphic perturbations of non-generic algebraic maps; the greater flexibility of this setup makes it possible to choose more easily computable local models. For example, the direct calculation of the monodromy invariants becomes possible for all linear systems of the type $\pi^*O(p,q)$ on double couvers of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along connected smooth algebraic curves of arbitrary degree [ADKY]. It also becomes possible to obtain a general "degree doubling" formula, describing explicitly the monodromy invariants associated to the linear system $L^{\otimes 2k}$ in terms of those associated to the linear system $L^{\otimes k}$ (when $k \gg 0$), both for branched covering maps to $\mathbb{CP}^2$ and for 4-dimensional Lefschetz pencils [AK2].

However, in spite of these successes, a serious obstacle restricts the practical applications of monodromy invariants: in general, they cannot be used efficiently to distinguish homeomorphic symplectic manifolds, because no algorithm exists to decide whether two words in a braid group or mapping class group are equivalent to each other via Hurwitz operations. Even if an algorithm could be found, another difficulty is due to the large amount of combinatorial data to be handled: on a typical interesting example, the braid monodromy data can already consist of $\sim 10^4$ factors in a braid group on ~ 100 strings for very small values of the parameter k, and the amount of data grows polynomially with k.

Hence, even when monodromy invariants can be computed, they cannot be *compared*. This theoretical limitation makes it necessary to search for other ways to exploit monodromy data, e.g. by considering invariants that contain less information than braid monodromy but are easier to use in practice.

4.6. **Fundamental groups of branch curve complements.** Given a singular plane curve $D \subset \mathbb{CP}^2$, e.g. the branch curve of a covering, it is natural to study the fundamental group $\pi_1(\mathbb{CP}^2 - D)$. The study of this group for various types of algebraic curves is a classical subject going back to the work of Zariski, and has undergone a lot of development in the 80's and 90's, in part thanks to the work of Moishezon and Teicher [Mo2, Mo3, Te1]. The relation to braid monodromy invariants is a very direct one: the Zariski-van Kampen theorem provides an explicit presentation of the group $\pi_1(\mathbb{CP}^2 - D)$ in terms of the braid monodromy morphism $\rho : \pi_1(\mathbb{C} - \{\text{pts}\}) \to B_d$. However, if one is interested in the case of symplectic branch curves, it is important to observe that the introduction or the cancellation of pairs of nodes affects the fundamental group of the complement, so that it cannot be used directly to define an invariant associated to a symplectic branched covering. In the symplectic world, the fundamental group of the branch curve complement must be replaced by a suitable quotient, the *stabilized fundamental group* [ADKY].

Using the same notations as in §4.3, the inclusion $i : \ell - (\ell \cap D_k) \to \mathbb{CP}^2 - D_k$ of the reference fiber of the linear projection π induces a surjective morphism on fundamental groups; the images of the standard generators of the free group $\pi_1(\ell - (\ell \cap D_k))$ and their conjugates form a subset $\Gamma_k \subset \pi_1(\mathbb{CP}^2 - D_k)$ whose elements are called *geometric generators*. Recall that the images of the geometric generators by the monodromy morphism θ_k are transpositions in S_N. The creation of a pair of nodes in the curve D_k amounts to quotienting $\pi_1(\mathbb{CP}^2 - D_k)$ by a relation of the form $[\gamma_1, \gamma_2] \sim 1$, where $\gamma_1, \gamma_2 \in \Gamma_k$; however, this creation of nodes can be carried out by deforming the branched covering map f_k only if the two transpositions $\theta_k(\gamma_1)$ and $\theta_k(\gamma_2)$ have disjoint supports. Let K_k be the normal subgroup of $\pi_1(\mathbb{CP}^2 - D_k)$ generated by all such commutators $[\gamma_1, \gamma_2]$. Then we have the following result [ADKY]:

Theorem 4.8 (A.-D.-K.-Y.). *For given $k \gg 0$, the* stabilized fundamental group *$\bar{G}_k = \pi_1(\mathbb{CP}^2 - D_k)/K_k$ is an invariant of the symplectic manifold (X^4, ω).*

This invariant can be calculated explicitly for the various examples where monodromy invariants are computable ($\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$, some Del Pezzo and K3 complete intersections, Hirzebruch surface $\mathbb{F}_1$, double covers of $\mathbb{CP}^1 \times \mathbb{CP}^1$); namely, the extremely complicated presentations given by the Zariski-van Kampen theorem in terms of braid monodromy data can be simplified in order to obtain a manageable description of the fundamental group of the branch curve complement. These examples lead to various observations and conjectures.

A first remark to be made is that, for all known examples, when the parameter k is sufficiently large the stabilization operation becomes trivial, i.e. geometric generators associated to disjoint transpositions already commute in $\pi_1(\mathbb{CP}^2 - D_k)$, so that $K_k = \{1\}$ and $\bar{G}_k = \pi_1(\mathbb{CP}^2 - D_k)$. For example, in the case of $X = \mathbb{CP}^2$ with its standard Kähler form, we have $\bar{G}_k = \pi_1(\mathbb{CP}^2 - D_k)$ for all $k \geq 3$. Therefore, when $k \gg 0$ no information seems to be lost when quotienting by K_k (the situation for small values of k is very different).

The following general structure result can be proved for the groups $\bar{G}_k$ (and hence for $\pi_1(\mathbb{CP}^2 - D_k)$) [ADKY]:

Theorem 4.9 (A.-D.-K.-Y.)**.** *Let $f : (X, \omega) \to \mathbb{CP}^2$ be a symplectic branched covering of degree N, with braided branch curve D of degree d, and let $\bar{G} = \pi_1(\mathbb{CP}^2 - D)/K$ be the stabilized fundamental group of the branch curve complement. Then there exists a natural exact sequence*

$$1 \longrightarrow G^0 \longrightarrow \bar{G} \longrightarrow S_N \times \mathbb{Z}_d \longrightarrow \mathbb{Z}_2 \longrightarrow 1.$$

Moreover, if X is simply connected then there exists a natural surjective homomorphism $\phi : G^0 \twoheadrightarrow (\mathbb{Z}^2/\Lambda)^{n-1}$, where

$$\Lambda = \{(c_1(K_X) \cdot \alpha, [f^{-1}(\bar{\ell})] \cdot \alpha),\ \alpha \in H_2(X, \mathbb{Z})\}.$$

In this statement, the two components of the morphism $\bar{G} \to S_N \times \mathbb{Z}_d$ are respectively the monodromy of the branched covering, $\theta : \pi_1(\mathbb{CP}^2 - D) \to S_N$, and the linking number (or abelianization, when D is irreducible) morphism

$$\delta : \pi_1(\mathbb{CP}^2 - D) \to \mathbb{Z}_d \ (\simeq H_1(\mathbb{CP}^2 - D, \mathbb{Z})).$$

The subgroup Λ of $\mathbb{Z}^2$ is entirely determined by the numerical properties of the canonical class $c_1(K_X)$ and of the hyperplane class (the homology class of the preimage of a line $\bar{\ell} \subset \mathbb{CP}^2$: in the case of the covering maps of Theorem 4.3 we have $[f^{-1}(\bar{\ell})] = c_1(L^{\otimes k}) = \frac{k}{2\pi}[\omega]$). The morphism ϕ is defined by considering the N lifts in X of a closed loop γ belonging to G^0, or more precisely their homology classes (whose sum is trivial) in the complement of a hyperplane section and of the ramification curve in X.

Moreover, in the known examples we have a much stronger result on the structure of the subgroups G^0_k for the branch curves of large degree covering maps (determined by sufficiently ample linear systems) [ADKY].

Say that the simply connected complex surface (X, ω) belongs to the class $(\mathcal{C})$ if it belongs to the list of computable examples: $\mathbb{CP}^1 \times \mathbb{CP}^1$, $\mathbb{CP}^2$, the Hirzebruch surface $\mathbb{F}_1$ (equipped with any Kähler form), a Del Pezzo or K3 surface (equipped with a Kähler form coming from a specific complete intersection realization), or a double cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along a connected smooth algebraic curve (equipped with a Kähler form in the class $\pi^* O(p, q)$ for $p, q \geq 1$). Then we have:

Theorem 4.10 (A.-D.-K.-Y.). *If (X,ω) belongs to the class $(\mathcal{C})$, then for all large enough k the homomorphism ϕ_k induces an isomorphism on the abelianized groups, i.e.* $\operatorname{Ab} G_k^0 \simeq (\mathbb{Z}^2/\Lambda_k)^{N_k-1}$, *while* $\operatorname{Ker}\phi_k = [G_k^0, G_k^0]$ *is a quotient of* $\mathbb{Z}_2 \times \mathbb{Z}_2$.

It is natural to make the following conjecture:

Conjecture 4.11. *If X is a simply connected symplectic 4-manifold, then for all large enough k the homomorphism ϕ_k induces an isomorphism on the abelianized groups, i.e.* $\operatorname{Ab} G_k^0 \simeq (\mathbb{Z}^2/\Lambda_k)^{N_k-1}$.

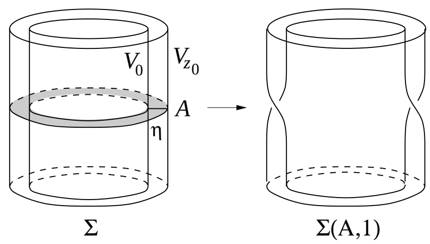
4.7. **Symplectic isotopy and non-isotopy.** While it has been well-known for many years that compact symplectic 4-manifolds do not always admit Kähler structures, it has been discovered more recently that symplectic curves (smooth or singular) in a given manifold can also offer a wider range of possibilities than complex curves. Proposition 4.2 and Theorem 4.3 establish a bridge between these two phenomena: indeed, a covering of $\mathbb{CP}^2$ (or any other complex surface) branched along a complex curve automatically inherits a complex structure. Therefore, starting with a non-Kähler symplectic manifold, Theorem 4.3 always yields branch curves that are not isotopic to any complex curve in $\mathbb{CP}^2$. The study of isotopy and non-isotopy phenomena for curves is therefore of major interest for our understanding of the topology of symplectic 4-manifolds.

The *symplectic isotopy problem* asks whether, in a given complex surface, every symplectic submanifold representing a given homology class is isotopic to a complex submanifold. The first positive result in this direction was due to Gromov, who showed using his compactness result for pseudo-holomorphic curves (Theorem 1.10) that, in $\mathbb{CP}^2$, a smooth symplectic curve of degree 1 or 2 is always isotopic to a complex curve. Successive improvements of this technique have made it possible to extend this result to curves of higher degree in $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$; the currently best known result is due to Siebert and Tian, and makes it possible to handle the case of smooth curves in $\mathbb{CP}^2$ up to degree 17 [ST]. Isotopy results are also known for sufficiently simple singular curves (Barraud, Shevchishin [Sh], …).

Contrarily to the above examples, the general answer to the symplectic isotopy problem appears to be negative. The first counterexamples among smooth connected symplectic curves were found by Fintushel and Stern [FS2], who constructed by a *braiding* process infinite families of mutually non-isotopic symplectic curves representing a same homology class (a multiple of the fiber) in elliptic surfaces, and to Smith, who used the same construction in higher genus. However, these two constructions are preceded by a result of Moishezon [Mo4], who established in the early 90's a result implying the existence in $\mathbb{CP}^2$ of infinite families of pairwise non-isotopic singular symplectic curves of given degree with given numbers of node and cusp singularities. A reformulation of Moishezon's construction makes it possible to see that it also relies on braiding; moreover, the braiding construction can be related to a surgery operation along a Lagrangian torus in a symplectic 4-manifold, known as *Luttinger surgery* [ADK]. This reformulation makes it possible to vastly simplify Moishezon's argument, which was based on lengthy and delicate calculations of fundamental groups of curve complements, while relating it with various constructions developed in 4-dimensional topology.

Given an embedded Lagrangian torus T in a symplectic 4-manifold (X,ω), a homotopically non-trivial embedded loop $\gamma \subset T$ and an integer k, Luttinger surgery is an operation that consists in cutting out from X a tubular neighborhood of T, foliated by parallel Lagrangian tori, and gluing it back in such a way that the new meridian loop differs from the old one by k twists along the loop γ (while longitudes are not affected), yielding a new symplectic manifold $(\tilde{X}, \tilde{\omega})$. This relatively little-known construction, which e.g. makes it possible to turn a product $T^2 \times \Sigma$ into any surface bundle over T^2, or to transform an untwisted fiber sum into a twisted one, can be used to described in a unified manner numerous examples of exotic symplectic 4-manifolds constructed in the past few years.

Meanwhile, the braiding construction of symplectic curves starts with a (possibly singular) symplectic curve $\Sigma \subset (Y^4, \omega_Y)$ and two symplectic cylinders embedded in Σ, joined by a Lagrangian annulus contained in the complement of Σ, and consists in performing k half-twists between these two cylinders in order to obtain a new symplectic curve $\tilde{\Sigma}$ in Y.



When Σ is the branch curve of a symplectic branched covering $f : X \to Y$, the following result holds [ADK]:

Proposition 4.12. *The covering of Y branched along the symplectic curve $\tilde{\Sigma}$ obtained by braiding Σ along a Lagrangian annulus $A \subset Y - \Sigma$ is naturally symplectomorphic to the manifold $\tilde{X}$ obtained from the branched cover X by Luttinger surgery along a Lagrangian torus $T \subset X$ formed by the union of two lifts of A.*

Hence, once an infinite family of symplectic curves has been constructed by braiding, it is sufficient to find invariants that distinguish the corresponding branched covers in order to conclude that the curves are not isotopic. In the Fintushel-Stern examples, the branched covers are distinguished by their Seiberg-Witten invariants, whose behavior is well understood in the case of elliptic fibrations and their surgeries.

In the case of Moishezon's examples, a braiding construction makes it possible to construct, starting from complex curves $\Sigma_{p,0} \subset \mathbb{CP}^2$ ($p \geq 2$) of degree $d_p = 9p(p-1)$ with $\kappa_p = 27(p-1)(4p-5)$ cusps and $\nu_p = 27(p-1)(p-2)(3p^2+3p-8)/2$ nodes, symplectic curves $\Sigma_{p,k} \subset \mathbb{CP}^2$ for all $k \in \mathbb{Z}$, with the same degree and numbers of singular points. By Proposition 4.12, these curves can be viewed as the branch curves of symplectic coverings whose total spaces $X_{p,k}$ differ by Luttinger surgeries along a Lagrangian torus $T \subset X_{p,0}$. The effect of these surgeries on the canonical class and on the symplectic form can be described explicitly, which makes it possible to distinguish the manifolds $X_{p,k}$: the canonical class of $(X_{p,k}, \omega_{p,k})$ is given by $p\, c_1(K_{p,k}) = (6p-9)[\omega_{p,k}] + (2p-3)k\, PD([T])$. Moreover, $[T] \in H_2(X_{p,k}, \mathbb{Z})$ is not a torsion class, and if $p \not\equiv 0 \mod 3$ or $k \equiv 0 \mod 3$ then it is a primitive class [ADK]. This implies that infinitely many of the curves $\Sigma_{p,k}$ are pairwise non-isotopic.

It is to be observed that the argument used by Moishezon to distinguish the curves $\Sigma_{p,k}$, which relies on a calculation of the fundamental groups $\pi_1(\mathbb{CP}^2 - \Sigma_{p,k})$ [Mo4], is related to the one in [ADK] by means of Conjecture 4.11, of which it can be concluded a posteriori that it is satisfied by the given branched covers $X_{p,k} \to \mathbb{CP}^2$: in particular, the fact that $\pi_1(\mathbb{CP}^2 - \Sigma_{p,k})$ is infinite for $k = 0$ and finite for $k \neq 0$ is consistent with the observation that the canonical class of $X_{p,k}$ is proportional to its symplectic class iff $k = 0$.

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