FUNDAMENTAL GROUPS OF COMPLEMENTS OF PLANE CURVES AND SYMPLECTIC INVARIANTS

D. AUROUX, S.K. DONALDSON, L. KATZARKOV, AND M. YOTOV

Abstract. Introducing the notion of stabilized fundamental group for the complement of a branch curve in $\mathbb{CP}^2$, we define effectively computable invariants of symplectic 4-manifolds that generalize those previously introduced by Moishezon and Teicher for complex projective surfaces. Moreover, we study the structure of these invariants and formulate conjectures supported by calculations on new examples.

1. Introduction

Using approximately holomorphic techniques first introduced in [5], it was shown in [2] (see also [1]) that compact symplectic 4-manifolds with integral symplectic class can be realized as branched covers of $\mathbb{CP}^2$ and can be investigated using the braid group techniques developed by Moishezon and subsequently by Moishezon and Teicher for the study of complex surfaces (see e.g. [13]):

Theorem 1.1 ([2])**.** *Let (X,ω) be a compact symplectic 4-manifold, and let L be a line bundle with $c_1(L) = \frac{1}{2\pi}[\omega]$. Then there exist branched covering maps $f_k : X \to \mathbb{CP}^2$ defined by approximately holomorphic sections of $L^{\otimes k}$ for all large enough values of k; the corresponding branch curves $D_k \subset \mathbb{CP}^2$ admit only nodes (both orientations) and complex cusps as singularities, and give rise to well-defined braid monodromy invariants. Moreover, up to admissible creations and cancellations of pairs of nodes in the branch curve, for large k the topology of f_k is a symplectic invariant.*

This makes it possible to associate to (X,ω) a sequence of invariants (indexed by $k \gg 0$) consisting of two objects: the braid monodromy characterizing the branch curve D_k, and the *geometric monodromy representation* $\theta_k : \pi_1(\mathbb{CP}^2 - D_k) \to S_n$ ($n = \deg f_k$) characterizing the n-fold covering of $\mathbb{CP}^2 - D_k$ induced by f_k [2]. These invariants are extremely powerful (from them one can recover (X,ω) up to symplectomorphism) but too complicated to handle in practical cases.

In the study of complex surfaces, Moishezon and Teicher have shown that the fundamental group $\pi_1(\mathbb{CP}^2 - D)$ (or, restricting to an affine subset, $\pi_1(\mathbb{C}^2 - D)$) can be computed explicitly in some simple examples; generally speaking, this group has been expected to provide a valuable invariant for distinguishing diffeomorphism types of complex surfaces of general type. However, in the symplectic case, it is affected by creations and cancellations of pairs of nodes and cannot be used immediately as an invariant.

We will introduce in §2 a certain quotient G_k (resp. $\bar{G}_k$) of $\pi_1(\mathbb{C}^2 - D_k)$ (resp. $\pi_1(\mathbb{CP}^2 - D_k)$), the *stabilized fundamental group*, which remains invariant under creations and cancellations of pairs of nodes. As an immediate corollary of the construction and of Theorem 1.1, we obtain the following

Theorem 1.2. *For large enough k, the stabilized groups $G_k = G_k(X,\omega)$ (resp. $\bar{G}_k(X,\omega)$) and their reduced subgroups $G_k^0 = G_k^0(X,\omega)$ are symplectic invariants of the manifold (X,ω).*

These invariants can be computed explicitly in various examples, some due to Moishezon, Teicher and Robb, others new; these examples will be presented in §4, and a brief overview of the techniques involved in the computations is given in §6 and §7. The new examples include double covers of $\mathbb{CP}^1\times\mathbb{CP}^1$ branched along arbitrary complex curves (Theorem 4.6 and §7); similar methods should apply to other double covers as well, thus providing results for both types of so-called Horikawa surfaces. The calculations described in §7, which rely on various innovative tools in addition to a suitable reformulation of the methods developed by Moishezon and Teicher, go well beyond the scope of results accessible using only the previously known techniques, and may present interest of their own for applications in algebraic geometry.

The available data suggest several conjectures about the structure of the stabilized fundamental groups.

First of all, it appears that in most examples the stabilization operation does not actually affect the fundamental group. The only known exceptions are given by "small" linear systems with insufficient ampleness properties, where the stabilization is a quotient by a non-trivial subgroup (see §4). Therefore we have the following

Conjecture 1.3. *Assume that (X,ω) is a complex surface, and let D_k be the branch curve of a generic projection to $\mathbb{CP}^2$ of the projective embedding of X given by the linear system $|kL|$. Then, provided that k is large enough, the stabilization operation is trivial, i.e. $G_k(X,\omega)\simeq\pi_1(\mathbb{C}^2-D_k)$ and $\bar{G}_k(X,\omega)\simeq\pi_1(\mathbb{CP}^2-D_k)$.*

An important class of fundamental groups for which the conjecture holds will be described in §3.

Moreover, the structure of the stabilized fundamental groups seems to be remarkably simple, at least when the manifold X is simply connected; in all known examples they are extensions of a symmetric group by a solvable group, while there exist plane curves with much more complicated complements [4, 6]. In fact these groups seem to be largely determined by intersection pairing data in $H_2(X,\mathbb{Z})$. More precisely, the following result will be proved in §5:

Definition 1.4. *Let Λ_k be the image of the map $\lambda_k : H_2(X,\mathbb{Z})\to\mathbb{Z}^2$ defined by $\lambda_k(\alpha) = (\alpha\cdot L_k, \alpha\cdot R_k)$, where $L_k = k\,c_1(L)$ and $R_k = c_1(K_X)+3L_k$ are the classes in $H^2(X,\mathbb{Z})$ Poincaré dual to a hyperplane section and to the ramification curve respectively.*

Theorem 1.5. *If the symplectic manifold X is simply connected, then there exists a natural surjective homomorphism $\phi_k : \mathrm{Ab}\,G_k^0(X,\omega)\to(\mathbb{Z}^2/\Lambda_k)\otimes\mathcal{R}_{n_k}\simeq(\mathbb{Z}^2/\Lambda_k)^{n_k-1}$, where $n_k = \deg f_k = L_k\cdot L_k$, and $\mathcal{R}_{n_k}$ is the reduced regular representation of S_{n_k} (isomorphic to $\mathbb{Z}^{n_k-1}$).*

The map ϕ_k is (G_k, S_{n_k})-equivariant, in the sense that $\phi_k(g^{-1}\gamma g) = \theta_k(g)\cdot\phi_k(\gamma)$ for any elements $g\in G_k(X,\omega)$ and $\gamma\in\mathrm{Ab}\,G_k^0(X,\omega)$ (cf. also Lemma 5.2).

In the examples discussed in §4, the group G_k^0 is always close to being abelian, and ϕ_k is always an isomorphism. It seems likely that the injectivity of ϕ_k can be proved using techniques similar to those described in §6–7. Therefore, it makes sense to formulate the following

Conjecture 1.6. *If the symplectic manifold X is simply connected and k is large enough, then* $\mathrm{Ab}\, G^0_k(X,\omega) \simeq (\mathbb{Z}^2/\Lambda_k)\otimes\mathcal{R}_{n_k}$*, and the commutator subgroup* $[G^0_k, G^0_k]$ *is a quotient of* $(\mathbb{Z}_2)^2$.

Conjectures 1.3 and 1.6 provide an almost complete tentative description of the structure of fundamental groups of branch curve complements in high degrees. In relation with the property $(*)$ introduced in §3, they also provide a framework to explain various observations and conjectures made in [14] and [12].

The obtained results seem to indicate that fundamental groups of branch curve complements cannot be used as invariants to symplectically distinguish homeomorphic manifolds. This is in sharp contrast with the braid monodromy data, which completely determines the symplectomorphism type of (X,ω) [2] ; how to introduce effectively computable invariants retaining more of the information contained in the braid monodromy remains an open question.

2. Braid monodromy and stabilized fundamental groups

Let D_k be the branch curve of a covering map $f_k : X \to \mathbb{CP}^2$ as in Theorem 1.1. Braid monodromy invariants are defined by considering a generic projection $\pi : \mathbb{CP}^2 - \{\mathrm{pt}\} \to \mathbb{CP}^1$: the pole of the projection lies away from D_k, and a generic fiber of π intersects D_k in $d = \deg D_k$ distinct points, the only exceptions being fibers through cusps or nodes of D_k, or fibers that are tangent to D_k at one of its smooth points ("vertical tangencies"). Moreover we can assume that the special points (cusps, nodes and vertical tangencies) of D_k all lie in different fibers of π.

By restricting ourselves to an affine subset $\mathbb{C}^2 \subset \mathbb{CP}^2$, choosing a base point and trivializing the fibration π, we can view the monodromy of $\pi_{|D_k}$ as a group homomorphism from $\pi_1(\mathbb{C} - \{q_i\})$ (where q_i are the images by π of the special points of D_k) to the braid group B_d. More precisely, the monodromy around a vertical tangency is a half-twist (a braid that exchanges two of the d intersection points of the fiber with D_k by rotating them around each other counterclockwise along a certain path); the monodromy around a positive (resp. negative) node is the square (resp. the inverse of the square) of a half-twist; the monodromy around a cusp is the cube of a half-twist [13, 2].

It is sometimes convenient to choose an ordered system of generating loops for $\pi_1(\mathbb{C} - \{q_i\})$ (one loop going around each q_i), and to express the monodromy as a *braid factorization*, i.e. a decomposition of the central braid Δ^2 (the monodromy around the fiber at infinity, due to the non-triviality of the fibration π over $\mathbb{CP}^1$) into the product of the monodromies along the chosen generating loops. However, this braid factorization is only well-defined up to simultaneous conjugation of all factors (i.e., a change in the choice of the identification of the fibers with $\mathbb{R}^2$) and *Hurwitz equivalence* (i.e., a rearrangement of the factors due to a different choice of the system of generating loops).

The braid monodromy determines in a very explicit manner the fundamental groups $\pi_1(\mathbb{C}^2 - D_k)$ and $\pi_1(\mathbb{CP}^2 - D_k)$. Indeed, consider a generic fiber $\ell \simeq \mathbb{C} \subset \mathbb{CP}^2$ of the projection π (e.g. the fiber containing the base point), intersecting D_k in d distinct points. The free group $\pi_1(\ell - (\ell \cap D_k)) = F_d$ is generated by a system of d loops going around the various points in $\ell \cap D_k$. The inclusion map $i : \ell - (\ell \cap D_k) \to \mathbb{C}^2 - D_k$ induces a surjective homomorphism $i_* : F_d \to \pi_1(\mathbb{C}^2 - D_k)$.

Definition 2.1. *The images of the standard generators of the free group F_d and their conjugates are called geometric generators of $\pi_1(\mathbb{C}^2 - D_k)$; the set of all geometric generators will be denoted by Γ_k.*

By the Zariski-Van Kampen theorem, $\pi_1(\mathbb{C}^2 - D_k)$ is realized as a quotient of F_d by relations corresponding to the various special points (vertical tangencies, nodes, cusps) of D_k; these relations express the fact that the action of the braid monodromy on F_d induces a trivial action on $\pi_1(\mathbb{C}^2 - D_k)$. To each factor in the braid factorization one can associate a pair of elements $\gamma_1, \gamma_2 \in \Gamma_k$ (small loops around the two portions of D_k that meet at the special point), well-determined up to simultaneous conjugation. The relation corresponding to a tangency is $\gamma_1 \sim \gamma_2$; for a node (of either orientation) it is $[\gamma_1, \gamma_2] \sim 1$; for a cusp it becomes $\gamma_1\gamma_2\gamma_1 \sim \gamma_2\gamma_1\gamma_2$. Taking into account all the special points of D_k (i.e. considering the entire braid monodromy), we obtain a presentation of $\pi_1(\mathbb{C}^2 - D_k)$. Moreover, $\pi_1(\mathbb{CP}^2 - D_k)$ is obtained from $\pi_1(\mathbb{C}^2 - D_k)$ just by adding the extra relation $g_1 \dots g_d \sim 1$, where g_i are the images of the standard generators of F_d under the inclusion.

It follows from this discussion that the creation or cancellation of a pair of nodes in D_k may affect $\pi_1(\mathbb{C}^2 - D_k)$ and $\pi_1(\mathbb{CP}^2 - D_k)$ by adding or removing commutation relations between geometric generators. Although it is reasonable to expect that negative nodes can always be cancelled in the branch curves given by Theorem 1.1, the currently available techniques are insufficient to prove such a statement. Instead, a more promising approach is to compensate for these changes in the fundamental groups by considering certain quotients where one stabilizes the group by adding commutation relations between geometric generators. The resulting group is in some sense more natural than $\pi_1(\mathbb{C}^2 - D_k)$ from the symplectic point of view, and as a side benefit it is often easier to compute (see §7). Moreover, it also turns out that, in many cases, no information is lost in the stabilization process (see §3).

In order to define the stabilized group G_k, first observe that, because the branching index of f_k above a smooth point of D_k is always 2, the geometric monodromy representation morphism $\theta_k : \pi_1(\mathbb{CP}^2 - D_k) \to S_n$ describing the topology of the covering above $\mathbb{CP}^2 - D_k$ maps all geometric generators to transpositions in S_n. As seen above, to each nodal point of D_k one can associate geometric generators $\gamma_1, \gamma_2 \in \Gamma_k$, one for each of the two intersecting portions of D_k, so that the corresponding relation in $\pi_1(\mathbb{C}^2 - D_k)$ is $[\gamma_1, \gamma_2] \sim 1$. Since the branching occurs in disjoint sheets of the cover, the two transpositions $\theta_k(\gamma_1)$ and $\theta_k(\gamma_2)$ are necessarily disjoint (i.e. they are distinct and commute). Therefore, adding or removing pairs of nodes amounts to adding or removing relations given by commutators of geometric generators associated to disjoint transpositions.

Definition 2.2. *Let K_k (resp. $\bar{K}_k$) be the normal subgroup of $\pi_1(\mathbb{C}^2 - D_k)$ (resp. $\pi_1(\mathbb{CP}^2 - D_k)$) generated by all commutators $[\gamma_1, \gamma_2]$ where $\gamma_1, \gamma_2 \in \Gamma_k$ are such that $\theta_k(\gamma_1)$ and $\theta_k(\gamma_2)$ are disjoint transpositions. The* stabilized fundamental group *is defined as $G_k = \pi_1(\mathbb{C}^2 - D_k)/K_k$, resp. $\bar{G}_k = \pi_1(\mathbb{CP}^2 - D_k)/\bar{K}_k$.*

Certain natural subgroups of G_k and $\bar{G}_k$ will play an important role in the following sections. Define the *linking number* homomorphism $\delta_k : \pi_1(\mathbb{C}^2 - D_k) \to \mathbb{Z}$ by $\delta_k(\gamma) = 1$ for every $\gamma \in \Gamma_k$; similarly one can define $\bar{\delta}_k : \pi_1(\mathbb{CP}^2 - D_k) \to \mathbb{Z}_d$. When D_k is irreducible (which is the general case), these can also be thought of as abelianization maps from the fundamental groups to the homology groups $H_1(\mathbb{C}^2 - D_k, \mathbb{Z}) \simeq \mathbb{Z}$ and $H_1(\mathbb{CP}^2 - D_k, \mathbb{Z}) \simeq \mathbb{Z}_d$.

Lemma 2.3. $\operatorname{Ker} \delta_k \simeq \operatorname{Ker} \bar{\delta}_k$.

Proof. Since $\pi_1(\mathbb{CP}^2 - D_k) = \pi_1(\mathbb{C}^2 - D_k)/\langle g_1 \dots g_d \rangle$ and $\delta_k(g_1 \dots g_d) = d$, it is sufficient to show that the product $g_1 \dots g_d$ belongs to the center of $\pi_1(\mathbb{C}^2 - D_k)$. Observe that the relation in $\pi_1(\mathbb{C}^2 - D_k)$ coming from a special point of D_k can be rewritten in the form $g \sim b_* g \ \forall g \in F_d$, where $b \in B_d$ is the braid monodromy around the given special point, acting on F_d. In particular, if we consider the braid monodromy as a factorization $\Delta^2 = \prod b_i$, we obtain that $g \sim (\prod b_i)_* g = (\Delta^2)_* g$ for any element g. However the action of the braid Δ^2 on F_d is exactly conjugation by $g_1 \dots g_d$; we conclude that $g_1 \dots g_d$ commutes with any element of $\pi_1(\mathbb{C}^2 - D_k)$, hence the result. ∎

The homomorphisms δ_k and $\bar{\delta}_k$ are obviously surjective. Moreover, θ_k is also surjective, because of the connectedness of X: the subgroup $\operatorname{Im} \theta_k \subseteq S_n$ is generated by transpositions and acts transitively on $\{1, \dots, n\}$, so it is equal to S_n. However, the image of $\theta_k^+ = (\theta_k, \delta_k) : \pi_1(\mathbb{C}^2 - D_k) \to S_n \times \mathbb{Z}$ is the index 2 subgroup $\{(\sigma, i) : \operatorname{sgn}(\sigma) \equiv i \bmod 2\}$, and similarly for $\bar{\theta}_k^+ = (\theta_k, \bar{\delta}_k) : \pi_1(\mathbb{CP}^2 - D_k) \to S_n \times \mathbb{Z}_d$ (note that d is always even). Since $K_k \subseteq \operatorname{Ker} \theta_k^+$, we can make the following definition:

Definition 2.4. *Let* $H_k^0 = \operatorname{Ker} \theta_k^+ \simeq \operatorname{Ker} \bar{\theta}_k^+$. *The* reduced subgroup *of* G_k *is* $G_k^0 = H_k^0 / K_k$. *We have the following exact sequences:*

$$1 \longrightarrow G_k^0 \longrightarrow G_k \longrightarrow S_n \times \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 1,$$
$$1 \longrightarrow G_k^0 \longrightarrow \bar{G}_k \longrightarrow S_n \times \mathbb{Z}_d \longrightarrow \mathbb{Z}_2 \longrightarrow 1.$$

Theorem 1.2 is now obvious from the definitions and from Theorem 1.1: since creating a pair of nodes amounts to adding a relation of the form $[\gamma_1, \gamma_2] \sim 1$ where $[\gamma_1, \gamma_2] \in K_k$ (resp. $\bar{K}_k$), by construction it does not affect the groups G_k, $\bar{G}_k$ and G_k^0, which are therefore symplectic invariants for k large enough.

3. $\tilde{B}_n$-groups and their stabilizations

Denote by B_n (resp. P_n, $P_{n,0}$) the braid group on n strings (resp. the subgroups of pure braids and pure braids of degree 0), and denote by $X_1, \dots, X_{n-1}$ the standard generators of B_n. Recall that X_i is a half-twist along a segment joining the points i and $i+1$, and that the relations among these generators are $[X_i, X_j] = 1$ if $|i-j| \geq 2$ and $X_i X_{i+1} X_i = X_{i+1} X_i X_{i+1}$.

Let $\tilde{B}_n$ be the quotient of B_n by the commutator of half-twists along two paths intersecting transversely in one point: $\tilde{B}_n = B_n / [X_2, X_3^{-1} X_1^{-1} X_2 X_1 X_3]$. The maps $\sigma : B_n \to S_n$ (induced permutation) and $\delta : B_n \to \mathbb{Z}$ (degree) factor through $\tilde{B}_n$, so one can define the subgroups $\tilde{P}_n = \operatorname{Ker} \sigma$ and $\tilde{P}_{n,0} = \operatorname{Ker} (\sigma, \delta)$. The structure of $\tilde{B}_n$ and its subgroups is described in detail in §1 of [9]; unlike P_n and $P_{n,0}$ which are quite complicated, these groups are fairly easy to understand: $\tilde{P}_{n,0}$ is solvable, its commutator subgroup is $[\tilde{P}_{n,0}, \tilde{P}_{n,0}] \simeq \mathbb{Z}_2$ and its abelianization is $\operatorname{Ab}(\tilde{P}_{n,0}) \simeq \mathbb{Z}^{n-1}$ (it can in fact be identified naturally with the reduced regular representation $\mathcal{R}_n$ of S_n). More precisely, we have:

Lemma 3.1 (Moishezon)**.** *Let x_i be the image of X_i in $\tilde{B}_n$, and define $s_1 = x_1^2$, $\eta = [x_1^2, x_2^2]$, $u_i = [x_i^{-1}, x_{i+1}^2]$ for $1 \le i \le n-2$, and $u_{n-1} = [x_{n-2}^2, x_{n-1}]$. Then $\tilde{P}_{n,0}$ is generated by $u_1, \dots, u_{n-1}$, and $\tilde{P}_n$ is generated by $s_1, u_1, \dots, u_{n-1}$.*

The relations among these elements are $[u_i, u_j] = 1$ if $|i-j| \ge 2$, $[u_i, u_{i+1}] = \eta$, $[s_1, u_i] = 1$ if $i \ne 2$, and $[s_1, u_2] = \eta$. The element η is central in $\tilde{B}_n$, has order 2 (i.e. $\eta^2 = 1$), and generates the commutator subgroups $[\tilde{P}_{n,0}, \tilde{P}_{n,0}] = [\tilde{P}_n, \tilde{P}_n] \simeq \mathbb{Z}_2$ (in particular, for any two adjacent half-twists x and y we have $[x^2, y^2] = \eta$). As a consequence, $\mathrm{Ab}(\tilde{P}_n) \simeq \mathbb{Z}^n$ and $\mathrm{Ab}(\tilde{P}_{n,0}) \simeq \mathbb{Z}^{n-1}$.

Moreover, the action of $\tilde{B}_n$ on $\tilde{P}_n$ by conjugation is given by the following formulas: $x_i^{-1} s_1 x_i = s_1$ if $i \ne 2$, $x_2^{-1} s_1 x_2 = s_1 u_2^{-1}$; $x_i^{-1} u_j x_i = u_j$ if $|i-j| \ge 2$, $x_i^{-1} u_j x_i = u_i u_j$ if $|i-j| = 1$, and $x_i^{-1} u_i x_i = u_i^{-1}\eta$.

Proof. Most of the statement is a mere reformulation of Definition 8 and Theorem 1 in §1.5 of [9]. The only difference is that we define u_i directly in terms of the generators of $\tilde{B}_n$, while Moishezon defines $u_1 = (x_2 x_1^2 x_2^{-1}) x_2^{-2} = x_1^{-1} x_2^2 x_1 x_2^{-2}$ and constructs the other u_i by conjugation. In fact, $u_i = x^2 y^{-2}$ whenever x and y are two adjacent half-twists having respectively i and $i+1$ among their end points and such that $xyx^{-1} = x_i$; our definition of u_i corresponds to the choice $x = x_i^{-1} x_{i+1} x_i$ and $y = x_{i+1}$ for $i \le n-2$, and $x = x_{n-2}$ and $y = x_{n-1} x_{n-2} x_{n-1}^{-1}$ for $i = n-1$. Also note that Moishezon's formula for $x_2^{-1} s_1 x_2$ is inconsistent, due to a mistake in equation (1.25) of [9]; the formula we give is corrected. □

Intuitively speaking, the reason why $\tilde{B}_n$ is a fairly small group is that, due to the extra commutation relations, very little is remembered about the path supporting a given half-twist, namely just its two endpoints and the total number of times that it circles around the $n-2$ other points. This can be readily checked on simple examples (e.g., half-twists exchanging the first two points along a path that encircles only one of the $n-2$ other points: since these differ by conjugation by half-twists along paths presenting a single transverse intersection, they represent the same element in $\tilde{B}_n$). More generally, we have the following fact:

Lemma 3.2. *The elements of $\tilde{B}_n$ corresponding to half-twists exchanging the first two points are exactly those of the form $x_1 u_1^k \eta^{k(k-1)/2}$ for some integer k.*

Proof. Any half-twist exchanging the first two points can be put in the form $\gamma x_1 \gamma^{-1}$, where $\gamma \in \tilde{P}_n$ can be expressed as $\gamma = s_1^\alpha u_1^{\beta_1} \cdots u_{n-1}^{\beta_{n-1}} \eta^\epsilon$. Using Lemma 3.1, we have $x_1^{-1} \gamma x_1 = s_1^\alpha (u_1^{-1}\eta)^{\beta_1} (u_1 u_2)^{\beta_2} u_3^{\beta_3} \cdots u_{n-1}^{\beta_{n-1}} \eta^\epsilon$. Since $(u_1 u_2)^{\beta_2} = \eta^{\beta_2(\beta_2-1)/2} u_1^{\beta_2} u_2^{\beta_2}$, we can rewrite this equality as $x_1^{-1}\gamma x_1 = u_1^{-2\beta_1} \eta^{\beta_1} u_1^{\beta_2} \eta^{\beta_2(\beta_2-1)/2} \gamma = u_1^k \eta^{k(k-1)/2} \gamma$, where $k = \beta_2 - 2\beta_1$. Multiplying by x_1 on the left and γ^{-1} on the right we obtain $\gamma x_1 \gamma^{-1} = x_1 u_1^k \eta^{k(k-1)/2}$. □

Lemma 3.3. *Let $x, y \in \tilde{B}_n$ be elements corresponding to half-twists along paths with mutually disjoint endpoints. Then $[x, y] = 1$.*

Proof. The result is trivial when the paths corresponding to x and y are disjoint or intersect only once. In general, after conjugation we can assume that $x = \gamma x_1 \gamma^{-1}$ for some $\gamma \in \tilde{P}_n$, and $y = x_3$. By Lemma 3.2, $x = x_1 u_1^k \eta^{k(k-1)/2}$ for some integer k. Since x_1, u_1 and η all commute with x_3, we conclude that $[x, y] = 1$ as desired. □

Lemma 3.4. *Let $x, y \in \tilde{B}_n$ be elements corresponding to half-twists along paths with one common endpoint. Then $xyx = yxy$.*

Proof. After conjugation we can assume that $x = x_1$ and $y = \gamma x_2 \gamma^{-1}$ for some $\gamma \in \tilde{P}_n$. By the classification of half-twists in $\tilde{B}_n$ (Lemma 3.2), there exists an integer k such that $y = x_2 u_2^k \eta^{k(k-1)/2} = x_2(s_1 u_2^{-1})^{-k} s_1^k = s_1^{-k} x_2 s_1^k$. Therefore $xyx = x_1 s_1^{-k} x_2 s_1^k x_1 = s_1^{-k}(x_1x_2x_1)s_1^k = s_1^{-k}(x_2x_1x_2)s_1^k = yxy$. □

It must be noted that Lemmas 3.3 and 3.4 have also been obtained by Robb [12].

Lemma 3.5. *The group $\tilde{B}_n$ admits automorphisms ϵ_i such that $\epsilon_i(x_i) = x_i u_i$ and $\epsilon_i(x_j) = x_j$ for every $j \neq i$. Moreover, $\epsilon_i(u_i) = u_i \eta$ and $\epsilon_i(u_j) = u_j$ $\forall j \neq i$.*

Proof. By Lemmas 3.3 and 3.4, the half-twists $x_1, \dots, x_{i-1}, (x_i u_i), x_{i+1}, \dots, x_{n-1}$ satisfy exactly the same relations as the standard generators of $\tilde{B}_n$. So ϵ_i is a well-defined group homomorphism from $\tilde{B}_n$ to itself, and it is injective. The formulas for $\epsilon_i(u_i)$ and $\epsilon_i(u_j)$ are easily checked. The surjectivity of ϵ_i follows from the identity $\epsilon_i(x_i u_i^{-1}\eta) = x_i$. □

The following definition is motivated by the very particular structure of the fundamental groups of branch curve complements computed by Moishezon for generic projections of $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{CP}^2$ [9, 10], which seems to be a feature common to a much larger class of examples (see §4):

Definition 3.6. *Define $\tilde{B}_n^{(2)} = \{(x,y) \in \tilde{B}_n \times \tilde{B}_n,\ \sigma(x) = \sigma(y) \text{ and } \delta(x) = \delta(y)\}$. We say that the group $\pi_1(\mathbb{C}^2 - D_k)$ satisfies property $(*)$ if there exists an isomorphism ψ from $\pi_1(\mathbb{C}^2 - D_k)$ to a quotient of $\tilde{B}_n^{(2)}$ such that, for any geometric generator $\gamma \in \Gamma_k$, there exist two half-twists $x, y \in \tilde{B}_n$ such that $\sigma(x) = \sigma(y) = \theta_k(\gamma)$ and $\psi(\gamma) = (x, y)$.*

In other words, $\pi_1(\mathbb{C}^2 - D_k)$ satisfies property $(*)$ if there exists a surjective homomorphism from $\tilde{B}_n^{(2)}$ to $\pi_1(\mathbb{C}^2 - D_k)$ which maps pairs of half-twists to geometric generators, in a manner compatible with the S_n-valued homomorphisms σ and θ_k.

Remark 3.7. *If $\pi_1(\mathbb{C}^2 - D_k)$ satisfies property $(*)$, then the kernel of the homomorphism $\theta_k^+ : \pi_1(\mathbb{C}^2 - D_k) \to S_n \times \mathbb{Z}$ is a quotient of $\tilde{P}_{n,0} \times \tilde{P}_{n,0}$ and therefore a solvable group; in particular its commutator subgroup is a quotient of $(\mathbb{Z}_2)^2$, and its abelianization is a quotient of $\mathbb{Z}^2 \otimes \mathcal{R}_n \simeq (\mathbb{Z} \oplus \mathbb{Z})^{n-1}$.*

As an immediate consequence of Definition 3.6 and Lemma 3.3, we have:

Proposition 3.8. *If $\pi_1(\mathbb{C}^2 - D_k)$ satisfies property $(*)$, then the stabilization operation is trivial, i.e. $K_k = \{1\}$, $G_k = \pi_1(\mathbb{C}^2 - D_k)$, and $G_k^0 = \operatorname{Ker} \theta_k^+$.*

Proof. Let $\gamma, \gamma' \in \Gamma_k$ be such that $\theta_k(\gamma)$ and $\theta_k(\gamma')$ are disjoint transpositions. Consider the isomorphism ψ given by Definition 3.6: there exist half-twists $x, x', y, y' \in \tilde{B}_n$ such that $\psi(\gamma) = (x, y)$ and $\psi(\gamma') = (x', y')$. Since $\theta_k(\gamma) = \sigma(x) = \sigma(y)$ and $\theta_k(\gamma') = \sigma(x') = \sigma(y')$ are disjoint transpositions, x and x' have disjoint endpoints, and similarly for y and y'. Therefore, by Lemma 3.3 we have $[x, x'] = 1$ and $[y, y'] = 1$, so that $[\psi(\gamma), \psi(\gamma')] = 1$, and therefore $[\gamma, \gamma'] = 1$. We conclude that $K_k = \{1\}$, which ends the proof. □

Let $D_{p,q}$ be the branch curve of a generic polynomial map $\mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^2$ of bidegree (p, q), $p, q \geq 2$. As will be shown in §4, it follows from the computations in [9] that $\pi_1(\mathbb{C}^2 - D_{p,q})$ satisfies property $(*)$. This property also holds for the complement of the branch curve of a generic polynomial map from $\mathbb{CP}^2$ to itself in degree ≥ 3, as follows from the calculations in [10] (see also [15]), and in various other examples as well (see §4). It is an interesting question to determine whether this remarkable structure of branch curve complements extends to generic high-degree projections of arbitrary algebraic surfaces; this would tie in nicely with a conjecture of Teicher about the virtual solvability of these fundamental groups [14], and would also imply Conjecture 1.3.

4. Examples

As follows from pp. 696–700 of [5], if the symplectic manifold X happens to be Kähler, then all approximately holomorphic constructions can actually be carried out using genuine holomorphic sections of $L^{\otimes k}$ over X, and as a consequence the $\mathbb{CP}^2$-valued maps given by Theorem 1.1 coincide up to isotopy with projective maps defined by generic holomorphic sections of $L^{\otimes k}$; therefore, in the case of complex projective surfaces all calculations can legitimately be performed within the framework of complex algebraic geometry.

The fundamental groups of complements of branch curves have already been computed for generic projections of various complex projective surfaces. In many cases, these computations only hold for specific linear systems, and do not apply to the high degree situation that we wish to consider.

Nevertheless, it is worth mentioning that, if $D \subset \mathbb{CP}^2$ is the branch curve of a generic linear projection of a hypersurface of degree n in $\mathbb{CP}^3$, then it has been shown by Moishezon that $\pi_1(\mathbb{C}^2 - D) \simeq B_n$ [7]. In fact, in this specific case there is a well-defined geometric monodromy representation morphism θ_B with values in the braid group B_n rather than in the symmetric group S_n as usual, because the n preimages of any point in $\mathbb{CP}^2 - D$ lie in a fiber of the projection $\mathbb{CP}^3 - \{\mathrm{pt}\} \to \mathbb{CP}^2$, which after trivialization over an affine subset can be identified with $\mathbb{C}$. Moishezon's computations then show that $\theta_B : \pi_1(\mathbb{C}^2 - D) \to B_n$ is an isomorphism. An attempt to quotient out B_n by commutators as in the definition of stabilized fundamental groups yields $\tilde{B}_n$: in this case the stabilization operation is non-trivial. However this situation is specific to the linear system $O(1)$, and one expects the fundamental groups of branch curve complements to behave differently when one instead considers projections given by sections of $O(k)$ for $k \gg 0$.

Moishezon's result about hypersurfaces in $\mathbb{CP}^3$ has been extended by Robb to the case of complete intersections (still considering only linear projections to $\mathbb{CP}^2$ rather than arbitrary linear systems) [12]. The result is that, if D is the branch curve for a complete intersection of degree n in $\mathbb{CP}^m$ ($m \geq 4$), then the group $\pi_1(\mathbb{C}^2 - D)$ is isomorphic to $\tilde{B}_n$. It is worth noting that, in this example, the stabilization operation is trivial. In fact, the groups $\pi_1(\mathbb{C}^2 - D)$ can be shown to have property $(*)$ (observe that $\tilde{B}_n$ is the quotient of $\tilde{B}_n^{(2)}$ by its subgroup $1 \times \tilde{P}_{n,0}$).

Conjecture 1.6 holds for $k = 1$ in these two families of examples: we have $\mathrm{Ab}\, G^0 \simeq \mathbb{Z}^{n-1}$ and $[G^0, G^0] \simeq \mathbb{Z}_2$ in both cases, while $\mathbb{Z}^2/\Lambda_1 \simeq \mathbb{Z}$ because the canonical class is proportional to the hyperplane class which is primitive.

More interestingly for our purposes, the calculations have also been carried out in the case of arbitrarily positive linear systems by Moishezon for two fundamental examples: $\mathbb{CP}^1 \times \mathbb{CP}^1$ [9], and $\mathbb{CP}^2$ [10] (unpublished, see also [15] for a summary).

Theorem 4.1 (Moishezon). *Let $D_{p,q}$ be the branch curve of a generic polynomial map $\mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^2$ of bidegree (p,q), $p,q \geq 2$. Then the group $\pi_1(\mathbb{C}^2 - D_{p,q})$ satisfies property $(*)$, and its subgroup $H^0_{p,q} = \mathrm{Ker}\, \theta^+_{p,q}$ has the following structure:*

$\mathrm{Ab}\, H^0_{p,q}$ is isomorphic to $(\mathbb{Z}_2 \oplus Z_{p-q})^{n-1}$ if p and q are even, and $(\mathbb{Z}_{2(p-q)})^{n-1}$ if p or q is odd (here $n = 2pq$); the commutator subgroup $[H^0_{p,q}, H^0_{p,q}]$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ when p and q are even, and $\mathbb{Z}_2$ if p or q is odd.

In fact, Moishezon identifies $\pi_1(\mathbb{C}^2 - D_{p,q})$ with a quotient of the semi-direct product $\tilde{B}_n \ltimes \tilde{P}_{n,0}$, where $\tilde{B}_n$ acts from the right on $\tilde{P}_{n,0}$ by conjugation [9]. However it is easy to observe that the map $\kappa : \tilde{B}_n \ltimes \tilde{P}_{n,0} \to \tilde{B}^{(2)}_n$ defined by $\kappa(x, u) = (x, xu)$ is a group isomorphism (recall the group structure on $\tilde{B}_n \ltimes \tilde{P}_{n,0}$ is given by $(x, u)(x', u') = (xx', x'^{-1}ux'u')$). The factor $\tilde{P}_{n,0}$ of the semi-direct product corresponds to the normal subgroup $1 \times \tilde{P}_{n,0}$ of $\tilde{B}^{(2)}_n$, while the factor $\tilde{B}_n$ corresponds to the diagonally embedded subgroup $\tilde{B}_n = \{(x, x)\} \subset \tilde{B}^{(2)}_n$.

Moreover, by carefully going over the various formulas identifying a set of geometric generators for $\pi_1(\mathbb{C}^2 - D_{p,q})$ with certain specific elements in $\tilde{B}_n \ltimes \tilde{P}_{n,0}$ (Propositions 8 and 10 of [9]; cf. also §1.4, Definition 24 and Remarks 28–29 of [9]), or equivalently in $\tilde{B}^{(2)}_n$ after applying the isomorphism κ, it is relatively easy to check that each geometric generator corresponds to a pair of half-twists with the expected end points in $\tilde{B}^{(2)}_n$ (see also §6 for more details). Therefore, property $(*)$ and Conjecture 1.3 hold for these groups.

Conjecture 1.6 also holds for $\mathbb{CP}^1 \times \mathbb{CP}^1$. Indeed, $H_2(\mathbb{CP}^1 \times \mathbb{CP}^1, \mathbb{Z})$ is generated by classes α and β corresponding to the two factors; the hyperplane section class is $L = p\alpha + q\beta$, while the ramification curve is $R = 3L + K = (3p - 2)\alpha + (3q - 2)\beta$. Therefore, the subgroup $\Lambda_{p,q}$ of $\mathbb{Z}^2$ is generated by $(\alpha \cdot L, \alpha \cdot R) = (q, 3q - 2)$ and $(\beta \cdot L, \beta \cdot R) = (p, 3p - 2)$. An easy computation shows that the quotient $\mathbb{Z}^2/\Lambda_{p,q} = \mathbb{Z}^2/\langle(q, 3q-2), (p, 3p-2)\rangle \simeq \mathbb{Z}^2/\langle(q, 2), (p, 2)\rangle$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_{p-q}$ when p and q are even, and to $\mathbb{Z}_{2(p-q)}$ otherwise.

It is worth noting that this nice description for $p, q \geq 2$ completely breaks down in the insufficiently ample case $p = 1$, where it follows from computations of Zariski [17] that $\pi_1(\mathbb{C}^2 - D_{1,q}) \simeq B_{2q}$. So both Conjecture 1.3 and Conjecture 1.6 require a sufficient amount of ampleness in order to hold ($p, q \geq 2$).

Theorem 4.2 (Moishezon). *Let D_k be the branch curve of a generic polynomial map $\mathbb{CP}^2 \to \mathbb{CP}^2$ of degree $k \geq 3$. Then the group $\pi_1(\mathbb{C}^2 - D_k)$ satisfies property $(*)$, and its subgroup $H^0_k = \mathrm{Ker}\,\theta^+_k$ has the following structure: $\mathrm{Ab}\, H^0_k$ is isomorphic to $(\mathbb{Z} \oplus \mathbb{Z}_3)^{n-1}$ if k is a multiple of 3, and to $\mathbb{Z}^{n-1}$ otherwise (here $n = k^2$); the commutator subgroup $[H^0_k, H^0_k]$ is trivial for k even and isomorphic to $\mathbb{Z}_2$ for k odd.*

In this case too, Moishezon in fact identifies $\pi_1(\mathbb{C}^2 - D_k)$ with a quotient of $\tilde{B}_n \ltimes \tilde{P}_{n,0}$ [10] (see also [15]). Property $(*)$ and Conjecture 1.3 hold for $\mathbb{CP}^2$ when $k \geq 3$, but for $k = 2$ the group $\pi_1(\mathbb{C}^2 - D_2)$ is much larger.

Since $H_2(\mathbb{CP}^2, \mathbb{Z})$ is generated by the class of a line, Λ_k is the subgroup of $\mathbb{Z}^2$ generated by $(k, 3k - 3)$, and $\mathbb{Z}^2/\Lambda_k$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_3$ when k is a multiple of 3 and to $\mathbb{Z}$ otherwise. Therefore Conjecture 1.6 holds for $\mathbb{CP}^2$ when $k \geq 3$.

Results for certain projections of Del Pezzo and K3 surfaces have also been announced by Robb in [12].

Theorem 4.3 (Robb). *Let X be either a cubic hypersurface in $\mathbb{CP}^3$ or a $(2, 2)$ complete intersection in $\mathbb{CP}^4$, and let D_k be the branch curve of a generic algebraic map $X \to \mathbb{CP}^2$ given by sections of $O(kH)$, where H is the hyperplane section and $k \geq 2$. Then the subgroup $H^0_k = \mathrm{Ker}\,\theta^+_k$ of $\pi_1(\mathbb{C}^2 - D_k)$ has abelianization $\mathrm{Ab}\, H^0_k \simeq \mathbb{Z}^{n-1}$.*

Theorem 4.4 (Robb)**.** *Let X be a K3 surface realized either as a degree 4 hypersurface in $\mathbb{CP}^3$, a $(3,2)$ complete intersection in $\mathbb{CP}^4$ or a $(2,2,2)$ complete intersection in $\mathbb{CP}^5$, and let D_k be the branch curve of a generic algebraic map $X \to \mathbb{CP}^2$ given by sections of $O(kH)$, where H is the hyperplane section and $k \geq 2$. Then the subgroup $H_k^0 = \operatorname{Ker} \theta_k^+$ of $\pi_1(\mathbb{C}^2 - D_k)$ has abelianization $\operatorname{Ab} H_k^0 \simeq (\mathbb{Z} \oplus \mathbb{Z}_k)^{n-1}$.*

Although to our knowledge no detailed proofs of Theorems 4.3 and 4.4 have appeared yet, it appears very likely from the sketch of argument given in [12] that property $(*)$ and Conjecture 1.3 will hold for these examples as well. In any case we can compare Robb's results with the answers predicted by Conjecture 1.6.

In the case of the Del Pezzo surfaces, the hyperplane class H is primitive, and $K = -H$ (so $R_k = (3k-1)H$), so that the subgroup $\Lambda_k \subset \mathbb{Z}^2$ is generated by $(k, 3k-1)$, and $\mathbb{Z}^2/\Lambda_k \simeq \mathbb{Z}$, which is in agreement with Theorem 4.3. In the case of the K3 surfaces, the hyperplane class H is again primitive, but $K = 0$ and $R_k = 3kH$, so that Λ_k is now generated by $(k, 3k)$, and $\mathbb{Z}^2/\Lambda_k \simeq \mathbb{Z} \oplus \mathbb{Z}_k$, in agreement with Theorem 4.4.

The following result for the Hirzebruch surface $\mathbb{F}_1 = \mathbb{P}(O_{\mathbb{CP}^1} \oplus O_{\mathbb{CP}^1}(1))$ is new to our knowledge; however partial results about this surface have been obtained by Moishezon, Robb and Teicher [11, 16], and an ongoing project of Teicher and coworkers is expected to yield another proof of the same result.

Theorem 4.5. *Let $D_{p,q}$ be the branch curve of a generic algebraic map $\mathbb{F}_1 \to \mathbb{CP}^2$ given by three sections of the linear system $O(pF + qE)$, where F is the class of a fiber, E is the exceptional section, and $p > q \geq 2$. Then the group $\pi_1(\mathbb{C}^2 - D_{p,q})$ satisfies property $(*)$, and its subgroup $H_{p,q}^0 = \operatorname{Ker} \theta_{p,q}^+$ has the following structure: $\operatorname{Ab} H_{p,q}^0 \simeq (\mathbb{Z}_{3q-2p})^{n-1}$, where $n = (2p-q)q$, and the commutator subgroup $[H_{p,q}^0, H_{p,q}^0]$ is isomorphic to $\mathbb{Z}_2$ if p is odd and q even, and trivial in all other cases.*

The proof relies on the observation that $\mathbb{F}_1$ is the blow-up of $\mathbb{CP}^2$ at one point. Recalling the interpretation of a symplectic (or Kähler) blow-up as the collapsing of an embedded ball, it is easy to check that $\mathbb{F}_1$ can be degenerated to a union of planes in a manner similar to $\mathbb{CP}^2$, only with some components missing; most of the calculations performed by Moishezon in [10] for $\mathbb{CP}^2$ can then be re-used in this context, with the only changes occurring along the exceptional curve E. More details are given in §6.2.

As a consequence of property $(*)$, Conjecture 1.3 holds for this example. So does Conjecture 1.6: indeed, $H_2(\mathbb{F}_1, \mathbb{Z})$ is generated by F and E. Recalling that $F \cdot F = 0$, $F \cdot E = 1$, $E \cdot E = -1$, and letting $L_{p,q} = pF + qE$ and $R_{p,q} = 3L_{p,q} + K = (3p-3)F + (3q-2)E$, we obtain that $\Lambda_{p,q} \subset \mathbb{Z}^2$ is generated by $(F \cdot L_{p,q}, F \cdot R_{p,q}) = (q, 3q-2)$ and $(E \cdot L_{p,q}, E \cdot R_{p,q}) = (p-q, 3p-3q-1)$. Therefore $\mathbb{Z}^2/\Lambda_k \simeq \mathbb{Z}^2/\langle (q, 3q-2), (p-q, 3p-3q-1) \rangle \simeq \mathbb{Z}_{3q-2p}$.

A much wider class of examples, including an infinite family of surfaces of general type, can be investigated if one brings approximately holomorphic techniques into the picture, although this makes it only possible to obtain results about the stabilized fundamental groups of branch curve complements (cf. §2) rather than the actual fundamental groups.

Theorem 4.6. *For given integers $a, b \geq 1$ and $p, q \geq 2$, let $X_{a,b}$ be the double cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along a smooth algebraic curve of degree $(2a, 2b)$, and let $L_{p,q}$ be the linear system over $X_{a,b}$ defined as the pullback of $O_{\mathbb{P}^1 \times \mathbb{P}^1}(p, q)$ via the double cover. Let $D_{p,q}$ be the branch curve of a generic approximately holomorphic* *perturbation of an algebraic map $X_{a,b} \to \mathbb{CP}^2$ given by three sections of $L_{p,q}$. Then the* stabilized *fundamental group $G_{p,q}(X_{a,b}) = \pi_1(\mathbb{C}^2 - D_{p,q})/K_{p,q}$ satisfies property $(*)$, and its reduced subgroup $G^0_{p,q}(X_{a,b}) = \operatorname{Ker}\theta^+_{p,q}/K_{p,q}$ has the following structure: $\operatorname{Ab} G^0_{p,q}(X_{a,b}) \simeq (\mathbb{Z}^2/\langle(p, a-2), (q, b-2)\rangle)^{n-1}$, where $n = 4pq$, and the commutator subgroup $[G^0_{p,q}(X_{a,b}), G^0_{p,q}(X_{a,b})]$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ if a, b, p, q are all even, trivial if a or b is odd and $a + p$ or $b + q$ is odd, and isomorphic to $\mathbb{Z}_2$ in all other cases.*

More precisely, the setup that we consider starts with a holomorphic map from $X_{a,b}$ to $\mathbb{CP}^2$ that factors through the double cover $X_{a,b} \to \mathbb{CP}^1 \times \mathbb{CP}^1$. Such a map is of course not generic in any sense; however there is a natural explicit way to perturb it in the approximately holomorphic category (see §7), giving rise to the branch curves $D_{p,q}$ that we consider. The map can also be perturbed in the holomorphic category, which at least for p and q large enough yields a branch curve that is equivalent to $D_{p,q}$ up to creations and cancellations of pairs of nodes. So, on the level of stabilized groups, our result does give an answer that is relevant from both the symplectic and algebraic points of view. Moreover, it is expected that, at least for p and q large enough, the fundamental groups themselves (rather than their stabilized quotients) should satisfy property $(*)$.

Theorem 4.6 implies that Conjecture 1.6 holds for the manifolds $X_{a,b}$. Indeed, $X_{a,b}$ can also be described topologically as follows: in $\mathbb{CP}^1 \times \mathbb{CP}^1$ consider $2a$ curves of the form $\mathbb{CP}^1 \times \{\mathrm{pt}\}$ and $2b$ curves of the form $\{\mathrm{pt}\} \times \mathbb{CP}^1$, and blow up their $4ab$ intersection points to obtain a manifold $Y_{a,b}$ containing disjoint rational curves $C_1, \dots, C_{2a}$ (of square $-2b$) and $C'_1, \dots, C'_{2b}$ (of square $-2a$). Then $X_{a,b}$ is the double cover of $Y_{a,b}$ branched along $C_1 \cup \dots \cup C_{2a} \cup C'_1 \cup \dots \cup C'_{2b}$. Now, consider the preimages $\tilde{C}_i = \pi^{-1}(C_i)$ and $\tilde{C}'_i = \pi^{-1}(C'_i)$, and let $L_{p,q} = p\pi^*\alpha + q\pi^*\beta$ and $R_{p,q} = 3L_{p,q} + K_{X_{a,b}} = (3p + a - 2)\pi^*\alpha + (3q + b - 2)\pi^*\beta$, where α and β are the homology generators corresponding to the two factors of $\mathbb{CP}^1 \times \mathbb{CP}^1$. We have $(\tilde{C}_i \cdot L_{p,q}, \tilde{C}_i \cdot R_{p,q}) = (q, 3q + b - 2)$ and $(\tilde{C}'_i \cdot L_{p,q}, \tilde{C}'_i \cdot R_{p,q}) = (p, 3p + a - 2)$. It is easily shown that these two elements of $\mathbb{Z}^2$ generate the subgroup $\Lambda_{p,q}$; therefore $\mathbb{Z}^2/\Lambda_{p,q} = \mathbb{Z}^2/\langle(q, 3q + b - 2), (p, 3p + a - 2)\rangle \simeq \mathbb{Z}^2/\langle(p, a - 2), (q, b - 2)\rangle$.

The techniques involved in the proof of Theorem 4.6, which will be discussed in §7, extend to double covers of other examples for which the answer is known, possibly including iterated double covers of $\mathbb{CP}^1 \times \mathbb{CP}^1$. One example of particular interest is that of double covers of Hirzebruch surfaces branched along disconnected curves, for which we make the following conjecture:

Conjecture 4.7. *Given integers $m, a \geq 1$, let $X_{2m,a}$ be the double cover of the Hirzebruch surface $\mathbb{F}_{2m}$ branched along the union of the exceptional section Δ_∞ and a smooth algebraic curve in the homology class $(2a - 1)[\Delta_0]$ (where Δ_0 is the zero section, of square $2m$). Given integers $p, q \geq 2$ such that $p > 2mq$, let $L_{p,q}$ be the linear system over $X_{2m,a}$ defined as the pullback of $O_{\mathbb{F}_{2m}}(pF + q\Delta_\infty)$ via the double cover. Let $D_{p,q}$ be the branch curve of a generic approximately holomorphic perturbation of an algebraic map $X_{2m,a} \to \mathbb{CP}^2$ given by three sections of $L_{p,q}$. Then the reduced stabilized fundamental group $G^0_{p,q}(X_{2m,a}) = \operatorname{Ker}\theta^+_{p,q}/K_{p,q}$ has abelianization $\operatorname{Ab} G^0_{p,q}(X_{2m,a}) \simeq (\mathbb{Z}^2/\langle(p - 2mq, m - 2), (2q, 2a - 4)\rangle)^{n-1}$.*

5. Stabilized fundamental groups and homological data

Consider a compact symplectic 4-manifold X such that $H_1(X,\mathbb{Z}) = 0$ and a branched covering map $f_k : X \to \mathbb{CP}^2$ determined by three sections of $L^{\otimes k}$, with branch curve $D_k \subset \mathbb{CP}^2$ and geometric monodromy representation morphism $\theta_k : \pi_1(\mathbb{C}^2 - D_k) \to S_n$. The purpose of this section is to construct a natural morphism $\psi_k : \operatorname{Ker}\theta_k \to (\mathbb{Z}^2/\Lambda_k)\otimes\bar{\mathcal{R}}_n \simeq (\mathbb{Z}^2/\Lambda_k)^n$ (where $\bar{\mathcal{R}}_n \simeq \mathbb{Z}^n$ is the regular representation of S_n) and use its properties to prove Theorem 1.5.

Fix a base point p_0 in $\mathbb{C}^2 - D_k$, and let $p_1,\dots,p_n$ be its preimages by f_k. Let $\gamma \in \pi_1(\mathbb{C}^2 - D_k)$ be a loop in the complement of D_k such that $\theta_k(\gamma) = \mathrm{Id}$. Since the monodromy of the branched cover f_k along γ is trivial, $f_k^{-1}(\gamma)$ is the union of n disjoint closed loops in X. Denote by γ_i the lift of γ that starts at the point p_i. Since $H_1(X,\mathbb{Z}) = 0$, there exists a surface (or rather a 2-chain) $S_i \subset X$ such that $\partial S_i = \gamma_i$. Since $\gamma \subset \mathbb{C}^2 - D_k$, the loop γ_i intersects neither the ramification curve R_k nor the preimage L_k of the line at infinity in $\mathbb{CP}^2$. Therefore, there exist well-defined algebraic intersection numbers $\lambda_i = S_i \cdot L_k$ and $\rho_i = S_i \cdot R_k \in \mathbb{Z}$. However, there are various possible choices for the surface S_i, and the relative cycle $[S_i]$ is only well-defined up to an element of $H_2(X,\mathbb{Z})$. Therefore, the pair $(\lambda_i,\rho_i) \in \mathbb{Z}^2$ is only defined up to an element of the subgroup Λ_k.

Definition 5.1. *With the above notations, we denote by $\psi_k : \operatorname{Ker}\theta_k \to (\mathbb{Z}^2/\Lambda_k)^n$ the morphism defined by $\psi_k(\gamma) = ((S_i\cdot L_k, S_i\cdot R_k))_{1\le i\le n}$.*

In fact, there is no canonical ordering of the preimages of p_0, and ψ_k more naturally takes values in $(\mathbb{Z}^2/\Lambda_k)\otimes\bar{\mathcal{R}}_n$, as evidenced by Lemma 5.2 below.

Definition 5.1 can naturally be extended to the case $H_1(X,\mathbb{Z}) \neq 0$ by instead considering the morphism $\tilde{\psi}_k : \operatorname{Ker}\theta_k \to H_1(X - L_k - R_k, \mathbb{Z})^n$ which maps a loop γ to the homology classes of its lifts γ_i in $X - L_k - R_k$. However, the properties to be expected of this morphism in general are not entirely clear, due to the lack of available non-simply connected examples (even though the techniques in §6–7 could probably be applied to the 4-manifold $\Sigma\times\mathbb{CP}^1$ for any Riemann surface Σ).

We now investigate the various properties of ψ_k.

Lemma 5.2. *For every $\gamma \in \operatorname{Ker}\theta_k$ and $g \in \pi_1(\mathbb{C}^2 - D_k)$, $\psi_k(g^{-1}\gamma g) = \theta_k(g)\cdot\psi_k(\gamma)$, where S_n acts on $(\mathbb{Z}^2/\Lambda_k)^n$ by permuting the factors (i.e., ψ_k is equivariant).*

Proof. Denoting by σ the permutation $\theta_k(g)$, observe that the lifts of $g^{-1}\gamma g$ are freely homotopic to those of γ, and more precisely that the lift of $g^{-1}\gamma g$ through $p_{\sigma(i)}$ is freely homotopic to the lift of γ through p_i. Therefore, the $\sigma(i)$-th component of $\psi_k(g^{-1}\gamma g)$ is equal to the i-th component of $\psi_k(\gamma)$. □

Lemma 5.3. *$K_k \subset \operatorname{Ker}\psi_k$, i.e. ψ_k factors through the stabilized group.*

Proof. Recall from Definition 2.2 that K_k is generated by commutators $[\gamma_1,\gamma_2]$ of geometric generators that are mapped to disjoint transpositions by θ_k. If γ_1 is a geometric generator, then $n-2$ of its lifts to X are contractible closed loops in $X - L_k - R_k$, while the two other lifts are not closed; and similarly for γ_2. However, if $\theta_k(\gamma_1)$ and $\theta_k(\gamma_2)$ are disjoint, then all the lifts of $[\gamma_1,\gamma_2]$ are contractible loops in $X - L_k - R_k$; therefore $[\gamma_1,\gamma_2] \in \operatorname{Ker}\psi_k$. □

It is worth noting that, similarly, if γ_1 and γ_2 are geometric generators mapped by θ_k to adjacent (non-commuting) transpositions, then $(\gamma_1\gamma_2\gamma_1)(\gamma_2\gamma_1\gamma_2)^{-1} \in \operatorname{Ker}\psi_k$

(only one of the lifts of this loop is possibly non-trivial, but its algebraic linking numbers with L_k and R_k are both equal to zero).

Lemma 5.4. *For any $\gamma \in \operatorname{Ker}\theta_k$, the n-tuple $\psi_k(\gamma) = ((\lambda_i, \rho_i))_{1\le i\le n}$ has the property that $(\sum \lambda_i, \sum \rho_i) \equiv (0, \delta_k(\gamma)) \mod \Lambda_k$.*

Proof. $\gamma \in \pi_1(\mathbb{C}^2 - D_k)$ is homotopically trivial in $\mathbb{C}^2$, so there exists a topological disk $\Delta \subset \mathbb{C}^2$ such that $\partial\Delta = \gamma$. Now observe that $\partial(f_k^{-1}(\Delta)) = \sum \gamma_i$; therefore $(\sum \lambda_i, \sum \rho_i)$ is equal (mod Λ_k) to the algebraic intersection numbers of $f_k^{-1}(\Delta)$ with L_k and R_k. We have $f_k^{-1}(\Delta)\cdot L_k = 0$ since $f_k^{-1}(\Delta) \subset f_k^{-1}(\mathbb{C}^2) = X - L_k$, and $f_k^{-1}(\Delta)\cdot R_k = \Delta\cdot D_k = \delta_k(\gamma)$. □

Lemma 5.5. *For any geometric generator $\gamma \in \Gamma_k$, $\psi_k(\gamma^2) = ((\lambda_i, \rho_i))_{1\le i\le n}$ is given by $(\lambda_i, \rho_i) = (0, 1)$ if i is one of the two indices exchanged by the transposition $\theta_k(\gamma)$, and $(\lambda_i, \rho_i) = (0, 0)$ otherwise.*

Proof. All lifts of γ^2 are homotopically trivial, except for two of them which are freely homotopic to each other and circle once around the ramification curve R_k. □

Lemma 5.6. *There exist two geometric generators $\gamma_1, \gamma_2 \in \Gamma_k$ such that $\theta_k(\gamma_1) = \theta_k(\gamma_2)$ and $\psi_k(\gamma_1\gamma_2) = ((-1, 0), (1, 2), (0, 0), \dots, (0, 0))$.*

Proof. Consider a generic line $\ell \subset \mathbb{CP}^2$ intersecting D_k transversely in $d = \deg D_k$ points, and let $\Sigma = f_k^{-1}(L)$. The restriction $f_{k|\Sigma} : \Sigma \to \ell = \mathbb{CP}^1$ is a connected simple branched cover of degree n with d branch points, with monodromy described by the morphism $\theta_k \circ i_* : \pi_1(\ell - \{d \text{ points}\}) \to S_n$. It is a classical fact that the moduli space of all connected simple branched covers of $\mathbb{CP}^1$ with fixed degree and number of branch points is connected, i.e. up to a suitable reordering of the branch points we can assume that the monodromy of $f_{k|\Sigma}$ is described by any given standard S_n-valued morphism.

So we can find an ordered system of generators $\gamma_1, \dots, \gamma_d$ of the free group $\pi_1(\ell \cap (\mathbb{C}^2 - D_k))$ such that $\theta_k(\gamma_1) = \theta_k(\gamma_2) = (12)$ and all the other transpositions $\theta_k(\gamma_i)$ for $i \ge 3$ are elements of $S_{n-1} = \operatorname{Aut}\{2, \dots, n\}$. The loop $\gamma_1\gamma_2$ then belongs to $\operatorname{Ker}\theta_k$, and admits only two non-trivial lifts g_1 and g_2 in Σ, those which start in the first two sheets of the branched cover. The loops g_1 and g_2 bound a topological annulus A which intersects R_k in two points (projecting to the first two intersection points of ℓ with D_k). This annulus separates Σ into two components, a "large" component consisting of the sheets numbered from 2 to n, and a disk Δ corresponding to the first sheet of the cover, which does not intersect R_k but contains one of the n preimages of the intersection point of ℓ with the line at infinity in $\mathbb{CP}^2$. The lift g_1 bounds Δ with reversed orientation; since $\Delta\cdot R_k = 0$ and $\Delta\cdot L_k = 1$, the first component of $\psi_k(\gamma_1\gamma_2)$ is $(-1, 0)$. The lift g_2 bounds $\Delta \cup A$; since $A\cdot R_k = 2$ and $A\cdot L_k = 0$, the second component of $\psi_k(\gamma_1\gamma_2)$ is $(1, 2)$. □

Proof of Theorem 1.5. By Lemma 5.4, ψ_k maps the kernel of $\theta_k^+ : \pi_1(\mathbb{C}^2 - D_k) \to S_n \times \mathbb{Z}$ into the subgroup $\Gamma = \{(\lambda_i, \rho_i),\ \sum \lambda_i = \sum \rho_i = 0\} \simeq (\mathbb{Z}^2/\Lambda_k) \otimes \mathcal{R}_n$ of $(\mathbb{Z}^2/\Lambda_k)^n$. By Lemma 5.3, ψ_k factors through the quotient $\operatorname{Ker}\theta_k^+/K_k = G_k^0(X, \omega)$, and gives rise to a map $\phi_k : G_k^0(X, \omega) \to \Gamma \simeq (\mathbb{Z}^2/\Lambda_k) \otimes \mathcal{R}_n \simeq (\mathbb{Z}^2/\Lambda_k)^{n-1}$. Since Γ is abelian, $[G_k^0, G_k^0] \subset \operatorname{Ker}\phi_k$, so ϕ_k factors through the abelianization $\operatorname{Ab} G_k^0(X, \omega)$, as announced in the statement of Theorem 1.5.

We now show that ϕ_k is surjective, i.e. that ψ_k maps $\operatorname{Ker}\theta_k^+$ *onto* Γ. First, let γ and γ' be two geometric generators of $\pi_1(\mathbb{C}^2-D_k)$ corresponding to adjacent transpositions in S_n: then $\gamma^2\gamma'^{-2}\in\operatorname{Ker}\theta_k^+$, and Lemma 5.5 implies that $\psi_k(\gamma^2\gamma'^{-2})$ has only two non-zero entries, one equal to $(0,1)$ and the other equal to $(0,-1)$. Recalling from §2 that θ_k is surjective, and using Lemma 5.2, by considering suitable conjugates of $\gamma^2\gamma'^{-2}$ we can find elements g_{ij} of $\operatorname{Ker}\theta_k^+$ such that $\psi_k(g_{ij})$ has only two non-zero entries, $(0,1)$ at position i and $(0,-1)$ at position j.

Next, consider the geometric generators γ_1,γ_2 given by Lemma 5.6: the element $\gamma_1\gamma_2^{-1}$ belongs to $\operatorname{Ker}\theta_k^+$, and $\psi_k(\gamma_1\gamma_2^{-1})=((-1,-1),(1,1),(0,0),\dots,(0,0))$. Therefore $\psi_k(g_{12}\gamma_1\gamma_2^{-1})=((-1,0),(1,0),(0,0),\dots,(0,0))$. So, using the surjectivity of θ_k and Lemma 5.2, we can find elements g'_{ij} of $\operatorname{Ker}\theta_k^+$ such that $\psi_k(g'_{ij})$ has only two non-zero entries, $(1,0)$ at position i and $(-1,0)$ at position j. We now conclude that $\psi_k(\operatorname{Ker}\theta_k^+)=\Gamma$ by observing that the $2n-2$ elements $\psi_k(g_{in})$ and $\psi_k(g'_{in})$, $1\le i\le n-1$, generate Γ. □

We finish this section by mentioning two conjectures related to Conjecture 1.6. First of all, we mention that Conjecture 1.6 implies a result about the fundamental groups of Galois covers associated to branched covers of $\mathbb{CP}^2$. More precisely, given a complex surface X and a generic projection $X\to\mathbb{CP}^2$ of degree n with branch curve D_k, the associated Galois cover $\tilde{X}_k$ is obtained by compactification of the n-fold fibered product of X with itself above $\mathbb{CP}^2$: the complex surface $\tilde{X}_k$ is a degree $n!$ cover of $\mathbb{CP}^2$ branched along D_k. Moishezon and Teicher have constructed many interesting examples of complex surfaces by this method, and computed their fundamental groups (see e.g. [13], [16], [11]). Given an ordered system of geometric generators $\gamma_1,\dots,\gamma_d$ of $\pi_1(\mathbb{C}^2-D_k)$, the fundamental group $\pi_1(\tilde{X}_k)$ is known to be isomorphic to the quotient of $\operatorname{Ker}(\theta:\pi_1(\mathbb{C}^2-D_k)\to S_n)$ by the subgroup generated by $\gamma_1^2,\dots,\gamma_d^2$, and $\prod\gamma_i$ (see e.g. [16], §4).

By Lemma 5.5, the elements γ_i^2 and their conjugates map under ψ_k to elements of $(\mathbb{Z}^2/\Lambda_k)^n$ with only two non-trivial entries $(0,1)$; therefore, assuming Conjecture 1.6, quotienting by all squares of geometric generators leads to quotienting the image of ψ_k by $\{(0,\rho_i),\ \sum\rho_i \text{ is even}\}\subset(\mathbb{Z}^2/\Lambda_k)^n$. Because of Lemma 5.4, and observing that δ_k takes only even values on $\operatorname{Ker}\theta_k$, we are left with only the first factor in each summand $\mathbb{Z}^2/\Lambda_k$. Moreover, one easily checks that $\psi_k(\prod\gamma_i)=((1,0),(1,0),\dots,(1,0))\equiv((1,0),\dots,(1,0),(1-n,d))\mod\Lambda_k$; and by Lemma 5.4, the sum of the first factors is always zero, so we end up with a group isomorphic to $(\mathbb{Z}_{ks})^{n-2}$, where ks is the divisibility of L_k in $H_2(X,\mathbb{Z})$. Moreover, if we also assume that property $(*)$ holds in addition to Conjecture 1.6, it can easily be checked that the commutator subgroup $[G_k^0,G_k^0]$ is contained in the subgroup generated by the γ_i^2. Therefore, we have the following conjecture, satisfied by the examples in §4:

Conjecture 5.7. *If X is a simply connected complex surface and k is large enough, then the fundamental group of the Galois cover $\tilde{X}_k$ associated to a generic projection $f_k:X\to\mathbb{CP}^2$ defined by sections of $L^{\otimes k}$ is $\pi_1(\tilde{X}_k)=(\mathbb{Z}_{ks})^{n_k-2}$, where ks is the divisibility of L_k in $H_2(X,\mathbb{Z})$ and $n_k=\deg f_k$.*

Also, a careful observation of the examples in §4 suggests the following possible structure for the commutator subgroup $[G_k^0,G_k^0]$, which is worth mentioning in spite of the rather low amount of supporting evidence:

Conjecture 5.8. *If the symplectic manifold X is simply connected and k is large enough, then the commutator subgroup $[G_k^0,G_k^0]$ is isomorphic to $\Gamma_1\times\Gamma_2$, where*

$\Gamma_1 = \mathbb{Z}_2$ *if* X *is spin and* 1 *otherwise, and* $\Gamma_2 = \mathbb{Z}_2$ *if* $L_k \equiv K_X \mod 2$ *and* 1 *otherwise.*

6. Moishezon-Teicher techniques for ruled surfaces

6.1. **Overview of Moishezon-Teicher techniques.** Moishezon and Teicher have developed a general strategy, consisting of two main steps [8, 9, 13], in order to compute the group $\pi_1(\mathbb{C}^2 - D)$ when D is the branch curve of a generic projection to $\mathbb{CP}^2$ of a given projective surface $X \subset \mathbb{CP}^N$. First, one computes the braid factorization (see §2) associated to the curve D. This calculation involves a degeneration of the surface X to a singular configuration X_0 consisting of a union of planes intersecting along lines in $\mathbb{CP}^N$, and a careful analysis of the "regeneration" process which produces the generic branch curve D out of the singular configuration [8]. As explained in §2, the braid factorization explicitly provides, via the Zariski-Van Kampen theorem, a (rather complicated) presentation of the group $\pi_1(\mathbb{C}^2 - D)$. In a second step, one attempts to obtain a simpler description by reorganizing the relations in a more orderly fashion and by constructing morphisms between subgroups of $\pi_1(\mathbb{C}^2 - D)$ and groups related to $\tilde{B}_n$. This process is carried out in [9] for the case $X \simeq \mathbb{CP}^1 \times \mathbb{CP}^1$, and in subsequent papers for other examples.

6.1.1. *Degenerations and braid monodromy calculations.* The starting point of the calculation is a degeneration of the projective surface $X \subset \mathbb{CP}^N$ to an arrangement X_0 of planes in $\mathbb{CP}^N$ intersecting along lines. The degeneration process in the case of manifolds like $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{CP}^2$ is described in detail in [8]. For example, in the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$ embedded by the linear system $O(p,q)$, one first degenerates the surface X of degree $2pq$ to a sum of q copies of $\mathbb{CP}^1 \times \mathbb{CP}^1$ embedded by $O(p,1)$ (each of degree $2p$) inside $\mathbb{CP}^N$; then each of these surfaces is degenerated into p quadric surfaces ($\mathbb{CP}^1 \times \mathbb{CP}^1$ embedded by $O(1,1)$); finally, each of the pq quadric surfaces is degenerated into a union of two planes intersecting along a line. The resulting arrangement can be represented by the diagram in Figure 1.

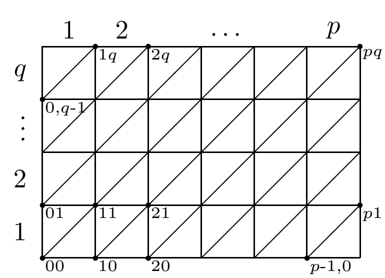

Figure 1

Each triangle in the diagram represents a plane. Each edge separating two triangles represents an intersection line L_i between the corresponding planes; note that the outer edges of the diagram are not part of the configuration. The branch curve for the projection $X_0 \to \mathbb{CP}^2$ is an arrangement of lines in $\mathbb{CP}^2$ (the projections of the various intersection lines L_i); however, in the regeneration process each of these lines acquires multiplicity 2, and the vertices where two or more lines intersect in X_0 turn into certain standard local configurations.

Therefore the braid factorization for D can be computed by looking at the local contributions of the various vertices in the diagram. Since the regeneration process turns a local configuration into a branch curve of degree $2m$, where m is the number of edges meeting at the given vertex, the local contribution of a vertex is naturally described by a word in the braid group B_{2m}. Moreover, because projecting X_0 to $\mathbb{CP}^2$ creates extra intersection points between the projections of the lines L_i whenever they do not intersect in X_0 (i.e. when they do not correspond to edges with a common vertex in the diagram), the branch curve D contains a number of additional nodes besides the local vertex configurations.

The major difficulty is to arrange the various local configurations and the additional nodes into a single braid factorization describing the curve D: given a linear projection $\pi : \mathbb{CP}^2 - \{\mathrm{pt}\} \to \mathbb{CP}^1$, one needs to fix a base point in $\mathbb{CP}^1$ and to choose an ordered system of loops in $\mathbb{CP}^1 - \mathrm{crit}\,\pi_{|D}$ in order to obtain a braid factorization. This choice determines in particular how the local braid monodromy (in B_{2m}) for each vertex of the grid is embedded into the braid monodromy of D (in B_d, $d = \deg D$). A careless setup leads to local embeddings $B_{2m} \hookrightarrow B_d$ that may be extremely difficult to determine.

An important observation of Moishezon is that the construction has sufficient flexibility to allow the images in $\mathbb{CP}^2$ of the various lines and intersection points to be chosen freely. This makes it possible to use the following very convenient setup [8]. First choose an ordering of the vertices in the diagram describing X_0; for example, for $\mathbb{CP}^1 \times \mathbb{CP}^1$ Moishezon chooses an ordering first by row, then by column, starting from the lower-left corner of the diagram: 00, 10, 20, ..., 01, 11, ..., pq. This determines a lexicographic ordering of the edges of the diagram: observing that each line L_i passes through two vertices v_i and v'_i ($v_i < v'_i$), the ordering is given by $L_i < L_j$ iff either $v'_i < v'_j$, or $v'_i = v'_j$ and $v_i < v_j$. It is then possible to choose a configuration where the projections of the lines L_i are given by equations with real coefficients, with slopes increasing according to the chosen lexicographic ordering, so that the intersection of the arrangement of lines in $\mathbb{CP}^2$ with a real slice $\mathbb{R}^2$ looks as in Figure 2.

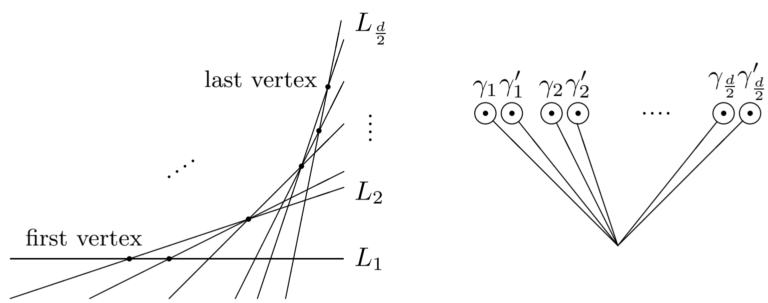

Figure 2

The choice of the slopes of the lines ensures that the intersection points of D with the reference fiber of π (chosen to be $\{x = A\}$ for some real number $A \gg 0$) are ordered in the natural way along the real axis, thus yielding a natural set of geometric generators $\{\gamma_i, \gamma'_i\}$ for $\pi_1(\mathbb{C}^2 - D)$, as shown on the right of Figure 2; recall that each line L_i has multiplicity 2 and hence yields two generators, and note that the correct ordering of these generators counterclockwise around the base point is $\gamma'_{d/2}, \gamma_{d/2}, \dots, \gamma'_1, \gamma_1$. Moreover, the various vertices of the diagram describing X_0 appear, in sequence, for increasing values of x (from left to right).

Since all the contributions to the braid monodromy of D are now localized along the real x-axis, it is a fairly straightforward task to choose a set of generating loops in the base $\mathbb{CP}^1$ of the fibration π and enumerate accordingly the various contributions to the braid monodromy of D (standard configurations at the vertices of the diagram and extra nodes coming from pairs of edges without a common vertex). Going through the list of vertices in decreasing sequence ("from right to left") yields the simplest formula (Proposition 1 of [8]):

Proposition 6.1 (Moishezon)**.** *With the above setup, the braid monodromy of D is given by the factorization $\prod_{i=\nu}^{1}(C_i \cdot F_i)$, where ν is the number of vertices in the diagram, C_i is a product of contributions from nodal intersections between parts of D corresponding to non-adjacent edges, and F_i is the braid monodromy corresponding to the i-th vertex, obtained as the image of a standard local configuration under the embedding $B_{2m_i} \hookrightarrow B_d$ which maps the standard half-twists generating B_{2m_i} to half-twists along arcs that remain* below *the real axis.*

Proposition 6.1 makes it fairly simple to obtain a presentation of $\pi_1(\mathbb{C}^2 - D)$ in terms of the "global" generators $\{\gamma_i, \gamma'_i\}$: the nature of the local embeddings $B_{2m} \hookrightarrow B_d$ implies that the relations coming from each vertex are obtained from standard "local" relations (determined by the local braid monodromy) simply by renaming each of the $2m$ local geometric generators into the corresponding global generator. Additionally, the extra nodes yield various commutation relations among geometric generators.

The local configurations for the various types of vertices have been analyzed by Moishezon in [8], leading to explicit formulas for the local contributions to the braid factorization. The easiest case is that of "2-points" such as the corner points 00 and pq in the diagram for $\mathbb{CP}^1 \times \mathbb{CP}^1$. The only line that passes through the vertex locally regenerates to a conic in $\mathbb{C}^2$, presenting a single vertical tangency near the origin; hence the local braid monodromy is a single half-twist in B_2, giving rise to an equality relation between the two corresponding geometric generators of $\pi_1(\mathbb{C}^2 - D)$.

The next case is that of "3-points" such as those occurring on the boundary of the diagram for $\mathbb{CP}^1 \times \mathbb{CP}^1$. During the first step of "regeneration", which turns X_0 into a union of pq quadric surfaces, the lines corresponding to the diagonal edges are replaced by conics (the branch curve of a bidegree $(1,1)$ map from $\mathbb{CP}^1 \times \mathbb{CP}^1$ to $\mathbb{CP}^2$). For the vertices along the top and right sides of the diagram (labelled pj or iq), the partially regenerated configuration in $\mathbb{CP}^2$ therefore consists of a portion of conic tangent to a line, with the line having the greatest slope; after further regeneration, the line acquires multiplicity 2 and the tangent intersection is replaced by three cusps. The local contribution to braid monodromy can therefore be expressed by the product $\tilde{Z}^3_{1'2} \cdot Z^3_{1'2'} \cdot Z^3_{1'2} \cdot \hat{Z}_{11'}$, where the various factors are powers of half-twists along the paths represented in Figure 3 (cf. [8] and equation (2.4) in [9]). The first three factors correspond to cusps arising from the tangent intersection between the conic and the line, while the last factor corresponds to the vertical tangency of the conic.

The 3-points on the bottom and left sides of the diagram give rise to a very similar local configuration, except for the ordering of the various components. Finally, the interior vertices of the diagram for $\mathbb{CP}^1 \times \mathbb{CP}^1$ are all of the same type ("6-points"

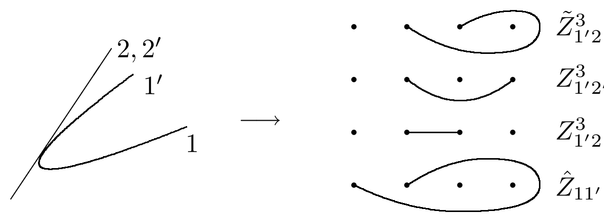

Figure 3

in Moishezon's terminology); a careful analysis of their regeneration yields a certain braid factorization in B_{12}, accounting for the 6 vertical tangencies, 24 nodes and 24 cusps in the local model, as described in [8]. The local contributions to the relations defining $\pi_1(\mathbb{C}^2 - D)$ have also been calculated by Moishezon for these various standard models in §2 of [9] (see also below).

6.1.2. *Fundamental group calculations.* The setup described in §6.1.1 provides an explicit presentation of $\pi_1(\mathbb{C}^2 - D)$ in terms of geometric generators $\{\gamma_i, \gamma_i'\}$, $i = 1, \dots, \frac{d}{2}$. By Proposition 6.1, the relations consist on one hand of standard relations given by local models for the various vertices of the diagram describing the degenerated surface X_0, and on the other hand of commutation relations coming from non-adjacent edges of the diagram. The goal is then to simplify this presentation and ultimately identify $\pi_1(\mathbb{C}^2 - D)$ with a certain quotient of $\tilde{B}_n^{(2)}$ (or $\tilde{B}_n \ltimes \tilde{P}_{n,0}$). In the remainder of this section, we describe the recipes used by Moishezon for the case $X = \mathbb{CP}^1 \times \mathbb{CP}^1$, following §3 of [9]; these methods also apply to other complex surfaces admitting similar degenerations, such as $X = \mathbb{CP}^2$ [10] or $X = \mathbb{F}_1$ (§6.2).

A first observation of Moishezon is that, after a slight change in the choice of generators, many of the local relations at the vertices can be expressed in terms of half of the generators only. More precisely, for each value of i, define a *twisting* action ρ_i on the two generators γ_i, γ_i' by the formula $\rho_i(\gamma_i) = \gamma_i'$ and $\rho_i(\gamma_i') = \gamma_i'\gamma_i\gamma_i'^{-1}$. Choose integers l_i satisfying the following compatibility conditions: if $i < j$ are the labels of the two diagonal edges meeting at a 6-point vertex of the diagram, then $l_j = l_i - 1$; if $i < j$ are the labels of the two vertical edges meeting at a 6-point, then $l_j = l_i + 1$; finally, if $i < j$ are the labels of the two horizontal edges meeting at a 6-point, then $l_j = l_i$. Now let $e_i = \rho_i^{l_i}(\gamma_i)$ and $e_i' = \rho_i^{l_i}(\gamma_i')$. Because of the *invariance properties* of the local models [8], the local relations corresponding to 2-points and 3-points have the same expressions in terms of $\{e_i, e_i'\}$ as in terms of $\{\gamma_i, \gamma_i'\}$, independently of the amount of twisting, and those for 6-points are also independent of the l_i as long as the compatibility relations hold. On the other hand, if $i_1 < \dots < i_6$ are the labels of the edges meeting at a 6-point (i_1 and i_6 are the two diagonal edges), then it is possible to eliminate either e_{i_1} or e_{i_6} from the list of generators, because the local relations imply that

$$e_{i_6} = (e_{i_3}e_{i_2}e_{i_4}^{-1}e_{i_5}^{-1})^{-1}e_{i_1}(e_{i_3}e_{i_2}e_{i_4}^{-1}e_{i_5}^{-1}). \tag{6.1}$$

The second important observation of Moishezon is that, in many cases (assuming the diagram is "large enough", i.e. in the case of a bidegree (p, q) linear system on $\mathbb{CP}^1 \times \mathbb{CP}^1$ that $p, q \geq 2$), the relations coming from cusps and nodes of D can all be reformulated into a very nice pattern (cf. Lemma 14 of [9]). If the two edges i and j bound a common triangle in the diagram, then the local relations at their common vertex imply that

$$e_ie_je_i = e_je_ie_j,\ e_ie'_je_i = e'_je_ie'_j,\ e'_ie_je'_i = e_je'_ie_j,\ \text{and } e'_ie'_je'_i = e'_je'_ie'_j. \tag{6.2}$$

Otherwise, if there is no triangle having i and j as edges, or equivalently if the two transpositions $\theta(e_i) = \theta(e'_i)$ and $\theta(e_j) = \theta(e'_j) \in S_n$ are disjoint, then we have

$$[e_i, e_j] = [e_i, e'_j] = [e'_i, e_j] = [e'_i, e'_j] = 1. \tag{6.3}$$

Looking at $e_1, \dots, e_{\frac{d}{2}}$, among which there are only $n-1$ independent generators (by (6.1), many of the e_i corresponding to diagonal edges can be expressed in terms of the others), a first consequence of the relations (6.2–6.3) is the following (Proposition 8 of [9]):

Lemma 6.2 (Moishezon)**.** *In the case of the linear system $O(p,q)$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$ $(p,q \geq 2)$, the subgroup $\mathcal{B}$ of $\pi_1(\mathbb{C}^2 - D)$ generated by $e_1, \dots, e_{d/2}$ is isomorphic to a quotient of $\tilde{B}_n$ $(n = 2pq)$. More precisely, there exists a surjective morphism $\tilde{\alpha} : \tilde{B}_n \to \mathcal{B}$ with the property that each e_i is the image of a half-twist in $\tilde{B}_n$, and $\theta \circ \tilde{\alpha} = \sigma$ (i.e. the end points of the half-twists agree with the transpositions $\theta(e_i)$).*

We now need to add to this description the other generators e'_i, or equivalently the elements $a_i = e'_ie_i^{-1}$. In the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$, we relabel these elements as d_{ij} for the diagonal edge in position ij $(1 \leq i \leq p,\ 1 \leq j \leq q$, see Figure 1), v_{ij} for the vertical edge in position ij $(1 \leq i < p,\ 1 \leq j \leq q)$, and h_{ij} for the horizontal edge in position ij $(1 \leq i \leq p,\ 1 \leq j < q)$. We are especially interested in $a_2 = v_{11}$. Moishezon's next observation is that, as a consequence of relations (6.2–6.3) and of the local relations of the lower-left-most 6-point in the diagram, the subgroup generated by v_{11} and the conjugates $g^{-1}v_{11}g$, $g \in \mathcal{B}$, is naturally isomorphic to a quotient of $\tilde{P}_{n,0}$ ([9], Definition 5 and Lemma 17). Moreover, the subgroup of $\pi_1(\mathbb{C}^2 - D)$ generated by the e_i and by v_{11} is similarly isomorphic to a quotient of the semi-direct product $\tilde{B}_n \ltimes \tilde{P}_{n,0}$, or equivalently (as seen in §4) $\tilde{B}_n^{(2)}$.

The most important relations in $\pi_1(\mathbb{C}^2 - D)$ are those coming from the vertical tangencies of D, which we now list for the various types of vertices. If the edge labelled i passes through a 2-point, then the local relation $e_i = e'_i$ can be rewritten in the form $a_i = 1$. If $i < j$ are the labels of the two edges meeting at a 3-point, then we have $e'_i = e_j^{-1}{e'_j}^{-1}e_ie'_je_j$, or equivalently $e'_j = e_i^{-1}{e'_i}^{-1}e_je'_ie_i$. Using (6.2) this relation can be rewritten as

$$a_j = e_i^{-1}e_je'_ie_j^{-1}e_ie_j^{-1} = e_i^{-2}(e_ie_j)a_i(e_j^{-1}e_i^{-1})e_je_i^2e_j^{-1}. \tag{6.4}$$

Finally, if $i_1 < \dots < i_6$ are the labels of the edges meeting at a 6-point (according to the ordering rules, i_1 and i_6 are diagonal, i_2 and i_5 are vertical, and i_3 and i_4 are horizontal), then, besides (6.1), we also have

$$\begin{cases} a_{i_6} = (e_{i_3}e_{i_2}e_{i_4}^{-1}e_{i_5}^{-1})^{-1}a_{i_1}(e_{i_3}e_{i_2}e_{i_4}^{-1}e_{i_5}^{-1}) \\ a_{i_5} = (e_{i_1}^{-1}e_{i_3}e_{i_4}^{-1}e_{i_6})^{-1}a_{i_2}(e_{i_1}^{-1}e_{i_3}e_{i_4}^{-1}e_{i_6}) \\ a_{i_4} = (e_{i_1}^{-1}e_{i_2}e_{i_5}^{-1}e_{i_6})^{-1}a_{i_3}(e_{i_1}^{-1}e_{i_2}e_{i_5}^{-1}e_{i_6}) \end{cases} \tag{6.5}$$

$$\begin{cases} a_{i_3} = (e_{i_3}e_{i_1})^{-1}\,a_{i_2}a_{i_1}(e_{i_1}a_{i_2}^{-1}e_{i_1}^{-1})\,(e_{i_3}e_{i_1}) \\ a_{i_2} = (e_{i_2}e_{i_1})^{-1}\,a_{i_3}a_{i_1}(e_{i_1}a_{i_3}^{-1}e_{i_1}^{-1})\,(e_{i_2}e_{i_1}) \end{cases} \tag{6.6}$$

A first consequence of relations (6.4–6.6) is that, going inductively through the various vertices of the grid, all a_i can be expressed in terms of the $e_1, \dots, e_{d/2}$ and of $a_2 = v_{11}$. Therefore $\pi_1(\mathbb{C}^2 - D)$ is generated by the e_i and by v_{11}; hence it is isomorphic to a quotient of $\tilde{B}_n^{(2)}$. In other words, we have a surjective homomorphism $\alpha : \tilde{B}_n^{(2)} \to \pi_1(\mathbb{C}^2 - D)$, extending the morphism $\tilde{\alpha} : \tilde{B}_n \to \mathcal{B}$ of Lemma 6.2.

From this point on, the results in §3 make it possible to present Moishezon's argument in a simpler and more illuminating way. Observe that by Lemma 6.2 each e_i is the image by α of a half-twist in the diagonally embedded subgroup $\tilde{B}_n \subset \tilde{B}_n^{(2)}$. Moreover, it is a general fact about irreducible plane curves that all geometric generators are conjugate to each other in $\pi_1(\mathbb{C}^2 - D)$; therefore each of the geometric generators e_i, e'_i is the image of a pair of half-twists in $\tilde{B}_n^{(2)}$. Alternately this can be seen directly from the above-listed relations; these relations also imply that each a_i belongs to the normal subgroup of pure degree 0 elements $\alpha(\tilde{P}_{n,0} \times \tilde{P}_{n,0})$, and therefore that the half-twists corresponding to the geometric generators e'_i have the correct end points as prescribed by the S_n-valued monodromy representation morphism θ. Therefore $\pi_1(\mathbb{C}^2 - D)$ has the property (*) defined in §3.

In view of Lemmas 3.3 and 3.4, at this point in the argument we can discard all the relations in $\pi_1(\mathbb{C}^2 - D)$ coming from nodes and cusps of D since they automatically hold in quotients of $\tilde{B}_n^{(2)}$, and focus on the relations (6.4–6.6) instead.

By Lemma 3.2, pairs of half-twists in $\tilde{B}_n^{(2)}$ with fixed end points can be classified by two integers. More precisely, fix an ordering of the n sheets of the branched cover f, e.g. from left to right and from bottom to top in the diagram. This provides an ordering of the end points of the half-twists corresponding to e_i and e'_i; we can find an element $g \in \tilde{B}_n^{(2)}$ such that $e_i = \alpha(g^{-1}(x_1, x_1)g)$, with ordering of the end points preserved. Then by Lemma 3.2 there exist integers k and l such that $e'_i = \alpha(g^{-1}(x_1 u_1^{-k} \eta^{-k(-k-1)/2}, x_1 u_1^{-l} \eta^{-l(-l-1)/2})g)$, i.e. $a_i = \alpha(g^{-1}(u_1^k \eta^{k(k-1)/2}, u_1^l \eta^{l(l-1)/2})g)$. One easily checks by Lemma 3.1 that reversing the ordering of the end points changes k into $-k$ and l into $-l$.

Since α is a priori not injective, the integers k and l are not necessarily unique, and there may exist another pair of integers $(k', l') = (k + \kappa, l + \lambda)$ with the same property, i.e. such that $\mu = (u_1^\kappa \eta^{k'(k'-1)/2 - k(k-1)/2}, u_1^\lambda \eta^{l'(l'-1)/2 - l(l-1)/2}) \in \operatorname{Ker}\alpha$. If κ is odd, then the normal subgroup generated by μ contains the commutator of μ with $(u_2, 1)$, which is equal to $(\eta, 1)$; so $(\eta, 1) \in \operatorname{Ker}\alpha$. If κ is even, then $\eta^{k'(k'-1)/2 - k(k-1)/2} = \eta^{\kappa/2} = \eta^{\kappa(\kappa-1)/2}$ (recall that $\eta^2 = 1$). Similarly, if λ is odd then $(1, \eta) \in \operatorname{Ker}\alpha$, otherwise $\eta^{l'(l'-1)/2 - l(l-1)/2} = \eta^{\lambda(\lambda-1)/2}$. In both cases we arrive to the conclusion that $\tilde{\mu} = (u_1^\kappa \eta^{\kappa(\kappa-1)/2}, u_1^\lambda \eta^{\lambda(\lambda-1)/2}) \in \operatorname{Ker}\alpha$. In fact, μ and $\tilde{\mu}$ generate the same normal subgroups, so we also have the converse implication.

Therefore the set of all possible values for (κ, λ) forms a subgroup $\Lambda \subset \mathbb{Z}^2$; in fact $\Lambda = \{(\kappa, \lambda),\ (u_1^\kappa \eta^{\kappa(\kappa-1)/2}, u_1^\lambda \eta^{\lambda(\lambda-1)/2}) \in \operatorname{Ker}\alpha\}$, and the pair of integers (k, l) is only defined mod Λ. So, to e_i and e'_i we can associate an element $\bar{a}_i = (k, l) \in \mathbb{Z}^2/\Lambda$. This element $\bar{a}_i$ contains all the relevant information about e_i and e'_i apart from the end points. Indeed, because of Lemma 3.5, up to composition of α with an automorphism of $\tilde{B}_n^{(2)}$ we can assume e_i to be the image by α of any given pair of half-twists with the correct end points. And, by Lemma 3.2, if two half-twists $x, y \in \tilde{B}_n$ have the same end points, then $x^2 y^{-2} \in \{1, \eta\}$, so up to a factor of η the product $e'_i e_i = a_i e_i^2$ is determined by $\bar{a}_i$; that ambiguity can in fact be lifted by arguing that e_i and e'_i are images of half-twists.

The subgroup Λ can be determined by looking at the relations in $\pi_1(\mathbb{C}^2 - D)$ coming from vertical tangencies of D, which determine the kernel of α. We now reformulate these relations in terms of the $\bar{a}_i$. First, at a 2-point, the relation $a_i = 1$ becomes $\bar{a}_i = (0,0)$. What happens at a 3-point depends on the ordering of the sheets of f (i.e., of the triangles of the diagram): the relation (6.4) becomes

$$\pm\bar{a}_i + \pm\bar{a}_j = (1,1), \tag{6.7}$$

where the first sign is + if the triangle T which has both i and j among its edges comes *after* the other triangle bounded by the edge i and − otherwise, and the second sign is + if T comes *after* the other triangle bounded by the edge j and − otherwise. In the case of a 6-point with the standard ordering used by Moishezon, (6.5) and (6.6) become

$$\bar{a}_{i_6} = \bar{a}_{i_1}, \qquad \bar{a}_{i_5} = \bar{a}_{i_2}, \qquad \bar{a}_{i_4} = \bar{a}_{i_3}, \qquad \bar{a}_{i_1} - \bar{a}_{i_2} + \bar{a}_{i_3} = 0. \tag{6.8}$$

In the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$, denoting by $\bar{d}_{ij}$, $\bar{v}_{ij}$ and $\bar{h}_{ij}$ the elements of $\mathbb{Z}^2/\Lambda$ corresponding to d_{ij}, v_{ij} and h_{ij}, the relations become (listing the vertices from left to right and bottom to top): $\bar{d}_{1,1} = (0,0)$, $\bar{v}_{i,1} - \bar{d}_{i+1,1} = (1,1)$, $\bar{h}_{1,j} + \bar{d}_{1,j+1} = (1,1)$; $\bar{d}_{i+1,j+1} = \bar{d}_{i,j}$, $\bar{v}_{i,j+1} = \bar{v}_{i,j}$, $\bar{h}_{i+1,j} = \bar{h}_{i,j}$, $\bar{d}_{i,j} - \bar{v}_{i,j} + \bar{h}_{i,j} = 0$; $-\bar{d}_{p,j} - \bar{h}_{p,j} = (1,1)$, $\bar{d}_{i,q} - \bar{v}_{i,q} = (1,1)$, $\bar{d}_{p,q} = (0,0)$. Moreover, by construction $\bar{v}_{11} = (0,1)$ (because v_{11} was identified to a generator of $\tilde{P}_{n,0}$).

Working inductively from the lower-left corner of the diagram, these equations yield the formulas

$$\bar{d}_{i,j} = (j-i,0), \qquad \bar{v}_{i,j} = (1-i,1), \qquad \bar{h}_{i,j} = (1-j,1) \tag{6.9}$$

(compare with Proposition 10 of [9], recalling that the identification between $\tilde{B}_n \ltimes \tilde{P}_{n,0}$ and $\tilde{B}_n^{(2)}$ is given by $(x,u) \mapsto (x,xu)$). Moreover, we are left with the relations $(p-1,-1) = (1,1)$ and $(q-1,-1) = (1,1)$. In other words, Λ is the subgroup of $\mathbb{Z}^2$ generated by $(2-p,2)$ and $(2-q,2)$.

Because all relations in $\pi_1(\mathbb{C}^2 - D)$ coming from vertical tangencies correspond to equality relations between pairs of half-twists in $\tilde{B}_n^{(2)}$, by the above remarks $\operatorname{Ker}\alpha$ is the normal subgroup of $\tilde{B}_n^{(2)}$ generated by a certain number of elements of the form $(u_1^\kappa \eta^{\kappa(\kappa-1)/2}, u_1^\lambda \eta^{\lambda(\lambda-1)/2})$, and therefore it is completely determined by the subgroup $\Lambda \subset \mathbb{Z}^2$. In our case, $\operatorname{Ker}\alpha$ is the normal subgroup of $\tilde{B}_n^{(2)}$ generated by $(u_1^{2-p}\eta^{(2-p)(1-p)/2}, u_1^2\eta)$ and $(u_1^{2-q}\eta^{(2-q)(1-q)/2}, u_1^2\eta)$. We can now finish the proof of Theorem 4.1, observing that $H^0_{p,q} = (\tilde{P}_{n,0} \times \tilde{P}_{n,0})/\operatorname{Ker}\alpha$. Recalling from Lemma 3.1 that $\tilde{P}_{n,0}$ has commutator subgroup $\{1,\eta\} \simeq \mathbb{Z}_2$ and that $\operatorname{Ab}\tilde{P}_{n,0} \simeq \mathbb{Z}^{n-1}$, we have two cases to consider. First, if e.g. p is odd, then by considering the commutator of $(u_1^{2-p}\eta^{(2-p)(1-p)/2}, u_1^2\eta)$ with $(u_2,1)$ we obtain that $(\eta,1) \in \operatorname{Ker}\alpha$ (and similarly if q is odd); but one easily checks that $(1,\eta) \notin \operatorname{Ker}\alpha$. On the other hand, if p and q are both even, then no non-trivial element of $C = \{1,\eta\} \times \{1,\eta\}$ belongs to $\operatorname{Ker}\alpha$. Therefore, $[H^0_{p,q}, H^0_{p,q}] \simeq C/(C \cap \operatorname{Ker}\alpha)$ is isomorphic to $\mathbb{Z}_2$ if p or q is odd, and to $\mathbb{Z}_2 \times \mathbb{Z}_2$ if p and q are even. Moreover, we have $\operatorname{Ab} H^0_{p,q} \simeq (\tilde{P}_{n,0} \times \tilde{P}_{n,0})/\langle C, \operatorname{Ker}\alpha\rangle \simeq (\mathbb{Z}^2/\Lambda)^{n-1}$, which one easily shows to be isomorphic to $(\mathbb{Z}_2 \oplus \mathbb{Z}_{p-q})^{n-1}$ or $(\mathbb{Z}_{2(p-q)})^{n-1}$ depending on the parity of p and q. This completes the proof of Theorem 4.1. The computations for $\mathbb{CP}^2$ (Theorem 4.2) and other algebraic surfaces admitting similar degenerations can be carried out by the same method; for example, the case of the Hirzebruch surface $\mathbb{F}_1$ is treated in §6.2 below.

6.2. **The Hirzebruch surface $\mathbb{F}_1$.** In this section, we prove Theorem 4.5 using the method outlined in the preceding section. Consider the projective embedding

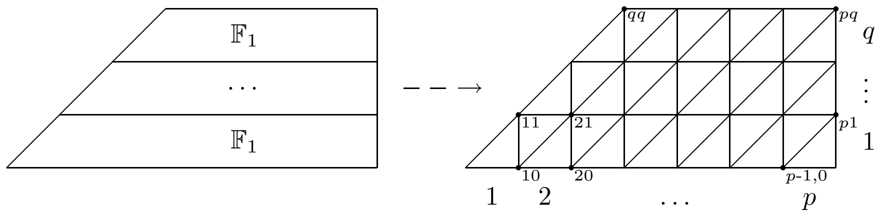

FIGURE 4

of $\mathbb{F}_1$ defined by sections of the linear system $O(pF+qE)$, $p > q \geq 2$ (recall F is the fiber and E is the exceptional section). This projective surface can be degenerated in the same manner as the Veronese surface of which it is a blow-up (the projective embedding of $\mathbb{CP}^2$ defined by sections of $O(p)$), following the procedure described in §3 of [8]. This surface of degree $n = (2p-q)q$ can be first degenerated into a sum of q Hirzebruch surfaces, of degrees respectively $2p-1$, $2p-3$, ... , $2(p-q)+1$. Each of these Hirzebruch surfaces can then be degenerated into the union of a plane and a certain number of quadric surfaces, which in turn can each be degenerated to two planes. The resulting diagram is pictured in the right half of Figure 4.

One uses the same setup as in §6.1.1, ordering the vertices from left to right and bottom to top, and the edges accordingly. The braid monodromy is given by Proposition 6.1. It follows from Moishezon's work that all vertices correspond to well-known configurations: the two vertices qq and pq are 2-points, while the other boundary vertices are 3-points and the interior vertices are 6-points.

As in §6.1.2, one replaces the natural set of geometric generators $\{\gamma_i, \gamma_i'\}$ by twisted generators $e_i = \rho_i^{l_i}(\gamma_i)$ and $e_i' = \rho_i^{l_i}(\gamma_i')$, where the integers l_i satisfy the required compatibility conditions, in order to have (6.1) at all 6-points. Moreover, relations (6.2) and (6.3) hold for all pairs of edges ((6.2) if the edges bound a common triangle, (6.3) otherwise), by the same argument as for $\mathbb{CP}^2$: the proof of Lemma 1 of [10] (see also Lemma 14 of [9]) applies almost without modification.

Eliminating redundant diagonal edges as allowed by (6.1), we are left with exactly $n-1$ independent generators among the e_i. As in the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$, relations (6.2) and (6.3) imply that the subgroup $\mathcal{B}$ generated by the e_i is isomorphic to a quotient of $\tilde{B}_n$, and Lemma 6.2 extends to the case of the Hirzebruch surface $\mathbb{F}_1$.

As previously, we let $a_i = e_i' e_i^{-1}$, and we relabel these elements as d_{ij}, v_{ij} and h_{ij}. We are now interested in $a_1 = v_{11}$: one can again show that the subgroup generated by v_{11} and the conjugates $g^{-1}v_{11}g$, $g \in \mathcal{B}$ is isomorphic to a quotient of $\tilde{P}_{n,0}$, by Lemma 5 of [10] (the argument is the same for $\mathbb{F}_1$ as for $\mathbb{CP}^2$); the subgroup of $\pi_1(\mathbb{C}^2 - D)$ generated by the e_i and by a_1 is again isomorphic to a quotient of $\tilde{B}_n \ltimes \tilde{P}_{n,0} \simeq \tilde{B}_n^{(2)}$.

Relations (6.4–6.6) imply that, going through the various 3-points and 6-points of the diagram, all the a_i can be expressed in terms of $e_1, \dots, e_{d/2}$ and $a_1 = v_{11}$; therefore $\pi_1(\mathbb{C}^2 - D)$ is generated by $e_1, \dots, e_{d/2}$ and a_1, so that we again obtain a surjective morphism $\alpha : \tilde{B}_n^{(2)} \to \pi_1(\mathbb{C}^2 - D)$. As in the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$, the various geometric generators are images by α of pairs of half-twists with correct end points, so that property (*) holds once more. Using the classification of half-twists in $\tilde{B}_n$ (Lemma 3.2), we can consider pairs of integers $\bar{a}_i$ instead of the elements a_i; once again, the $\bar{a}_i$ are only defined modulo a certain subgroup $\Lambda \subset \mathbb{Z}^2$.

The various relations between the $\bar{a}_i$ are now the following: $\bar{v}_{i,1} - \bar{d}_{i+1,1} = (1,1)$, $\bar{v}_{i,i} - \bar{h}_{i+1,i} = (1,1)$; $\bar{d}_{i+1,j+1} = \bar{d}_{i,j}$, $\bar{v}_{i,j+1} = \bar{v}_{i,j}$, $\bar{h}_{i+1,j} = \bar{h}_{i,j}$, $\bar{d}_{i,j} - \bar{v}_{i,j} + \bar{h}_{i,j} = 0$; $-\bar{d}_{p,j} - \bar{h}_{p,j} = (1,1)$, $\bar{v}_{q,q} = (0,0)$, $\bar{d}_{i,q} - \bar{v}_{i,q} = (1,1)$, $\bar{d}_{p,q} = (0,0)$. Moreover, $\bar{v}_{1,1} = (0,1)$. Therefore, $\bar{d}_{i,j} = (2j-2i+1, j-i+1)$, $\bar{v}_{i,j} = (2-2i, 2-i)$ and $\bar{h}_{i,j} = (1-2j, 1-j)$ (compare with Proposition 4 of [10]), and we are left with two additional relations: $(2p-2, p-2) = (1,1)$ and $(2-2q, 2-q) = (0,0)$. Therefore, Λ is the subgroup of $\mathbb{Z}^2$ generated by $(2p-3, p-3)$ and $(2q-2, q-2)$, and $\operatorname{Ker}\alpha$ is the normal subgroup of $\tilde{B}_n^{(2)}$ generated by $(u_1^{2p-3}\eta^{(2p-3)(2p-4)/2}, u_1^{p-3}\eta^{(p-3)(p-4)/2})$ and $(u_1^{2q-2}\eta^{(2q-2)(2q-3)/2}, u_1^{q-2}\eta^{(q-1)(q-2)/2})$.

Considering the commutator of the first generator with $(u_2, 1)$, we obtain that $(\eta, 1) \in \operatorname{Ker}\alpha$. Moreover, if either p is even or q is odd, then considering the commutator of one of the generators with $(1, u_2)$, we obtain that $(1, \eta) \in \operatorname{Ker}\alpha$. On the contrary, if p is odd and q is even then $(1,\eta) \notin \operatorname{Ker}\alpha$. We conclude that $[H^0_{p,q}, H^0_{p,q}] \simeq C/(C \cap \operatorname{Ker}\alpha)$ is trivial or isomorphic to $\mathbb{Z}_2$ depending on the parity of p and q, and that $\operatorname{Ab} H^0_{p,q} \simeq (\mathbb{Z}^2/\Lambda)^{n-1} \simeq (\mathbb{Z}^2/\langle (p,3),(q,2)\rangle)^{n-1} \simeq (\mathbb{Z}_{3q-2p})^{n-1}$.

7. Double covers of $\mathbb{CP}^1 \times \mathbb{CP}^1$

In this section, we sketch the proof of Theorem 4.6, which combines the methods described in §6 with ideas similar to those in [3].

7.1. **Generic perturbations of iterated branched covers.** Let C be a smooth algebraic curve of degree $(2a, 2b)$ in $Y = \mathbb{CP}^1 \times \mathbb{CP}^1$, and let $X_{a,b}$ be the double cover of Y branched along C. Then one can construct a map $f^0 : X_{a,b} \to \mathbb{CP}^2$ simply by composing the double cover $\pi : X_{a,b} \to Y$ with a generic projective map $g : Y \to \mathbb{CP}^2$ determined by sections of $O(p,q)$. The map f^0 is not generic : its ramification curve is the union of the ramification curve of π and the preimage by π of the ramification curve of g, and so the branch curve D^0 of f^0 is the union of $g(C)$ (with multiplicity 1) and the branch curve D_g of g (with multiplicity 2).

This situation is extremely similar to that considered in [3] for the composition of a generic map from a symplectic 4-manifold to $\mathbb{CP}^2$ with a quadratic map from $\mathbb{CP}^2$ to itself. The local behavior of the map f^0 is generic everywhere except at the intersection points of C with the ramification curve of g ; assuming that C and g are chosen generically, a local model for f^0 near these points is $(x,y) \mapsto (-x^2+y, -y^2)$, for which a generic local perturbation is given e.g. by $(x,y) \mapsto (-x^2+y, -y^2+\epsilon x)$ where ϵ is a small non-zero constant (cf. also [3]). There are several ways in which the map f^0 can be perturbed and made generic. If the linear system $\pi^* O(p,q)$ is sufficiently ample, then f^0 can be deformed within the holomorphic category into a generic projective map which no longer factors through the double cover π. Another possibility, if p and q are sufficiently large, is to use approximately holomorphic methods (Theorem 1.1) to deform f^0 into a map with generic local models (cf. [3]).

In both cases, the effect of the perturbation on the topology of the branch curve of f^0 is pretty much the same. First, the local model near an intersection point of C with the ramification curve of g is perturbed as described above (up to isotopy), which transforms a tangent intersection of $g(C)$ with the branch curve of g in $\mathbb{CP}^2$ into a standard configuration with three cusps [3]. Secondly, the two copies of the branch curve of g, which make up the multiplicity two component of D^0, are separated and made transverse to each other; this deformation of D_g is performed either within the holomorphic category or resorting to approximately holomorphic perturbations. In the second case, the perturbation process can be performed in a very flexible manner, which in some cases may create negative intersections ; restricting oneself to algebraic perturbations is a convenient way to avoid this phenomenon, but makes the global perturbation harder to describe explicitly. In any case, up to isotopy and creation or cancellation of pairs of intersections between the two deformed copies of the branch curve of g, the topology of the resulting generic branch curve D is uniquely determined and can be computed easily from that of D^0. In fact, the approximately holomorphic perturbation process can always be carried out, even for small values of p and q for which neither the holomorphic construction nor Theorem 1.1 are able to yield generic projective maps ; in this situation, we can still study the topology of the curve D, but Theorem 4.6 only describes a "virtual" generic projective map.

As in §6, the study of the curve D relies on a degeneration process: one first degenerates the curve C in $Y = \mathbb{CP}^1 \times \mathbb{CP}^1$ into a union of two sets of parallel lines, $2a$ along one factor and $2b$ along the other factor. Parallel lines are then merged, so that the resulting configuration $C_0 \subset Y$ consists of only two components, a $(1,0)$-line of multiplicity $2a$ and a $(0,1)$-line of multiplicity $2b$. Finally, one degenerates the projective embedding of Y given by the linear system $O(p,q)$ into an arrangement Y_0 of planes intersecting along lines, as in §6.1. The fully degenerated branch curve is a union of lines, some of which correspond to the intersections between the planes in Y_0 (each contributing with multiplicity 4, since the branch curve of g is counted with multiplicity 2), while the others are the images of the $p+q$ components into which C_0 degenerates (some of these components contribute with multiplicity $2a$, others with multiplicity $2b$).

The curve D can be recovered from this arrangement of lines by the converse "regeneration" process, which first yields the union $D_g \cup g(C_0)$ (by deforming Y_0 into the smooth surface Y), then $D_g \cup g(C) = D^0$ (by separating the multiple components of C_0 and smoothing the resulting curve), and finally D (by performing the prescribed local perturbation at the intersection points of the two ramification curves and by perturbing the two copies of D_g in a generic way).

7.2. **Braid monodromy calculations.** The braid monodromy for the curve $D_g \cup g(C_0)$ (and for the subsequent regenerations D^0 and D) can be computed using the same methods as in §6.1.1. The diagram describing the degenerated configuration is as represented on Figure 5, which differs from Figure 1 only by the addition of edges corresponding to C_0 along the top and right boundaries of the diagram.

Thanks to Proposition 6.1, we only need to understand the local behavior of the curves $D_g \cup g(C_0)$, D^0 and D near the various vertices of the diagram. At all vertices except those through which C_0 passes (top and right sides of the diagram), the local description of $D_g \cup g(C_0)$ and D^0 is exactly the same as that of D_g, which has already been discussed in §6.1 : the various vertices are standard 2-points, 3-points and 6-points as in Moishezon's work [9]. Moreover, the local configuration for D at such a vertex simply consists of two copies of the local configuration for D_g, shifted apart from each other by a generic translation. The two components, which correspond to the two preimages of the ramification curve of g under the branched cover π, may intersect at nodal points of either orientation ; we won't

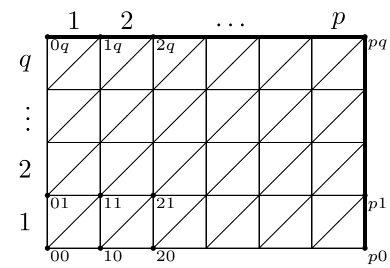

FIGURE 5

be overly concerned by the details of these intersections, since the various possible configurations only differ by isotopies and creations or cancellations of pairs of nodes, which do not affect the stabilized fundamental group in any way.

We now consider a vertex along the top boundary of the diagram, at position iq with $1 \leq i \leq p-1$. The local configuration for $D_g \cup g(C_0)$ at such a point is as shown on Figure 6. The parts labelled $1, 1', 2, 2'$ correspond to D_g, and form a standard 3-point (cf. §6.1.1 and Figure 3), presenting three cusp singularities near the point A. The parts labelled 3 and 4 correspond to $g(C_0)$, obtained by "regeneration" of the two lines associated to the horizontal edges of the diagram passing through the vertex. The curve $g(C_0)$ presents tangent intersections with the two lines 2 and $2'$ near the point B, and with the conic $1, 1'$ at the point C. The two intersections of the line labelled 4 with the conic $1, 1'$ in $\mathbb{CP}^2$ remain as nodes since the corresponding curves fail to intersect in Y.

The local description of the curve $D^0 = D_g \cup g(C)$ is obtained from that of $D_g \cup g(C_0)$ by separating C_0 into $2b$ parallel components ; this yields $2b$ copies of the lines labelled 3 and 4 in Figure 6, and the local configuration near the points B and C becomes as shown in the right half of Figure 6 (the pictures correspond to the case $b = 2$). Finally, in order to obtain D we must perturb D^0 in the manner explained in §7.1: the multiplicity two component $D_g \subset D^0$ (corresponding to the parts labelled $1, 1', 2, 2'$ in Figure 6) is separated into two distinct copies (in particular the point A is duplicated), while each tangent intersection of $g(C)$ with D_g (such as those near points B and C) gives rise to three cusps. It is then possible to write explicitly the local braid monodromy for D, with values in B_{4b+8} by enumerating carefully the $4b+2$ vertical tangencies, $18b+6$ cusps, and nodes of the local model

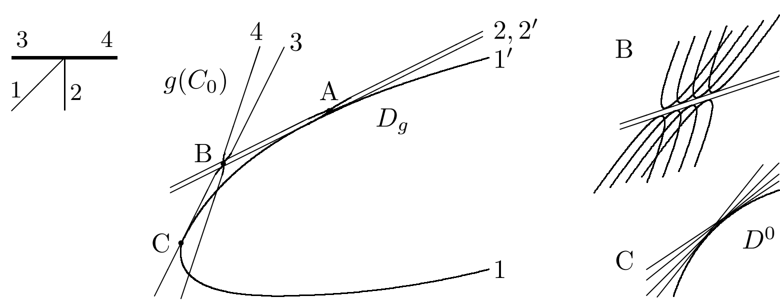

FIGURE 6

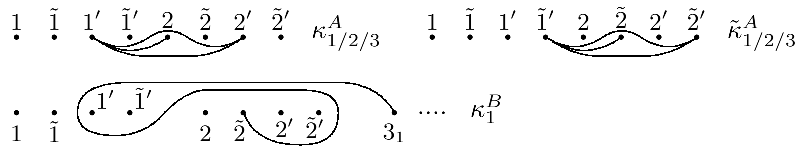

Figure 7

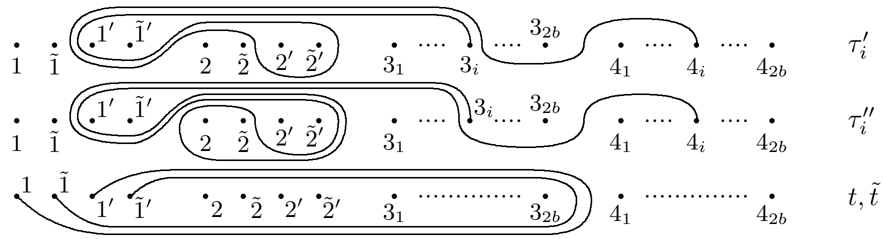

Figure 8

(the exact number of nodes depends on the choice of boundary values for the local perturbation of D^0).

In fact, since we only aim to compute *stabilized* fundamental groups of branch curve complements, we shall not concern ourselves with the nodes of D, since these only yield commutation relations which by definition always hold in the stabilized group. Moreover, for reasons that will be apparent later in the argument, the cusp points are also of limited relevance for our purposes; those which will play a role in the argument, namely the six cusps near point A and one of the $12b$ cusps near point B of Figure 6, give rise to braid monodromies equal to the cubes of the half-twists represented in Figure 7. Actually, the truly important information is contained in the vertical tangencies, which correspond to the half-twists $\tau'_1, \dots, \tau'_{2b}, \tau''_1, \dots, \tau''_{2b}, t, \tilde t \in B_{4b+8}$ represented in Figure 8. As in §6.1, the reference fiber of π is $\{x = A\}$ for A a large positive real constant, and the chosen generating paths in the base (x-plane) remain under the real axis except near their end points; the labels $1, 1', 2, 2', \tilde 1, \tilde 1', \tilde 2, \tilde 2'$ and $3_1, \dots, 3_{2b}, 4_1, \dots, 4_{2b}$ correspond respectively to the two copies of D_g and to $g(C)$.

We now turn to vertices along the right boundary of the diagram, at positions pj with $1 \le j \le q-1$. The local geometric configuration is very similar to that for the vertices along the top boundary, except for the local description of the curve $g(C)$ which now involves $2a$ parallel copies of $g(C_0)$ instead of $2b$. Another difference is that, due to the ordering of the vertices and edges of the diagram, the slope of some of the line components to which $g(C)$ degenerates becomes smaller than that of some of the components to which D_g degenerates, so that the braid monodromy has to be calculated again, with results very similar to those above. In fact, it can easily be checked that, up to a Hurwitz equivalence, the only effect of the change of ordering on the local braid monodromy is the simultaneous conjugation of all contributions by a braid that exchanges the groups of points labelled $2, \tilde 2, 2', \tilde 2'$ and $3_1, \dots, 3_{2a}$ by moving them around each other counterclockwise.

The last vertex that remains to be investigated is the corner vertex at position pq. The local configuration for $D^0 = D_g \cup g(C)$ is obtained from that represented

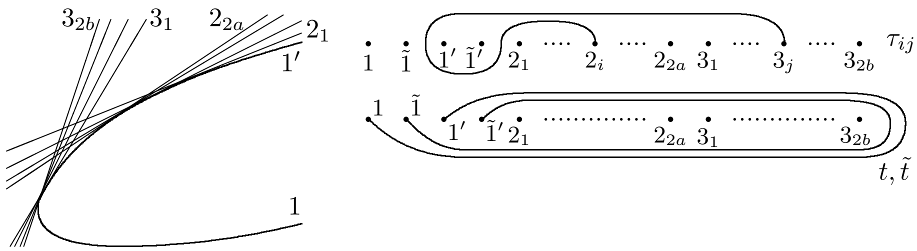

FIGURE 9

in Figure 9 (left) by smoothing the $4ab$ mutual intersections between the lines labelled $2_1, \dots, 2_{2a}$ and $3_1, \dots, 3_{2b}$. Indeed, the local configuration for D_g is simply a conic (labelled $1, 1'$ in Figure 9), while $g(C_0)$ consists of two lines tangent to that conic, and $g(C)$ is obtained by "thickening" these two lines into respectively $2a$ and $2b$ components ($2_1, \dots, 2_{2a}$ corresponding to the vertical edge of the diagram, and $3_1, \dots, 3_{2b}$ corresponding to the horizontal edge of the diagram) and smoothing their mutual intersections. The curve D is then obtained from D^0 by separating the multiplicity 2 component D_g into two distinct copies, while each tangent intersection of D_g with $g(C)$ gives rise to three cusps.

The braid monodromy for the corner vertex can be deduced explicitly from this description. We are particularly interested in the $8ab+2$ vertical tangencies of the local model, for which the corresponding half-twists τ_{ij} ($1 \le i \le 2a$, $1 \le j \le 2b$, each appearing twice), t and $\tilde{t}$ in $B_{2a+2b+4}$ are represented in Figure 9 (right).

7.3. **Fundamental group calculations.** As in §6, the Zariski-Van Kampen theorem provides an explicit presentation of $\pi_1(\mathbb{C}^2 - D)$ in terms of the braid monodromy. The main difference is that there are now four generators for each interior edge of the diagram (Figure 5), because the regeneration process involves two copies of the branch curve of g; we denote by γ_i, γ_i' and $\tilde{\gamma}_i, \tilde{\gamma}_i'$ the four generators corresponding to the i-th interior edge. Moreover, each edge along the top boundary of the diagram contributes $2b$ generators (denoted by $z_{i,1}, \dots, z_{i,2b}$ for the horizontal edge in position iq, where $1 \le i \le p$), and similarly each edge along the right boundary contributes $2a$ generators ($y_{j,1}, \dots, y_{j,2a}$ for the vertical edge in position pj, where $1 \le j \le q$).

We are in fact interested in the stabilized quotient G of $\pi_1(\mathbb{C}^2 - D)$ (see Definition 2.2), which can be expressed in terms of the same generators by adding suitable commutation relations. Let Γ be the subgroup of G generated by the γ_i, γ_i', and let $\tilde{\Gamma}$ be the subgroup generated by the $\tilde{\gamma}_i, \tilde{\gamma}_i'$. By definition, the elements of Γ always commute with those of $\tilde{\Gamma}$, because the images by the geometric monodromy representation θ of the geometric generators γ_i, γ_i' and $\tilde{\gamma}_i, \tilde{\gamma}_i'$ act on two disjoint sets of $n/2 = 2pq$ sheets of the branched cover f.

As in §6, we introduce twisted generators e_i, e_i' and $\tilde{e}_i, \tilde{e}_i'$ for Γ and $\tilde{\Gamma}$, by choosing integers l_i satisfying the same compatibility conditions at the inner vertices as in §6, and setting as previously $e_i = \rho_i^{l_i}(\gamma_i)$, $e_i' = \rho_i^{l_i}(\gamma_i')$, $\tilde{e}_i = \tilde{\rho}_i^{l_i}(\tilde{\gamma}_i)$ and $\tilde{e}_i' = \tilde{\rho}_i^{l_i}(\tilde{\gamma}_i')$, with the obvious definition for ρ_i and $\tilde{\rho}_i$. Even though this could be avoided by proving a suitable invariance property, we will assume that $l_i = 1$ for every diagonal edge in the top-most row or in the right-most column of the diagram (so $e_i = \gamma_i'$, $\tilde{e}_i = \tilde{\gamma}_i'$), and $l_j = 0$ for every vertical edge in the top-most row and every horizontal edge in the right-most column (so $e_j = \gamma_j$, $\tilde{e}_j = \tilde{\gamma}_j$). Finally, as in §6.1 we let $a_i = e'_i e_i^{-1}$ and $\tilde{a}_i = \tilde{e}'_i \tilde{e}_i^{-1}$, and we relabel these elements as d_{ij}, v_{ij}, h_{ij} (resp. $\tilde{d}_{ij}, \tilde{v}_{ij}, \tilde{h}_{ij}$) according to their position in the diagram.

Lemma 7.1. *The subgroup $\mathcal{B}_\Gamma \subset \Gamma$ generated by the e_i and the subgroup $\mathcal{B}_{\tilde{\Gamma}} \subset \tilde{\Gamma}$ generated by the $\tilde{e}_i$ are naturally isomorphic to quotients of $\tilde{B}_{n/2}$. Moreover, the subgroups Γ and $\tilde{\Gamma}$ of G are naturally isomorphic to quotients of $\tilde{B}^{(2)}_{n/2}$, with geometric generators corresponding to pairs of half-twists. Furthermore, Γ is generated by the elements of $\mathcal{B}_\Gamma$ and v_{11}, and $\tilde{\Gamma}$ is generated by the elements of $\mathcal{B}_{\tilde{\Gamma}}$ and $\tilde{v}_{11}$.*

Proof. We first look at relations corresponding to the interior vertices of the diagram (Figure 5) and to the vertices along the bottom and left boundaries. Since the local description of D at these vertices simply consists of two superimposed copies of D_g, and since the generators of Γ commute with those of $\tilde{\Gamma}$, one easily checks that the local configurations yield relations among the e_i, e'_i that are exactly identical to those discussed in §6 in the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$; additionally, an identical set of relations also holds among the $\tilde{e}_i, \tilde{e}'_i$.

Next we consider the local configuration at a vertex along the top boundary of the diagram, and more precisely the cusp singularities present near the point labelled A on Figure 6, as pictured on Figure 7. Denoting by i and j respectively the labels of the diagonal and vertical edges meeting at the given vertex, the relations corresponding to these six cusps are

$$\begin{aligned}(7.1)\quad &\gamma'_i\gamma_j\gamma'_i = \gamma_j\gamma'_i\gamma_j,\ \gamma'_i\gamma'_j\gamma'_i = \gamma'_j\gamma'_i\gamma'_j,\ \gamma'_i(\gamma_j^{-1}\gamma'_j\gamma_j)\gamma'_i = (\gamma_j^{-1}\gamma'_j\gamma_j)\gamma'_i(\gamma_j^{-1}\gamma'_j\gamma_j),\\ &\tilde{\gamma}'_i\tilde{\gamma}_j\tilde{\gamma}'_i = \tilde{\gamma}_j\tilde{\gamma}'_i\tilde{\gamma}_j,\ \tilde{\gamma}'_i\tilde{\gamma}'_j\tilde{\gamma}'_i = \tilde{\gamma}'_j\tilde{\gamma}'_i\tilde{\gamma}'_j,\ \tilde{\gamma}'_i(\tilde{\gamma}_j^{-1}\tilde{\gamma}'_j\tilde{\gamma}_j)\tilde{\gamma}'_i = (\tilde{\gamma}_j^{-1}\tilde{\gamma}'_j\tilde{\gamma}_j)\tilde{\gamma}'_i(\tilde{\gamma}_j^{-1}\tilde{\gamma}'_j\tilde{\gamma}_j).\end{aligned}$$

It can easily be checked that these relations satisfy a property of invariance under twisting similar to that of 3-points. In fact, replacing the various generators by their images under arbitrary powers of the twisting actions $\rho_i, \tilde{\rho}_i, \rho_j, \tilde{\rho}_j$ amounts to a conjugation of the relations (7.1) by braids belonging to the local monodromy (either the entire local monodromy, or two of the six cusps near A, or combinations thereof), and thus always yields valid relations.

Therefore, the twisted generators e_i, e'_i, e_j, e'_j of Γ satisfy the relations (6.2), and similarly for $\tilde{e}_i, \tilde{e}'_i, \tilde{e}_j, \tilde{e}'_j$ in $\tilde{\Gamma}$. One easily checks that a similar conclusion holds for pairs of inner edges meeting at a vertex along the right boundary of the diagram (recall that the local braid monodromy only differs by a simple conjugation). Finally, because we are looking at the stabilized fundamental group, the commutation relations discussed in §6 automatically hold in Γ and $\tilde{\Gamma}$.

So, except for the equality relations arising from vertical tangencies at the vertices along the top and right boundaries of the diagram, all the relations described in §6.1 for the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$ simultaneously hold in Γ and in $\tilde{\Gamma}$. Therefore, the structure of Γ and $\tilde{\Gamma}$ can be studied by the same argument as in the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$ ([9], see also §6), which yields the desired result. □

Lemma 7.2. *The equality $z_{r,i} = z_{r,1}$ holds for every $1 \le r \le p$, $1 \le i \le 2b$; similarly, $y_{r,i} = y_{r,1}$ for every $1 \le r \le q$, $1 \le i \le 2a$. Moreover, the $y_{r,i}$ and the $z_{r,i}$ are all conjugates of $y_{q,1}$ under the action of elements of $\mathcal{B}_\Gamma$ and $\mathcal{B}_{\tilde{\Gamma}}$.*

Proof. First consider the corner vertex at position pq, and more precisely the half-twists τ_{ij} arising from the vertical tangencies of the local model near this vertex

(Figure 9). Denoting by μ the label of the diagonal edge in position pq, the half-twist τ_{1i} yields the relation $(y_{q,1}^{-1}\dots y_{q,2a}^{-1}z_{p,1}^{-1}\dots z_{p,i-1}^{-1})z_{p,i}(z_{p,i-1}\dots z_{p,1}y_{q,2a}\dots y_{q,1}) = \tilde\gamma'_\mu\gamma'_\mu y_{q,1}{\gamma'_\mu}^{-1}{\tilde\gamma'_\mu}^{-1}$. It follows that the quantity $(z_{p,1}^{-1}\dots z_{p,i-1}^{-1})z_{p,i}(z_{p,i-1}\dots z_{p,1})$ is independent of i, which by an easy induction on i implies that $z_{p,i} = z_{p,1}$ for all i. Observing that $y_{q,1},\dots,y_{q,2a}$ and $z_{p,1},\dots,z_{p,2b}$ are mapped by θ to disjoint transpositions and hence commute in G, we in fact have $z_{p,i} = \tilde\gamma'_\mu\gamma'_\mu y_{q,1}{\gamma'_\mu}^{-1}{\tilde\gamma'_\mu}^{-1}$ for all i. Since by assumption the twisting parameter l_μ is equal to 1, the generators $\gamma'_\mu = e_\mu$ and $\tilde\gamma'_\mu = \tilde e_\mu$ belong to $\mathcal{B}_\Gamma$ and $\mathcal{B}_{\tilde\Gamma}$ respectively. This proves the claims made about the $z_{p,i}$.

Similarly comparing the relations corresponding to the half-twists τ_{i1}, it can be seen immediately that the quantity $(y_{q,1}^{-1}\dots y_{q,i-1}^{-1})y_{q,i}(y_{q,i-1}\dots y_{q,1})$ is independent of i, which implies that $y_{q,i} = y_{q,1}$ for all i.

We now proceed by induction : assume that $z_{r+1,i} = z_{r+1,1}$ for all i, and that $z_{r+1,1}$ is a conjugate of $y_{q,1}$ under the action of $\mathcal{B}_\Gamma$ and $\mathcal{B}_{\tilde\Gamma}$. Let μ and ν be the labels of the diagonal and vertical edges meeting at the vertex in position rq, and let $\psi_r = \tilde\gamma'_\nu\gamma'_\nu\tilde\gamma_\nu\gamma_\nu\tilde\gamma'_\mu\gamma'_\mu\gamma_\nu^{-1}\tilde\gamma_\nu^{-1}{\gamma'_\nu}^{-1}{\tilde\gamma'_\nu}^{-1}$. Define $\zeta_r = \psi_r\gamma_\nu\psi_r^{-1}$, $\zeta'_r = \psi_r\gamma'_\nu\psi_r^{-1}$, $\tilde\zeta_r = \psi_r\tilde\gamma_\nu\psi_r^{-1}$, and $\tilde\zeta'_r = \psi_r\tilde\gamma'_\nu\psi_r^{-1}$. Recalling that the elements of Γ commute with those of $\tilde\Gamma$, the relations (7.1) imply that $\zeta'_r = \gamma'_\nu\gamma_\nu\gamma'_\mu(\gamma_\nu^{-1}\gamma'_\nu\gamma_\nu){\gamma'_\mu}^{-1}\gamma_\nu^{-1}{\gamma'_\nu}^{-1} = \gamma'_\nu\gamma_\nu(\gamma_\nu^{-1}\gamma'_\nu\gamma_\nu)^{-1}\gamma'_\mu(\gamma_\nu^{-1}\gamma'_\nu\gamma_\nu)\gamma_\nu^{-1}{\gamma'_\nu}^{-1} = \gamma_\nu\gamma'_\mu\gamma_\nu^{-1} = {\gamma'_\mu}^{-1}\gamma_\nu\gamma'_\mu$. Similar calculations for the other elements yield that

(7.2)
$$\zeta_r = {\gamma'_\mu}^{-1}(\gamma_\nu^{-1}\gamma'_\nu\gamma_\nu)\gamma'_\mu,\ \tilde\zeta_r = {\tilde\gamma'_\mu}^{-1}(\tilde\gamma_\nu^{-1}\tilde\gamma'_\nu\tilde\gamma_\nu)\tilde\gamma'_\mu,\ \zeta'_r = {\gamma'_\mu}^{-1}\gamma_\nu\gamma'_\mu,\ \tilde\zeta'_r = {\tilde\gamma'_\mu}^{-1}\tilde\gamma_\nu\tilde\gamma'_\mu.$$

Due to the choice of twisting parameters $l_\mu = 1$ and $l_\nu = 0$, $\zeta'_r \in \mathcal{B}_\Gamma$ and $\tilde\zeta'_r \in \mathcal{B}_{\tilde\Gamma}$.

Since the $z_{r,i}$ commute with the $z_{r+1,i}$ in G (they are mapped to disjoint transpositions by θ), and since by assumption $z_{r+1,i} = z_{r+1,1}$ for all i, we have

$$(z_{r,1}^{-1}\dots z_{r,i}^{-1}z_{r+1,1}^{-1}\dots z_{r+1,i-1}^{-1})z_{r+1,i}(z_{r+1,i-1}\dots z_{r+1,1}z_{r,i}\dots z_{r,1}) = z_{r+1,1}$$

for all i. Therefore, the relation arising from the vertical tangency τ'_i (Figure 8) at the vertex rq can be written in the form

$$z_{r+1,1} = \tilde\zeta'_r\zeta'_r(z_{r,1}^{-1}\dots z_{r,i-1}^{-1})z_{r,i}(z_{r,i-1}\dots z_{r,1}){\zeta'_r}^{-1}{\tilde\zeta'_r}^{-1}.$$

In particular, the value of $(z_{r,1}^{-1}\dots z_{r,i-1}^{-1})z_{r,i}(z_{r,i-1}\dots z_{r,1})$ does not depend on i, which implies that $z_{r,i} = z_{r,1}$ for all i. Moreover, we have $z_{r,i} = {\zeta'_r}^{-1}{\tilde\zeta'_r}^{-1}z_{r+1,1}\tilde\zeta'_r\zeta'_r$. So, by induction on decreasing values of r, we obtain the desired results about $z_{r,i}$. The case of $y_{r,i}$ is handled using exactly the same argument, going inductively through the vertices along the right boundary of the diagram. Indeed, observe that the local braid monodromy at one of these vertices simply differs from that at a vertex along the top boundary by a conjugation which exchanges the positions of two groups of geometric generators ; however, because the corresponding transpositions in S_n are disjoint, these generators commute with each other in G, so that the relations induced by the local braid monodromy can be expressed in exactly the same form. □

Lemma 7.3. *The element $\tilde v_{11}$ belongs to the subgroup of G generated by Γ, $\mathcal{B}_{\tilde\Gamma}$, and $y_{q,1}$.*

Proof. Consider the local relations for the vertex at position $1q$, and more precisely the equality relation corresponding to the half-twist labelled τ''_1 in Figure 8 : with the same notations as in the proof of Lemma 7.2, we have $z_{2,1} = \zeta_1^{-1}\tilde{\zeta}_1^{-1} z_{1,1}\tilde{\zeta}_1\zeta_1$. Moreover, the cusp point with monodromy κ_1^B pictured on Figure 7 yields the relation $\tilde{\zeta}_1 z_{1,1}\tilde{\zeta}_1 = z_{1,1}\tilde{\zeta}_1 z_{1,1}$. It follows that $z_{2,1} = \zeta_1^{-1} z_{1,1}\tilde{\zeta}_1 z_{1,1}^{-1}\zeta_1$. Therefore, using formula (7.2) for $\tilde{\zeta}_1$, we obtain $\tilde{\gamma}'_\nu = \tilde{\gamma}_\nu\tilde{\gamma}'_\mu z_{1,1}^{-1}\zeta_1 z_{2,1}\zeta_1^{-1} z_{1,1}\tilde{\gamma}'^{-1}_\mu\tilde{\gamma}_\nu^{-1}$, where μ and ν are the labels of the two interior edges meeting at the considered vertex.

Observe that, since $l_\nu = 0$ and $l_\mu = 1$, the generators $\tilde{\gamma}_\nu = \tilde{e}_\nu$ and $\tilde{\gamma}'_\mu = \tilde{e}_\mu$ belong to $\mathcal{B}_{\tilde{\Gamma}}$. Moreover, it is obvious from (7.2) that $\zeta_1 \in \Gamma$. Using the result of Lemma 7.2 to express $z_{1,1}$ and $z_{2,1}$ in terms of $y_{q,1}$, it follows that $\tilde{\gamma}'_\nu = \tilde{e}'_\nu$ belongs to the subgroup of G generated by Γ, $\mathcal{B}_{\tilde{\Gamma}}$, and $y_{q,1}$. Therefore, $\tilde{v}_{1,q} = \tilde{e}'_\nu\tilde{e}_\nu^{-1}$ also belongs to this subgroup. Finally, the local relations analogous to (6.5) for the $\tilde{e}_i$ and $\tilde{a}_i$ at the vertex in position $1r$ imply that $\tilde{v}_{1,r}$ and $\tilde{v}_{1,r+1}$ are conjugates of each other under the action of elements of $\mathcal{B}_{\tilde{\Gamma}}$. Therefore, by induction $\tilde{v}_{1,1}$ can be expressed in terms of $\tilde{v}_{1,q}$ and elements of $\mathcal{B}_{\tilde{\Gamma}}$, which completes the proof. □

Lemma 7.4. *The subgroup $\mathcal{B}$ of G generated by $\mathcal{B}_\Gamma$, $\mathcal{B}_{\tilde{\Gamma}}$ and $y_{q,1}$ is naturally a quotient of $\tilde{B}_n$, with geometric generators corresponding to half-twists.*

Proof. We construct a surjective map $\alpha : \tilde{B}_n \to \mathcal{B}$ as follows (recall that $n = 4pq$). First observe that the subgroup of $\tilde{B}_n$ generated by the half-twists $x_1, \dots, x_{2pq-1}$ is naturally isomorphic to $\tilde{B}_{n/2}$, which by Lemma 7.1 admits a surjective homomorphism to $\mathcal{B}_\Gamma$ mapping half-twists to geometric generators. We use this homomorphism to define $\alpha(x_i)$ for $1 \le i \le 2pq - 1$. Any two half-twists in $\tilde{B}_{n/2}$ are conjugate to each other; therefore, after a suitable conjugation we can assume that $\alpha(x_{2pq-1}) = e_\mu$, where μ is the label of the diagonal edge at position pq in the diagram, and that the other $\alpha(x_i)$ ($i \le 2pq - 2$) are geometric generators mapped by θ to transpositions disjoint from $\theta(y_{q,1})$. Because of the stabilization process, this last requirement implies that $\alpha(x_i)$ commutes with $y_{q,1}$ for $i \le 2pq - 2$.

Similarly, the subgroup of $\tilde{B}_n$ generated by $x_{2pq+1}, \dots, x_{n-1}$ is naturally isomorphic to $\tilde{B}_{n/2}$ and admits a surjective homomorphism to $\mathcal{B}_{\tilde{\Gamma}}$, which we use to define $\alpha(x_i)$ for $2pq+1 \le i \le n-1$. Once again, without loss of generality we can assume that $\alpha(x_{2pq+1}) = \tilde{e}_\mu$ and that the other $\alpha(x_i)$ commute with $y_{q,1}$. Finally, we define $\alpha(x_{2pq}) = y_{q,1}$.

All that remains to be checked is that α can be made into a group homomorphism (obviously surjective by construction), i.e. that the relations defining $\tilde{B}_n$ are also satisfied by the chosen images $\alpha(x_i)$ in $\mathcal{B}$. Since α is built out of two group homomorphisms and since the elements of $\mathcal{B}_\Gamma$ commute with those of $\mathcal{B}_{\tilde{\Gamma}}$, the only relations to be checked are those involving x_{2pq}.

Consider the corner vertex at position pq in the diagram: the cusp singularities arising from the regeneration of the rightmost tangent intersection of D_g with $g(C)$ in Figure 9 imply the relations $\gamma'_\mu y_{q,1}\gamma'_\mu = y_{q,1}\gamma'_\mu y_{q,1}$ and $\tilde{\gamma}'_\mu y_{q,1}\tilde{\gamma}'_\mu = y_{q,1}\tilde{\gamma}'_\mu y_{q,1}$. Since $l_\mu = 1$, we have $\gamma'_\mu = e_\mu$ and $\tilde{\gamma}'_\mu = \tilde{e}_\mu$, so that these relations can be rewritten as $\alpha(x_{2pq-1})\alpha(x_{2pq})\alpha(x_{2pq-1}) = \alpha(x_{2pq})\alpha(x_{2pq-1})\alpha(x_{2pq})$ and $\alpha(x_{2pq+1})\alpha(x_{2pq})\alpha(x_{2pq+1}) = \alpha(x_{2pq})\alpha(x_{2pq+1})\alpha(x_{2pq})$. Finally, for all i such that $|i-2pq| \ge 2$, the relation $[\alpha(x_{2pq}), \alpha(x_i)] = 1$ holds by construction. Therefore, α defines a surjective group homomorphism from $\tilde{B}_n$ to $\mathcal{B}$, mapping half-twists to geometric generators. □

Proposition 7.5. *The morphism α extends to a surjective group homomorphism from $\tilde{B}_n^{(2)} \simeq \tilde{B}_n \ltimes \tilde{P}_{n,0}$ to G mapping pairs of half-twists to geometric generators. In particular, the group G has property $(*)$.*

Proof. Lemma 7.2 implies that G is generated by Γ, $\tilde{\Gamma}$, and $y_{q,1}$. Therefore, by Lemma 7.1, G is generated by $\mathcal{B}$, v_{11} and $\tilde{v}_{11}$, while Lemma 7.3 implies that $\tilde{v}_{11}$ can be eliminated from the list of generators. Since Lemma 7.4 identifies $\mathcal{B}$ with a quotient of $\tilde{B}_n$, the main remaining task is to check that the subgroup $\mathcal{P}$ generated by the $g^{-1}v_{11}g$, $g \in \mathcal{B}$, is naturally isomorphic to a quotient of $\tilde{P}_{n,0}$. This can be done by proving that $\mathcal{P}$ is a *primitive $\tilde{B}_n$-group* (Definition 5 of [9]), as it follows from the discussion in §1 of [9] that every such group is a quotient of $\tilde{P}_{n,0}$ (compare Propositions 1, 2, 3 of [9] with the presentation of $\tilde{P}_{n,0}$ given in Lemma 3.1).

As stated in Lemma 7.1, the arguments of [9] show that the subgroup generated by the $g^{-1}v_{11}g$, $g \in \mathcal{B}_\Gamma$, is a primitive $\tilde{B}_{n/2}$-group (and hence a quotient of $\tilde{P}_{n/2,0}$). The desired result about $\mathcal{P}$ then follows simply by observing that v_{11} commutes with $y_{q,1}$ and with the generators of $\mathcal{B}_{\tilde{\Gamma}}$ and using a criterion due to Moishezon (Proposition 6 of [9]); indeed, an obvious corollary of this criterion is that, upon enlarging the conjugation action from $\tilde{B}_{n/2}$ to $\tilde{B}_n$, it is sufficient to check that the additional half-twist generators act trivially on the given prime element (v_{11}).

Since G is obviously generated by its subgroups $\mathcal{B}$ and $\mathcal{P}$, and since $\mathcal{P}$ is normal, it is naturally a quotient of $\tilde{B}_n \ltimes \tilde{P}_{n,0} \simeq \tilde{B}_n^{(2)}$. Moreover, the geometric generators of G are all mutually conjugate (because the curve D is irreducible), and by construction the e_i (and $\tilde{e}_i$) correspond to pairs of half-twists in $\tilde{B}_n^{(2)}$, so the same is true of all geometric generators. Finally, by going carefully over the construction, it is not hard to check that the end points of the half-twists (x, y) corresponding to a given geometric generator γ are always the natural ones, in the sense that $\sigma(x) = \sigma(y) = \theta(\gamma)$. Therefore, G has property $(*)$. □

At this point, the only remaining task in the proof of Theorem 4.6 is to characterize the kernel of the surjective morphism $\alpha : \tilde{B}_n^{(2)} \to G$ given by Proposition 7.5. As a consequence of Lemmas 3.3 and 3.4, the commutation relations induced either by nodes in the branch curve D or by the stabilization process, as well as the relations induced by the cusp points of D, automatically hold, so that $\operatorname{Ker}\alpha$ is generated by equality relations between pairs of half-twists induced by the vertical tangencies of D. Moreover, as in §6.1.2 the classification of half-twists in $\tilde{B}_n$ (Lemma 3.2) allows us to associate to every a_i (resp. $\tilde{a}_i$) a pair of integers $\bar{a}_i$ (resp. $\bar{\tilde{a}}_i$), well-defined modulo the subgroup $\Lambda = \{(\kappa, \lambda),\ (u_1^\kappa \eta^{\kappa(\kappa-1)/2}, u_1^\lambda \eta^{\lambda(\lambda-1)/2}) \in \operatorname{Ker}\alpha\} \subset \mathbb{Z}^2$. Recall however from §6.1.2 that this construction requires us to choose an ordering of the $n = 4pq$ sheets of the branched cover; in our case, these split into two sets of $2pq$ sheets, the first one on which the $\theta(e_i), \theta(e'_i)$ act by permutations, and the second one on which the $\theta(\tilde{e}_i), \theta(\tilde{e}'_i)$ act by permutations. The ordering we will consider is obtained by enumerating first the first set of $2pq$ sheets, and then the second one. In each set, the sheets are naturally in correspondence with the $2pq$ triangles of the diagram in Figure 5: the ordering we choose for each of the two sets of $2pq$ sheets is obtained as in the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$ [9] by enumerating the $2pq$ triangles of the diagram from left to right and from bottom to top.

We have seen above that the relations coming from the vertical tangencies at the inner vertices of the diagram and at those along the lower and left boundaries are exactly the same as in the case of $\mathbb{CP}^1 \times \mathbb{CP}^1$, except they simultaneously apply to the generators of Γ and to those of $\tilde{\Gamma}$. Therefore, as in §6.1.2, these relations do not contribute to $\operatorname{Ker}\alpha$ by themselves, but they translate into equalities between the $\bar{a}_i$ (and similarly between the $\tilde{\bar{a}}_i$), which yield the following formulas (with the obvious notations) : $\bar{d}_{i,j} = \tilde{\bar{d}}_{i,j} = (j-i,0)$, $\bar{v}_{i,j} = \tilde{\bar{v}}_{i,j} = (1-i,1)$, $\bar{h}_{i,j} = \tilde{\bar{h}}_{i,j} = (1-j,1)$ (compare with (6.9)).

Next, we consider the corner vertex at position pq, for which the braid monodromy contribution of the vertical tangencies is represented in Figure 9. Recall that some of the half-twists τ_{ij} were used in the proof of Lemma 7.2 to eliminate $y_{q,2},\dots,y_{q,2a}$ and $z_{p,1},\dots,z_{p,2b}$ from the list of generators by expressing them in terms of $y_{q,1}$; however, since these relations imply that $y_{q,i} = y_{q,1}$ and $z_{p,i} = z_{p,1}$ (cf. Lemma 7.2), all the other relations coming from the τ_{ij} become redundant. Therefore these equality relations do not make any contributions to the kernel of α. We are left with the two half-twists $t,\tilde{t}$ of Figure 9. Denote by μ the label of the diagonal edge passing through the corner vertex. Because G has property $(*)$, and using the results of §3, we can find an element $g\in\tilde{B}^{(2)}_n$ such that $z_{p,1} = \alpha(g^{-1}(x_1,x_1)g)$, $e_\mu = \gamma'_\mu = \alpha(g^{-1}(x_2,x_2)g)$, and $y_{q,1} = \alpha(g^{-1}(x_3,x_3)g)$. Recalling that $\bar{d}_{p,q} = (q-p,0)$ and observing that the conjugation by g preserves the ordering of the end points for e_μ, by definition of $\bar{d}_{p,q}$ we have $e'_\mu = \alpha(g^{-1}(x_2u_2^{p-q}\eta^{(p-q)(p-q-1)/2},x_2)g)$, and therefore $\gamma_\mu = e_\mu^{-1}e'_\mu e_\mu = \alpha(g^{-1}(x_2u_2^{q-p}\eta^{(q-p)(q-p-1)/2},x_2)g)$. The half-twist t yields the relation $\gamma_\mu = z_{p,1}^{2b}y_{q,1}^{2a}\gamma'_\mu y_{q,1}^{-2a}z_{p,1}^{-2b}$; an easy computation shows that the right-hand side of this relation is equal to $\alpha(g^{-1}(x_2u_2^{a-b}\eta^{(a-b)(a-b-1)/2},x_2u_2^{a-b}\eta^{(a-b)(a-b-1)/2})g)$. Comparing the two formulas for γ_μ, we conclude that the relation introduced by the half-twist t is equivalent to the property that $(a-b+p-q,a-b)\in\Lambda$. A similar calculation shows that the relation introduced by $\tilde{t}$ can also be rewritten in the form $(a-b+p-q,a-b)\in\Lambda$.

We now consider the vertex at position rq $(1\le r\le p-1)$, and investigate in the same manner the equality relations coming from the vertical tangencies $\tau'_i,\tau''_i,t,\tilde{t}$ represented in Figure 8. Recall that the relations induced by τ'_i were used in the proof of Lemma 7.2 to show that $z_{r,i} = \zeta'^{-1}_r\tilde{\zeta}'^{-1}_r z_{r+1,1}\tilde{\zeta}'_r\zeta'_r$ and consequently eliminate the $z_{r,i}$ from the list of generators; these relations are therefore already accounted for. Next, we turn to the relation induced by τ''_i, which taking into account that $z_{r,i} = z_{r,1}$ and $z_{r+1,i} = z_{r+1,1}$ can be written in the form $z_{r+1,1} = \zeta_r^{-1}\tilde{\zeta}_r^{-1}z_{r,1}\tilde{\zeta}_r\zeta_r$. Using the expression of $z_{r,1}$ in terms of $z_{r+1,1}$, this identity can also be expressed by the commutation relation $[z_{r+1,1},\tilde{\zeta}'_r\zeta'_r\tilde{\zeta}_r\zeta_r]=1$. By (7.2), we have $\tilde{\zeta}'_r\zeta'_r\tilde{\zeta}_r\zeta_r = e_\mu^{-1}\tilde{e}_\mu^{-1}\tilde{e}'_\nu\tilde{e}_\nu e'_\nu e_\nu\tilde{e}_\mu e_\mu$, where μ and ν are the labels of the two interior edges meeting at position rq. Since $z_{r+1,1}$ commutes with e_μ and $\tilde{e}_\mu$, the relation can then be rewritten as $[z_{r+1,1},\tilde{e}'_\nu\tilde{e}_\nu e'_\nu e_\nu]=1$. Taking into account the ordering of the sheets of the branched cover, an easy calculation in $\tilde{B}^{(2)}_n$ shows that this relation automatically holds as a consequence of the equality $\bar{v}_{r,q} = \tilde{\bar{v}}_{r,q}$.

The relation induced by the half-twist t (Figure 8) can be expressed as $\gamma_\mu = z_{r,1}^{2b}\gamma'_\nu\gamma_\nu\gamma'_\mu\gamma_\nu^{-1}\gamma'^{-1}_\nu z_{r,1}^{-2b}$. Using property $(*)$ and recalling that $\bar{d}_{r,q} = (q-r,0)$ and $\bar{v}_{r,q} = (1-r,1)$, we can find $g\in\tilde{B}^{(2)}_n$, preserving the ordering of the end points for e_μ and e_ν, such that $z_{r,1} = \alpha(g^{-1}(x_1,x_1)g)$, $\gamma'_\mu = e_\mu = \alpha(g^{-1}(x_2,x_2)g)$, $\gamma_\mu = e_\mu^{-1}e'_\mu e_\mu = \alpha(g^{-1}(x_2u_2^{q-r}\eta^{(q-r)(q-r-1)/2},x_2)g)$, $\gamma_\nu = e_\nu = \alpha(g^{-1}(x_3,x_3)g)$, and $\gamma'_\nu = e'_\nu = \alpha(g^{-1}(x_3u_3^{r-1}\eta^{(r-1)(r-2)/2},x_3u_3^{-1}\eta)g)$. So $z_{r,1}^{2b}\gamma'_\nu\gamma_\nu\gamma'_\mu\gamma_\nu^{-1}\gamma'^{-1}_\nu z_{r,1}^{-2b}$

is equal to $\alpha(g^{-1}(x_2 u_2^{2-r-b}\eta^{(2-r-b)(1-r-b)/2}, x_2 u_2^{2-b}\eta^{(2-b)(1-b)/2})g)$. Comparing this with the expression for γ_μ, it becomes apparent that the relation induced by t is in fact equivalent to the condition $(q+b-2, b-2) \in \Lambda$. A similar calculation for the half-twist $\tilde{t}$ shows that the relation it induces can also be expressed in the form $(q+b-2, b-2) \in \Lambda$.

Finally, the case of the vertices along the right boundary of the diagram can be studied by exactly the same argument; the relations corresponding to the vertical tangencies of the local model can be expressed by the single requirement that $(p+a-2, a-2) \in \Lambda$.

Therefore, $\Lambda \subset \mathbb{Z}^2$ is the subgroup generated by $(p+a-2, a-2)$ and $(q+b-2, b-2)$, and $\operatorname{Ker}\alpha$ is the normal subgroup of $\tilde{B}_n^{(2)}$ generated by the two elements $g_1 = (u_1^{p+a-2}\eta^{\lambda(p+a-2)}, u_1^{a-2}\eta^{\lambda(a-2)})$ and $g_2 = (u_1^{q+b-2}\eta^{\lambda(q+b-2)}, u_1^{b-2}\eta^{\lambda(b-2)})$, where $\lambda(i) = i(i-1)/2$. Observe that $G^0_{p,q} = (\tilde{P}_{n,0} \times \tilde{P}_{n,0})/\operatorname{Ker}\alpha$, and recall from Lemma 3.1 that $[\tilde{P}_{n,0}, \tilde{P}_{n,0}] = \{1, \eta\} \simeq \mathbb{Z}_2$ and $\operatorname{Ab}\tilde{P}_{n,0} \simeq \mathbb{Z}^{n-1}$.

We first consider the commutator subgroup $[G^0_{p,q}, G^0_{p,q}] \simeq C/(C \cap \operatorname{Ker}\alpha)$, where $C = \{1,\eta\} \times \{1,\eta\}$. First of all, if $a+p$ is odd, then considering the commutator of g_1 with $(u_2, 1)$ we obtain that $(\eta, 1) \in \operatorname{Ker}\alpha$, and similarly if $b+q$ is odd; otherwise, one easily checks that $(\eta, 1) \notin \operatorname{Ker}\alpha$. Moreover, if a is odd, then considering the commutator of g_1 with $(1, u_2)$ we obtain that $(1, \eta) \in \operatorname{Ker}\alpha$, and similarly if b is odd; when a and b are both even, $(1,\eta) \notin \operatorname{Ker}\alpha$. Also, it is easy to check that $\operatorname{Ker}\alpha$ only contains (η,η) if it also contains $(\eta, 1)$ and $(1,\eta)$. The claim made in the statement of Theorem 4.6 about the structure of $[G^0_{p,q}, G^0_{p,q}]$ follows.

Finally, we have $\operatorname{Ab} G^0_{p,q} \simeq (\tilde{P}_{n,0} \times \tilde{P}_{n,0})/\langle C, \operatorname{Ker}\alpha\rangle \simeq (\mathbb{Z}^2/\Lambda)^{n-1}$. Observing that $\mathbb{Z}^2/\Lambda = \mathbb{Z}^2/\langle (p+a-2, a-2), (q+b-2, b-2)\rangle \simeq \mathbb{Z}^2/\langle (p, a-2), (q, b-2)\rangle$, this completes the proof of Theorem 4.6.

Acknowledgements. The authors wish to thank M. Gromov for prompting their interest in this problem and for useful advice and suggestions. The first and third authors wish to thank respectively Ecole Polytechnique and Imperial College for their hospitality.

References

[1] D. Auroux, *Symplectic maps to projective spaces and symplectic invariants*, Proc. 7th Gökova Geometry-Topology Conference (2000), International Press, 2001, pp. 1–42.

[2] D. Auroux, L. Katzarkov, *Branched coverings of $\mathbb{CP}^2$ and invariants of symplectic 4-manifolds*, Inventiones Math. **142** (2000) 631–673.

[3] D. Auroux, L. Katzarkov, *The degree doubling formula for braid monodromies and Lefschetz pencils*, preprint.

[4] G. Dethloff, S. Orevkov, M. Zaidenberg, *Plane curves with a big fundamental group of the complement*, Voronezh Winter Math. Schools, Amer. Math. Soc. Transl. Ser. 2, **184**, Amer. Math. Soc., 1998, pp. 63–84.

[5] S.K. Donaldson, *Symplectic submanifolds and almost-complex geometry*, J. Diff. Geom. **44** (1996) 666–705.

[6] L. Katzarkov, T. Pantev, *Fundamental groups of complements of curves in $\mathbb{CP}^2$ with wild topology*, preprint.

[7] B. Moishezon, *Stable branch curves and braid monodromies*, Algebraic Geometry (Chicago, 1980), Lecture Notes in Math. **862**, Springer, 1981, pp. 107–192.

[8] B. Moishezon, *Algebraic surfaces and the arithmetic of braids II*, Combinatorial methods in topology and algebraic geometry (Rochester, 1982), Contemp. Math. **44**, Amer. Math. Soc., 1985, pp. 311–344.

[9] B. Moishezon, *On cuspidal branch curves*, J. Algebraic Geom. **2** (1993) 309–384.

[10] B. Moishezon, *Topology of generic polynomial maps in complex dimension two*, preprint.
[11] B. Moishezon, A. Robb, M. Teicher, *On Galois covers of Hirzebruch surfaces*, Math. Annalen **305** (1996) 493–539.
[12] A. Robb, *On branch curves of algebraic surfaces*, Singularities and Complex Geometry (Beijing, 1994), AMS/IP Stud. Adv. Math. **5**, Amer. Math. Soc., 1997, pp. 193–221.
[13] M. Teicher, *Braid groups, algebraic surfaces and fundamental groups of complements of branch curves*, Algebraic Geometry (Santa Cruz, 1995), Proc. Sympos. Pure Math. **62** (part 1), Amer. Math. Soc., 1997, pp. 127–150.
[14] M. Teicher, *New invariants for surfaces*, Tel Aviv Topology Conference: Rothenberg Festschrift (1998), Contemp. Math. **231**, Amer. Math. Soc., 1999, pp. 271–281.
[15] M. Teicher, *The fundamental group of a $\mathbb{CP}^2$ complement of a branch curve as an extension of a solvable group by a symmetric group*, Math. Ann. **314** (1999) 19–38.
[16] M. Teicher, *Hirzebruch surfaces: degenerations, related braid monodromy, Galois covers*, Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), Contemp. Math. **241**, Amer. Math. Soc., 1999, pp. 305–325.
[17] O. Zariski, *On the Poincaré group of rational plane curves*, Amer. J. Math. **58** (1936) 607–619.

Department of Mathematics, M.I.T., Cambridge MA 02139, USA
E-mail address: `auroux@math.mit.edu`

Department of Mathematics, Imperial College, London SW7 2BZ, United Kingdom
E-mail address: `s.donaldson@ic.ac.uk`

Department of Mathematics, University of California, Irvine, CA 92697, USA
E-mail address: `lkatzark@math.uci.edu`

Department of Mathematics, University of California, Irvine, CA 92697, USA
E-mail address: `myotov@math.uci.edu`