FUKAYA CATEGORIES OF SYMMETRIC PRODUCTS AND BORDERED HEEGAARD-FLOER HOMOLOGY

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Abstract. The main goal of this paper is to discuss a symplectic interpretation of Lipshitz, Ozsváth and Thurston's bordered Heegaard-Floer homology [7] in terms of Fukaya categories of symmetric products and Lagrangian correspondences. More specifically, we give a description of the algebra $\mathcal{A}(F)$ which appears in the work of Lipshitz, Ozsváth and Thurston in terms of (partially wrapped) Floer homology for product Lagrangians in the symmetric product, and outline how bordered Heegaard-Floer homology itself can conjecturally be understood in this language.

1. Introduction

Lipshitz, Ozsváth and Thurston's *bordered Heegaard-Floer homology* [7] extends Heegaard-Floer homology to an invariant for 3-manifolds with parametrized boundary. Their construction associates to a (marked and parametrized) surface F a certain algebra $\mathcal{A}(F)$, and to a 3-manifold with boundary F a pair of (A_∞-) modules over $\mathcal{A}(F)$, which satisfy a TQFT-like gluing theorem. On the other hand, recent work of Lekili and Perutz [5] suggests another construction, whereby a 3-manifold with boundary yields an object in (a variant of) the Fukaya category of the symmetric product of F.

1.1. Lagrangian correspondences and Heegaard-Floer homology.

Given a closed 3-manifold Y, the Heegaard-Floer homology group $\widehat{HF}(Y)$ is classically constructed by Ozsváth and Szabó from a Heegaard decomposition by considering the Lagrangian Floer homology of two product tori in the symmetric product of the punctured Heegaard surface. Here is an alternative description of this invariant.

Equip Y with a Morse function (with only one minimum and one maximum, and with distinct critical values). Then the complement Y' of a ball in Y (obtained by deleting a neighborhood of a Morse trajectory from the maximum to the minimum) can be decomposed into a succession of elementary cobordisms Y'_i ($i = 1, \dots, r$) between connected Riemann surfaces with boundary $\Sigma_0, \Sigma_1, \dots, \Sigma_r$ (where $\Sigma_0 = \Sigma_r = D^2$, and the genus increases or decreases by 1 at each step). By a construction of Perutz [10], each Y'_i determines a Lagrangian correspondence $L_i \subset \mathrm{Sym}^{g_{i-1}}(\Sigma_{i-1}) \times \mathrm{Sym}^{g_i}(\Sigma_i)$ between symmetric products. The *quilted Floer homology* of the sequence

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$(L_1, \dots, L_r)$, as defined by Wehrheim and Woodward [16, 17], is then isomorphic to $\widehat{HF}(Y)$. (This relies on two results from the work in progress of Lekili and Perutz [5]: the first one concerns the invariance of this quilted Floer homology under exchanges of critical points, which allows one to reduce to the case where the genus first increases from 0 to g then decreases back to 0; the second one states that the composition of the Lagrangian correspondences from $\mathrm{Sym}^0(D^2)$ to $\mathrm{Sym}^g(\Sigma_g)$ is then Hamiltonian isotopic to the product torus considered by Ozsváth and Szabó.)

Given a 3-manifold Y with boundary $\partial Y \simeq F \cup_{S^1} D^2$ (where F is a connected genus g surface with one boundary component), we can similarly view Y as a succession of elementary cobordisms (from D^2 to F), and hence associate to it a sequence of Lagrangian correspondences $(L_1, \dots, L_r)$. This defines an object $\mathbb{T}_Y$ of the *extended Fukaya category* $\mathcal{F}^\sharp(\mathrm{Sym}^g(F))$, as defined by Ma'u, Wehrheim and Woodward [9] (see [16, 17] for the cohomology level version).

More generally, we can consider a cobordism between two connected surfaces F_1 and F_2 (each with one boundary component), i.e., a 3-manifold Y_{12} with connected boundary, together with a decomposition $\partial Y_{12} \simeq -F_1 \cup_{S^1} F_2$. The same construction associates to such Y a generalized Lagrangian correspondence (i.e., a sequence of correspondences) from $\mathrm{Sym}^{k_1}(F_1)$ to $\mathrm{Sym}^{k_2}(F_2)$, whenever $k_2 - k_1 = g(F_2) - g(F_1)$; by Ma'u, Wehrheim and Woodward's formalism, such a correspondence defines an A_∞-functor from $\mathcal{F}^\sharp(\mathrm{Sym}^{k_1}(F_1))$ to $\mathcal{F}^\sharp(\mathrm{Sym}^{k_2}(F_2))$.

To summarize, this suggests that we should associate:

- to a genus g surface F (with one boundary), the collection of extended Fukaya categories of its symmetric products, $\mathcal{F}^\sharp(\mathrm{Sym}^k(F))$ for $0 \le k \le 2g$;
- to a 3-manifold Y with boundary $\partial Y \simeq F \cup_{S^1} D^2$, an object of $\mathcal{F}^\sharp(\mathrm{Sym}^g(F))$ (namely, the generalized Lagrangian $\mathbb{T}_Y$);
- to a cobordism Y_{12} with boundary $\partial Y_{12} \simeq -F_1 \cup_{S^1} F_2$, a collection of A_∞-functors from $\mathcal{F}^\sharp(\mathrm{Sym}^{k_1}(F_1))$ to $\mathcal{F}^\sharp(\mathrm{Sym}^{k_2}(F_2))$.

These objects behave naturally under gluing: for example, if a closed 3-manifold decomposes as $Y = Y_1 \cup_{F \cup D^2} Y_2$, where $\partial Y_1 = F \cup D^2 = -\partial Y_2$, then we have a quasi-isomorphism

$$\hom_{\mathcal{F}^\sharp(\mathrm{Sym}^g(F))}(\mathbb{T}_{Y_1}, \mathbb{T}_{-Y_2}) \simeq \widehat{CF}(Y). \tag{1.1}$$

Our main goal is to relate this construction to bordered Heegaard-Floer homology. More precisely, our main results concern the relation between the algebra $\mathcal{A}(F)$ introduced in [7] and the Fukaya category of $\mathrm{Sym}^g(F)$. For 3-manifolds with boundary, we also propose (without complete proofs) a dictionary between the A_∞-module $\widehat{CFA}(Y)$ of [7] and the generalized Lagrangian submanifold $\mathbb{T}_Y$ introduced above.

Remark. The cautious reader should be aware of the following issue concerning the choice of a symplectic form on $\mathrm{Sym}^g(F)$. We can equip F with an exact area form, and choose exact Lagrangian representatives of all the simple closed curves that appear in Heegaard diagrams. By Corollary 7.2 in [11], the symmetric product $\mathrm{Sym}^g(F)$ carries an exact Kähler form for which the relevant product tori are exact Lagrangian. Accordingly, a sizeable portion of this paper, namely all the results which do not involve correspondences, can be understood in the exact setting. However, Perutz's construction of Lagrangian correspondences requires the Kähler form to be deformed by a negative multiple of the first Chern class (cf. Theorem A of [10]). Bubbling is not an issue in any case, because the symmetric product of F does not contain any closed holomorphic curves (also, we can arrange for all Lagrangian submanifolds and correspondences to be *balanced* and in particular monotone). Still, we will occasionally need to ensure that our results hold for the perturbed Kähler form on $\mathrm{Sym}^g(F)$ and not just in the exact case.

1.2. **Fukaya categories of symmetric products.** Let Σ be a double cover of the complex plane branched at n points. In Section 2, we describe the symmetric product $\mathrm{Sym}^k(\Sigma)$ as the total space of a Lefschetz fibration $f_{n,k}$, for any integer $k \in \{1, \dots, n\}$. The fibration $f_{n,k}$ has $\binom{n}{k}$ critical points, and the Lefschetz thimbles D_s ($s \subseteq \{1, \dots, n\}$, $|s| = k$) can be understood explicitly as products of arcs on Σ.

For the purposes of understanding bordered Heegaard-Floer homology, it is natural to apply these considerations to the case of the once punctured genus g surface F, viewed as a double cover of the complex plane branched at $2g+1$ points. However, the algebra $\mathcal{A}(F, k)$ considered by Lipshitz, Ozsváth and Thurston only has $\binom{2g}{k}$ primitive idempotents [7], whereas our Lefschetz fibration has $\binom{2g+1}{k}$ critical points.

In Section 3, we consider a somewhat easier case, namely that of a twice punctured genus $g-1$ surface F', viewed as a double cover of the complex plane branched at $2g$ points. We also introduce a subalgebra $\mathcal{A}_{1/2}(F', k)$ of $\mathcal{A}(F, k)$, consisting of collections of Reeb chords on a matched *pair* of pointed circles, and show that it has a natural interpretation in terms of the Fukaya category of the Lefschetz fibration $f_{2g,k}$ as defined by Seidel [14, 15]:

Theorem 1.1. *$\mathcal{A}_{1/2}(F', k)$ is isomorphic to the endomorphism algebra of the exceptional collection $\{D_s,\ s \subseteq \{1, \dots, 2g\},\ |s| = k\}$ in the Fukaya category $\mathcal{F}(f_{2g,k})$.*

By work of Seidel [15], the thimbles D_s generate the Fukaya category $\mathcal{F}(f_{2g,k})$; hence we obtain a derived equivalence between $\mathcal{A}_{1/2}(F', k)$ and $\mathcal{F}(f_{2g,k})$.

Next, in Section 4 we turn to the case of the genus g surface F, which we now regard as a surface with boundary, and associate a *partially wrapped* Fukaya category $\mathcal{F}_z$ to the pair $(\mathrm{Sym}^k(F), \{z\} \times \mathrm{Sym}^{k-1}(F))$ where z is a marked point on the boundary of F (see Definition 4.4). Viewing F' as a subsurface of F, we specifically consider the same collection of $\binom{2g}{k}$ product Lagrangians D_s, $s \subseteq \{1, \dots, 2g\}$, $|s| = k$ as in Theorem 1.1. Then we have:

Theorem 1.2. $\mathcal{A}(F, k) \simeq \bigoplus_{s,s'} \hom_{\mathcal{F}_z}(D_s, D_{s'})$.

As we will explain in Section 4.4, a similar result also holds when the algebra $\mathcal{A}(F,k)$ is defined using a different matching than the one used throughout the paper.

Our next result concerns the structure of the A_∞-category $\mathcal{F}_z$.

"Theorem" 1.3. *The partially wrapped Fukaya category $\mathcal{F}_z$ is generated by the $\binom{2g}{k}$ objects D_s, $s \subseteq \{1,\dots,2g\}$, $|s| = k$. In particular, the natural functor from the category of A_∞-modules over $\mathcal{F}_z$ to that of $\mathcal{A}(F,k)$-modules is an equivalence.*

Moreover, the same result still holds if we enlarge the category $\mathcal{F}_z$ to include compact closed "generalized Lagrangians" (i.e., sequences of Lagrangian correspondences) of the sort that arose in the previous section.

As we will see in Section 5, this result uses the existence of a "partial wrapping" A_∞-functor from the Fukaya category of $f_{2g+1,k}$ to $\mathcal{F}_z$, and requires a detailed understanding of the relations between various flavors of Fukaya categories. While the proof seems to be within reach of standard techniques, it would require a lengthy technical discussion which is beyond the scope of this paper; in this sense "Theorem" 1.3 is not quite a theorem.

1.3. **Yoneda embedding and $\widehat{CFA}$.** Let Y be a 3-manifold with parameterized boundary $\partial Y \simeq F \cup_{S^1} D^2$. Following [7], the manifold Y can be described by a bordered Heegaard diagram, i.e. a surface Σ of genus $\bar{g} \geq g$ with one boundary component, carrying:

- $\bar{g} - g$ simple closed curves $\alpha_1^c, \dots, \alpha_{\bar{g}-g}^c$, and $2g$ arcs $\alpha_1^a, \dots, \alpha_{2g}^a$;
- $\bar{g}$ simple closed curves $\beta_1, \dots, \beta_{\bar{g}}$;
- a marked point $z \in \partial\Sigma$.

As usual, the β-curves determine a product torus $T_\beta = \beta_1 \times \dots \times \beta_{\bar{g}}$ inside $\mathrm{Sym}^{\bar{g}}(\Sigma)$. As to the closed α-curves, using Perutz's construction they determine a Lagrangian correspondence T_α from $\mathrm{Sym}^g(F)$ to $\mathrm{Sym}^{\bar{g}}(\Sigma)$ (or, equivalently, $\bar{T}_\alpha$ from $\mathrm{Sym}^{\bar{g}}(\Sigma)$ to $\mathrm{Sym}^g(F)$). The object $\mathbb{T}_Y$ of the extended Fukaya category $\mathcal{F}^\sharp(\mathrm{Sym}^g(F))$ introduced in §1.1 is then isomorphic to the formal composition of T_β and $\bar{T}_\alpha$.

There is a contravariant Yoneda-type A_∞-functor $\mathcal{Y}$ from the extended Fukaya category of $\mathrm{Sym}^g(F)$ to the category of right A_∞-modules over $\mathcal{A}(F,g)$. Indeed, $\mathcal{F}^\sharp(\mathrm{Sym}^g(F))$ can be enlarged into a partially wrapped A_∞-category $\mathcal{F}_z^\sharp$ by adding to it the same non-compact objects (products of properly embedded arcs) as in $\mathcal{F}_z$. This allows us to associate to a generalized Lagrangian $\mathbb{L}$ the A_∞-module

$$\mathcal{Y}(\mathbb{L}) = \bigoplus\nolimits_s \hom_{\mathcal{F}_z^\sharp}(\mathbb{L}, D_s),$$

where the module maps are given by products in the partially wrapped Fukaya category. With this understood, the right A_∞-module constructed by Lipshitz, Ozsváth and Thurston [7] is simply the image of $\mathbb{T}_Y$ under the Yoneda functor $\mathcal{Y}$:

"Theorem" 1.4. *$\widehat{CFA}(Y) \simeq \mathcal{Y}(\mathbb{T}_Y)$.*

Since the Lagrangian correspondence T_α maps D_s to

$$T_\alpha(D_s) := \alpha_1^c \times \cdots \times \alpha_{\bar{g}-g}^c \times \prod_{i\in s} \alpha_i^a \subset \mathrm{Sym}^{\bar{g}}(\Sigma),$$

a more down-to-earth formulation of "Theorem" 1.4 is:

$$\widehat{CFA}(Y) \simeq \bigoplus\nolimits_s CF^*(T_\beta, T_\alpha(D_s)).$$

However the module structure is less apparent in this formulation.

Consider now a closed 3-manifold Y which decomposes as the union $Y_1 \cup_{F\cup D^2} Y_2$ of two manifolds with $\partial Y_1 = F \cup_{S^1} D^2 = -\partial Y_2$. Then we have:

"Theorem" 1.5. $\hom_{\mathcal{A}(F,g)\text{-}mod}(\widehat{CFA}(-Y_2), \widehat{CFA}(Y_1))$ *is quasi-isomorphic to* $\widehat{CF}(Y)$

This statement is equivalent to the pairing theorem in [7] via a duality property relating $\widehat{CFA}(-Y_2)$ to $\widehat{CFD}(Y_2)$ which is known to Lipshitz, Ozsváth and Thurston. Thus, it should be viewed not as a new result, but rather as a different insight into the main result in [7] (see also [3] and [8] for recent developments). Observe that the formulation given here does not involve $\widehat{CFD}$; this is advantageous since, even though the two types of modules contain equivalent information, $\widehat{CFA}$ is much more natural from our perspective.

The main ingredients in the proofs of "Theorems" 1.4 and 1.5 are presented in Section 6. Much of the technology on which the arguments rely is still being developed; therefore, full proofs are well beyond the scope of this paper.

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2. A Lefschetz fibration on $\mathrm{Sym}^k(\Sigma)$

Fix an ordered sequence of n real numbers $\theta_1 < \theta_2 < \cdots < \theta_n$, and consider the points $p_j = i\theta_j$ on the imaginary axis in the complex plane. Let Σ be the double cover of $\mathbb{C}$ branched at $p_1, \dots, p_n$: hence Σ is a Riemann surface of genus $\lfloor \frac{n-1}{2} \rfloor$ with one (resp. two) puncture(s) if n is odd (resp. even). We denote by $\pi : \Sigma \to \mathbb{C}$ the covering map, and let $q_j = \pi^{-1}(p_j) \in \Sigma$.

We consider the k-fold symmetric product of the Riemann surface Σ ($1 \le k \le n$), equipped with the product complex structure J, and the holomorphic map $f_{n,k}$: $\mathrm{Sym}^k(\Sigma) \to \mathbb{C}$ defined by $f_{n,k}([z_1, \dots, z_k]) = \pi(z_1) + \cdots + \pi(z_k)$.

Proposition 2.1. *$f_{n,k} : \mathrm{Sym}^k(\Sigma) \to \mathbb{C}$ is a Lefschetz fibration, whose $\binom{n}{k}$ critical points are the tuples consisting of k distinct points in $\{q_1, \dots, q_n\}$.*

Proof. Given $\underline{z} \in \mathrm{Sym}^k(\Sigma)$, denote by $z_1, \dots, z_r$ the distinct elements in the k-tuple $\underline{z}$, and by $k_1, \dots, k_r$ the multiplicities with which they appear. The tangent space $T_{\underline{z}}\mathrm{Sym}^k(\Sigma)$ decomposes into the direct sum of the $T_{[z_i,\dots,z_i]}\mathrm{Sym}^{k_i}(\Sigma)$, and $df_{n,k}(\underline{z})$ splits into the direct sum of the differentials $df_{n,k_i}([z_i, \dots, z_i])$. Thus $\underline{z}$ is a critical point of $f_{n,k}$ if and only if $[z_i, \dots, z_i]$ is a critical point of f_{n,k_i} for each $i \in \{1, \dots, r\}$.

By considering the restriction of f_{n,k_i} to the diagonal stratum, we see that $[z_i, \dots, z_i]$ cannot be a critical point of f_{n,k_i} unless z_i is a critical point of π. Assume now that z_i is a critical point of π, and pick a local complex coordinate w on Σ near z_i, in which $\pi(w) = w^2 + \text{constant}$. Then a neighborhood of $[z_i, \dots, z_i]$ in $\mathrm{Sym}^{k_i}(\Sigma)$ identifies with a neighborhood of the origin in $\mathrm{Sym}^{k_i}(\mathbb{C})$, with coordinates given by the elementary symmetric functions $\sigma_1, \dots, \sigma_{k_i}$. The local model for f_{n,k_i} is then

$$f_{n,k_i}([w_1, \dots, w_{k_i}]) = w_1^2 + \dots + w_{k_i}^2 + \text{constant} = \sigma_1^2 - 2\sigma_2 + \text{constant}.$$

Thus, for $k_i \geq 2$ the point $[z_i, \dots, z_i]$ is never a critical point of f_{n,k_i}. We conclude that the only critical points of $f_{n,k}$ are tuples of distinct critical points of π; moreover these critical points are clearly non-degenerate. □

We denote by $\mathcal{S}_k^n$ the set of all k-element subsets of $\{1, \dots, n\}$, and for $s \in \mathcal{S}_k^n$ we call $\vec{q}_s$ the critical point $\{q_j,\ j \in s\}$ of $f_{n,k}$.

We equip Σ with an area form σ, and equip $\mathrm{Sym}^k(\Sigma)$ with an exact Kähler form ω that coincides with the product Kähler form on Σ^k away from the diagonal strata (see e.g. Corollary 7.2 in [11]). The Kähler form ω defines a symplectic horizontal distribution on the fibration $f_{n,k}$ away from its critical points, given by the symplectic orthogonal to the fibers. Because $f_{n,k}$ is holomorphic, this horizontal distribution is spanned by the gradient vector fields for $\mathrm{Re}\, f_{n,k}$ and $\mathrm{Im}\, f_{n,k}$ with respect to the Kähler metric $g = \omega(\cdot, J\cdot)$.

Given a critical point $\vec{q}_s$ of $f_{n,k}$ and an embedded arc γ in $\mathbb{C}$ connecting $f_{n,k}(\vec{q}_s)$ to infinity, the *Lefschetz thimble* associated to $\vec{q}_s$ and γ is the properly embedded Lagrangian disc consisting of all points in $f_{n,k}^{-1}(\gamma)$ whose parallel transport along γ converges to the critical point $\vec{q}_s$ [14, 15]. In our case, we take γ to be the straight line $\gamma(\theta_s) = \mathbb{R}_{\geq 0} + i\theta_s$, where $\theta_s = \mathrm{Im}\, f_{n,k}(\vec{q}_s) = \sum_{j \in s} \theta_j$, and we denote by $D_s \subset \mathrm{Sym}^k(\Sigma)$ the corresponding Lefschetz thimble.

The thimbles D_s have a simple description in terms of the disjoint properly embedded arcs $\alpha_j = \pi^{-1}(\mathbb{R}_{\geq 0} + i\theta_j) \subset \Sigma$. Namely:

Lemma 2.2. $D_s = \prod_{j \in s} \alpha_j$.

Proof. Since γ_s is parallel to the real axis, parallel transport is given by the gradient flow of $\mathrm{Re}\, f_{n,k}$ with respect to the Kähler metric g. Away from the diagonal strata, g is

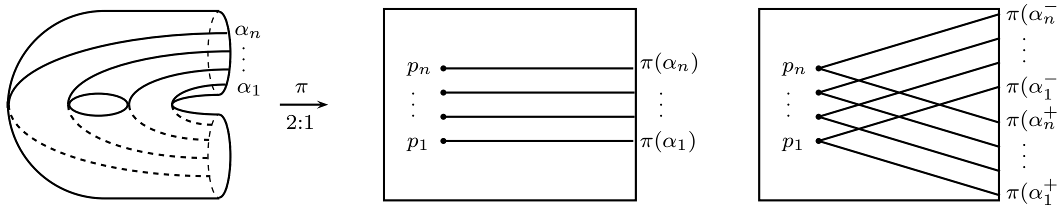


FIGURE 1. The arcs α_j and $\alpha_j^\pm$

a product metric, and so the components of the gradient vector of $\mathrm{Re}\, f_{n,k}$ at $[z_1, \dots, z_k]$ are $\nabla \mathrm{Re}\, \pi(z_1), \dots, \nabla \mathrm{Re}\, \pi(z_k)$. Thus parallel transport along γ_s decomposes into the product of the parallel transports along the arcs $\mathbb{R}_{\geq 0} + i\theta_j$. $\square$

In the subsequent discussion, we will also need to consider perturbed versions of the thimbles D_s. Fix a positive real number ϵ. Given $\theta \in \mathbb{R}$, we consider the arc $\gamma^\pm(\theta) = \{i\theta + (1 \mp i\epsilon)t,\ t \geq 0\}$ in the complex plane, connecting $i\theta$ to infinity. For $s \in \mathcal{S}_k^n$ we denote by $D_s^\pm \subset \mathrm{Sym}^k(\Sigma)$ the thimble associated to the arc $\gamma^\pm(\theta_s)$, and for $j \in \{1, \dots, n\}$ we set $\alpha_j^\pm = \pi^{-1}(\gamma^\pm(\theta_j)) \subset \Sigma$ (see Figure 1). The same argument as above then gives:

Lemma 2.3. $D_s^\pm = \prod_{j \in s} \alpha_j^\pm$.

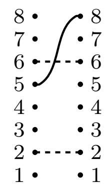
3. The algebra $\mathcal{A}_{1/2}(F', k)$ and the Fukaya category of $f_{2g,k}$

3.1. **The algebra $\mathcal{A}_{1/2}(F', k)$.** We start by briefly recalling the definition of the differential algebra $\mathcal{A}(F, k)$ associated to a genus g surface F with one boundary; the reader is referred to [7, §3] for details. Consider $4g$ points $a_1, \dots, a_{4g}$ along an oriented segment (thought of as the complement of a marked point in an oriented circle), carrying the labels $1, \dots, 2g, 1, \dots, 2g$ (we fix this specific matching throughout). The generators of $\mathcal{A}(F, k)$ are unordered k-tuples consisting of two types of items:

- ordered pairs (i, j) with $1 \leq i < j \leq 4g$, corresponding to Reeb chords connecting pairs of points on the marked circle; in the notation of [7] these are denoted by a column $\left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right]$, or graphically by an upwards strand connecting the i-th point to the j-th point;
- unordered pairs $\{i, j\}$ such that a_i and a_j carry the same label (i.e., in our case, i and j differ by $2g$), denoted by a column $\left[\, i \,\right]$, or graphically by two horizontal dotted lines.

The k source labels (i.e., the labels of the initial points) are moreover required to be all distinct, and similarly for the k target labels. We will think of $\mathcal{A}(F, k)$ as a finite category with objects indexed by k-element subsets of $\{1, \dots, 2g\}$, where, given $s, t \in \mathcal{S}_k := \mathcal{S}_k^{2g}$, $\hom(s, t)$ is the linear span of the generators with source labels the elements of s and target labels the elements of t. For instance, taking $g = k = 2$, the generator

$$\left[\begin{smallmatrix} 5 & 2 \\ 8 & \end{smallmatrix}\right] =$$



is viewed as a morphism from $\{1,2\}$ to $\{2,4\}$.

Composition in $\mathcal{A}(F,k)$ is given by concatenation of strand diagrams, provided that no two strands of the concatenated diagram cross more than once; otherwise the product is zero [7]. (Of course, the product also vanishes if the target and source labels fail to match up). The primitive idempotents of $\mathcal{A}(F,k)$ correspond to diagrams consisting only of dotted lines, which are the identity endomorphisms of the various objects. Finally, the differential in $\mathcal{A}(F,k)$ is described graphically as the sum of all the ways of resolving one crossing of the strand diagram (again excluding resolutions in which two strands intersect twice). In these operations, a pair of dotted lines should be treated as the sum of the corresponding arcs. For example,

$$\partial \left[\begin{smallmatrix} 5 & 2 \\ 8 & \end{smallmatrix}\right] = \left[\begin{smallmatrix} 5 & 6 \\ 6 & 8 \end{smallmatrix}\right]. \tag{3.1}$$

Definition 3.1. *We define $\mathcal{A}_{1/2}(F',k)$ to be the subalgebra of $\mathcal{A}(F,k)$ generated by the strand diagrams for which no strand crosses the interval* $[2g, 2g+1]$.

(This definition makes sense, as $\mathcal{A}_{1/2}(F',k)$ is clearly closed under both the differential and the product of $\mathcal{A}(F,k)$.)

Remark 3.2. It is useful to think of $\mathcal{A}_{1/2}(F',k)$ as the algebra associated to a *pair* of pointed circles, one of them carrying the $2g$ points $a_1, \dots, a_{2g}$ while the other carries $a_{2g+1}, \dots, a_{4g}$; in addition, each of the two circles is equipped with a marked point through which Reeb chords are not allowed to pass. Connecting two annuli by $2g$ bands in the manner prescribed by the labels and further attaching a pair of discs yields a twice punctured genus $g-1$ surface, which we denote by F'; as we will see in the rest of this section, the algebra $\mathcal{A}_{1/2}(F',k)$ can be understood in terms of the symplectic geometry of this surface and its symmetric products.

The algebra $\mathcal{A}_{1/2}(F',k)$ is significantly smaller than $\mathcal{A}(F,k)$: for instance, every object of $\mathcal{A}_{1/2}(F',k)$ is exceptional, i.e. $\hom(s,s) = \mathbb{Z}_2 \,\mathrm{id}_s$, while there are many more endomorphisms in $\mathcal{A}(F,k)$. Another feature distinguishing $\mathcal{A}_{1/2}(F',k)$ from $\mathcal{A}(F,k)$ is directedness. In fact, as will be clear from the rest of this paper, the relation between $\mathcal{A}_{1/2}(F',k)$ and $\mathcal{A}(F,k)$ is analogous to that between the directed Fukaya category of a Lefschetz fibration and a partially wrapped counterpart.

3.2. **The Fukaya category of $f_{n,k}$.** The Fukaya category of the Lefschetz fibration $f_{n,k}$ is a variant of the Fukaya category of $\mathrm{Sym}^k(\Sigma)$ which allows potentially non-compact Lagrangian submanifolds as long as they are *admissible*, i.e. invariant under the gradient flow of $\mathrm{Re}\, f_{n,k}$ outside of a compact subset. While the construction finds its roots in ideas of Kontsevich about homological mirror symmetry for Fano varieties, it has been most extensively studied by Seidel; see in particular [14, 15]. In order to make intersection theory for admissible non-compact Lagrangians well-defined, one needs to choose Hamiltonian perturbations that behave in a consistent manner near infinity. The description we give here is slightly different from that in Seidel's work, but can easily be checked to be equivalent; it is also closely related to the viewpoint given by Abouzaid in Section 2 of [1], except we place the base point at infinity.

Given a real number ν, we say that an exact Lagrangian submanifold L of $\mathrm{Sym}^k(\Sigma)$ is *admissible with slope* $\nu = \nu(L)$ if the restriction of $f_{n,k}$ to L is proper and, outside of a compact set, takes values in the half-line $i\theta + (1 + i\nu)\mathbb{R}_+$ for some $\theta \in \mathbb{R}$. A pair of admissible exact Lagrangians (L_1, L_2) is said to be *positive* if their slopes satisfy $\nu(L_1) > \nu(L_2)$.

Given two admissible Lagrangians L_1 and L_2, we can always deform them by Hamiltonian isotopies (among admissible Lagrangians) to a positive pair $(\tilde{L}_1, \tilde{L}_2)$. We define $\hom_{\mathcal{F}(f_{n,k})}(L_1, L_2) = CF^*(\tilde{L}_1, \tilde{L}_2)$, the Floer complex of the pair $(\tilde{L}_1, \tilde{L}_2)$, equipped with the Floer differential. Positivity ensures that the intersections of $\tilde{L}_1$ and $\tilde{L}_2$ remain in a bounded subset, and the maximum principle applied to $\mathrm{Re}\, f_{n,k}$ prevents sequences of holomorphic discs from escaping to infinity. Moreover, the Floer cohomology defined in this manner does not depend on the chosen Hamiltonian isotopies. The composition $\hom_{\mathcal{F}(f_{n,k})}(L_1, L_2) \otimes \hom_{\mathcal{F}(f_{n,k})}(L_2, L_3) \to \hom_{\mathcal{F}(f_{n,k})}(L_1, L_3)$ is similarly defined using the pair-of-pants product in Floer theory, after replacing each L_i by a Hamiltonian isotopic admissible Lagrangian $\tilde{L}_i$ in such a way that the pairs $(\tilde{L}_1, \tilde{L}_2)$ and $(\tilde{L}_2, \tilde{L}_3)$ are both positive; likewise for the higher compositions.

In order for this construction to be well-defined at the chain level, in general one needs to specify a procedure for perturbing Lagrangians towards positive position. If one considers a collection of Lefschetz thimbles as will be the case here, then there is a natural choice, for which the morphisms and A_∞ operations can be described in terms of Floer theory for the vanishing cycles inside the fiber of $f_{n,k}$ [14, 15]. (This dimensional reduction is one of the key features that make Seidel's construction computationally powerful; however, in the present case it is more efficient to consider the thimbles rather than the vanishing cycles).

Remark 3.3. We will work over $\mathbb{Z}_2$ coefficients to avoid getting into sign considerations, and to match with the construction in [7]; however, the Lefschetz thimbles D_s are contractible and hence carry canonical spin structures, which can be used to orient all the moduli spaces. Keeping track of orientations should give a procedure for defining the algebras $\mathcal{A}_{1/2}(F', k)$ and $\mathcal{A}(F, k)$ over $\mathbb{Z}$.

3.3. **Proof of Theorem 1.1.** We now specialize to the case $n = 2g$, and consider a twice punctured genus $g-1$ surface F' (viewed as a double cover of $\mathbb{C}$ branched at $2g$ points), and the Lefschetz fibration $f_{2g,k} : \mathrm{Sym}^k(F') \to \mathbb{C}$. Consider two k-element subsets $s, t \in \mathcal{S}_k = \mathcal{S}_k^{2g}$, and the thimbles $D_s, D_t \subset \mathrm{Sym}^k(F')$ defined in Section 2. Positivity can be achieved in a number of manners, e.g. we may consider any of the pairs (D_s^-, D_t^+), (D_s, D_t^+), or (D_s^-, D_t). We pick the first possibility. By Lemma 2.3,

$$D_s^- \cap D_t^+ = \Big(\prod_{i\in s} \alpha_i^-\Big) \cap \Big(\prod_{j\in t} \alpha_j^+\Big).$$

Proposition 3.4. *The chain complexes* $\hom_{\mathcal{F}(f_{2g,k})}(D_s, D_t)$ *and* $\hom_{\mathcal{A}_{1/2}(F',k)}(s,t)$ *are isomorphic.*

Proof. The intersections of D_s^- with D_t^+ consist of k-tuples of intersections between the arcs α_i^-, $i \in s$ and α_j^+, $j \in t$. These can be determined by looking at Figure 1. Namely, $\alpha_i^- \cap \alpha_j^+$ is empty if $i > j$, a single point (the branch point q_i) if $i = j$, and a pair of points if $i < j$. The preimage $\pi^{-1}(\{\mathrm{Re}\, z > 0\})$ consists of two distinct components, which we call V and V'; then for $i < j$ we call $q_{i^-j^+}$ (resp. $q'_{i^-j^+}$) the point of $\alpha_i^- \cap \alpha_j^+$ which lies in V (resp. V').

The dictionary between intersection points and generators of $\hom(s,t)$ is as follows:

- the point q_i corresponds to the column $\left[\,{}^{i}\,\right]$;
- the point $q_{i^-j^+}$ corresponds to the column $\left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right]$;
- the point $q'_{i^-j^+}$ corresponds to the column $\left[\begin{smallmatrix} 2g+i \\ 2g+j \end{smallmatrix}\right]$.

In both cases, we consider k-tuples of such items with the property that the labels in s and t each appear exactly once; thus we have a bijection between the generators of $\hom_{\mathcal{F}(f_{2g,k})}(D_s, D_t)$ and those of $\hom_{\mathcal{A}_{1/2}(F',k)}(s,t)$.

Next, we consider the Floer differential on $\hom_{\mathcal{F}(f_{2g,k})}(D_s, D_t) = CF^*(D_s^-, D_t^+)$. Since the thimbles $D_s^- = \prod_{i\in s} \alpha_i^-$ and $D_t^+ = \prod_{j\in t} \alpha_j^+$ are products of arcs in F', results from Heegaard-Floer theory can be used in this setting. The key observation is that the arcs α_i^- and α_j^+ form a *nice* diagram on F', in the sense that the bounded regions of F' delimited by the arcs α_i^- and α_j^+ are all rectangles (namely, the preimages of the bounded regions depicted on Figure 1 right). As observed by Sarkar and Wang, this implies that the Floer differential on $CF^*(D_s^-, D_t^+)$ counts empty embedded rectangles [13, Theorems 3.3 and 3.4].

(Recall that an *embedded rectangle* connecting $x \in D_s^- \cap D_t^+$ to $y \in D_s^- \cap D_t^+$ is an embedded rectangular domain R in the Riemann surface F', satisfying a local convexity condition, and with boundary on the arcs that make up the product Lagrangians D_s^- and D_t^+; the two corners where the boundary of R jumps from some α_i^- to some α_j^+ are two of the components of the k-tuple x, while the two other corners are components of y. The embedded rectangle R is said to be *empty* if the

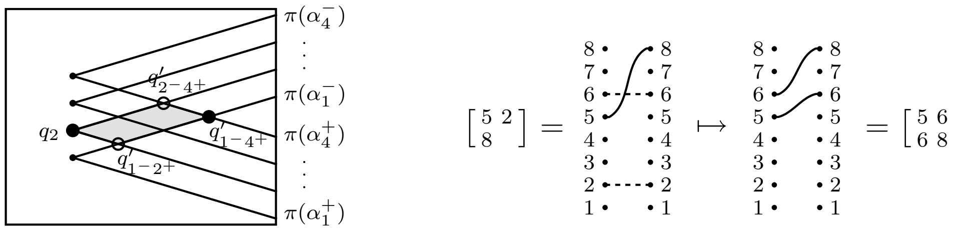


FIGURE 2. An empty rectangle and the corresponding differential.

other intersection points which make up the generators x and y all lie outside of R. Because the Maslov index of holomorphic strips in the symmetric product is given by their intersection number with the diagonal strata, any index 1 strip must project to an empty embedded rectangle in F'. See [13, Section 3].)

The embedded rectangles we need to consider lie either within the closure of V or within that of V'; thus they can be understood by looking at Figure 1 right. If a rectangle in V has its sides on $\alpha_i^-, \alpha_j^-, \alpha_l^+, \alpha_m^+$ ($i < j \le l < m$), then its "incoming" vertices are $q_{i^-m^+}$ and either $q_{j^-l^+}$ (if $j < l$) or q_j (if $j = l$), and its "outgoing" vertices are $q_{i^-l^+}$ and $q_{j^-m^+}$. Via the above dictionary, this corresponds precisely to resolving the crossing between a strand that connects a_i to a_m and a strand that connects a_j to a_l (the latter possibly dotted if $j = l$).

The rectangle bounded by $\alpha_i^-, \alpha_j^-, \alpha_l^+, \alpha_m^+$ in V is empty if and only if the generators under consideration do not include any of the intersection points $q_{v^-w^+}$ (or equivalently, strands connecting a_v to a_w) with $i < v < j$ and $l < w < m$; this forbidden configuration is precisely the case in which resolving the crossing would create a double crossing, which is excluded by the definition of the differential on $\mathcal{A}_{1/2}(F', k)$.

Empty rectangles in V' can be described similarly in terms of resolving crossings between strands that connect pairs of points in $\{a_{2g+1}, \dots, a_{4g}\}$. Thus the differential on $CF^*(D_s^-, D_t^+)$ agrees with that on $\hom_{\mathcal{A}_{1/2}(F',k)}(s, t)$. ☐

To illustrate the above construction, Figure 2 shows the image under π of the empty rectangle (contained in V') which determines (3.1).

Next we need to compare the products in $\mathcal{F}(f_{2g,k})$ and $\mathcal{A}_{1/2}(F', k)$. Given $s, t, u \in \mathcal{S}_k$, the composition $\hom(D_s, D_t) \otimes \hom(D_t, D_u) \to \hom(D_s, D_u)$ in $\mathcal{F}(f_{2g,k})$ is defined in terms of perturbations of the thimbles for which positivity holds: namely, we can consider the Floer pair-of-pants product

$$CF^*(D_s^-, D_t) \otimes CF^*(D_t, D_u^+) \to CF^*(D_s^-, D_u^+).$$

Proposition 3.5. *The isomorphism of Proposition 3.4 intertwines the product structures of $\mathcal{F}(f_{2g,k})$ and $\mathcal{A}_{1/2}(F', k)$.*

Proof. As before, we use the fact that the thimbles D_s^-, D_t and D_u^+ are products of arcs in F'. The image under π of the triple diagram formed by these arcs is depicted

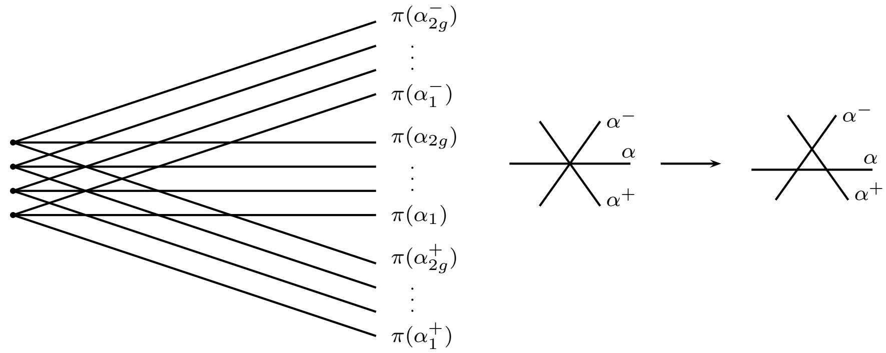


FIGURE 3. The projection of the triple diagram $(F', \alpha_i^-, \alpha_i, \alpha_i^+)$

on Figure 3 for convenience. This diagram has non-generic triple intersections, which can be perturbed as in Figure 3 right.

Pick generators $z \in D_s^- \cap D_t$, $x \in D_t \cap D_u^+$, and $y \in D_s^- \cap D_u^+$ (each viewed as k-tuples of intersections between arcs in the diagram), and consider the homotopy class ϕ of a holomorphic triangle contributing to the coefficient of y in the product $z \cdot x$. Projecting from the symmetric product to F', we can think of ϕ as a 2-chain in F' with boundary on the arcs of the diagram, staying within the bounded regions of the diagram. Then the Maslov index $\mu(\phi)$ and the intersection number $i(\phi)$ of ϕ with the diagonal divisor in $\mathrm{Sym}^k(F')$ are related to each other by the following formula due to Sarkar [12]:

$$\mu(\phi) = i(\phi) + 2e(\phi) - k/2, \tag{3.2}$$

where $e(\phi)$ is the *Euler measure* of the 2-chain ϕ, characterized by additivity and by the property that the Euler measure of an embedded m-gon with convex corners is $1 - \frac{m}{4}$. In our situation, we can draw the perturbed diagram in such a way that all intersections occur at 60-degree and 120-degree angles as in Figure 3 right. The Euler measure of a convex polygonal region of the diagram can then be computed by summing contributions from its vertices, namely $+\frac{1}{12}$ for every vertex with a 60-degree angle, and $-\frac{1}{12}$ for every vertex with a 120-degree angle; using additivity, $e(\phi)$ can be expressed as a sum of local contributions near the intersection points of the diagram covered by the 2-chain ϕ.

View the 2-chain ϕ as the image of a holomorphic map u from a Riemann surface S (with boundaries and strip-like ends) to F' (as in Lipshitz's approach to Heegaard-Floer theory), and fix an intersection point p in the triple diagram. If u hits p at an interior point of S, then the local contributions to the multiplicities of ϕ in the four regions that meet at p are all equal, hence the local contribution to the Euler measure is zero. Likewise, if u hits p at a point on the boundary of S, then (assuming u is unbranched at p) locally the image of u hits two of the four regions that meet at p, one making a 60-degree angle and the other making a 120-degree angle; in any case the local contributions to the Euler measure cancel out. On the other hand, consider a strip-like end of S where u converges to p (i.e. an actual *corner* of the 2-chain ϕ): looking at the local configurations of Figure 3 and remembering the ordering condition on the boundaries of S, we see that locally u maps into a region with a 60-degree angle at the vertex p (unless there is a nearby boundary branch point, in which case u locally maps into two 60-degree regions and one 120-degree region). Thus each corner of ϕ contributes $+\frac{1}{12}$ to the Euler measure. Summing over all $3k$ strip-like ends of S, we deduce that

$$e(\phi) = k/4.$$

The Floer product counts holomorphic discs such that $\mu(\phi) = 0$; by (3.2) these are precisely the discs for which $i(\phi) = 0$, i.e., using positivity of intersections, those which do not intersect the diagonal in $\mathrm{Sym}^k(F')$. Such holomorphic discs in $\mathrm{Sym}^k(F')$ can be viewed as k-tuples of holomorphic discs in F' (i.e., the domain S is a disjoint union of k discs), and the Maslov index for such a product of discs is easily seen to be the sum of the individual Maslov indices. Next, we recall that rigid holomorphic discs on a Riemann surface are immersed polygonal regions with convex corners; i.e., there are no branch points. (This conclusion can also be reached by using equation (6) of [6] which expresses the Maslov index in terms of the Euler measure and the total number of branch points.)

Hence, the conclusion is the same as if our triple diagram had been "nice" in the sense of [6, 12]: the Floer product counts k-tuples of immersed holomorphic triangles in F' such that the corresponding map to $\mathrm{Sym}^k(F')$ does not hit the diagonal.

Moreover, closer inspection of the triple diagram shows that immersed triangles are actually embedded, and are contained either in a small neighborhood $\mathcal{V}$ of V or in a small neighborhood $\mathcal{V}'$ of V'. (Recall that V, V' are the two components of $\pi^{-1}(\{\mathrm{Re}\, z > 0\})$; in the limit where we consider the unperturbed diagram of Figure 3 with triple intersections at the branch points q_i the triangles cannot cross over the branch locus to jump from V to V', hence after perturbation they are contained in a small neighborhood of either V or V'.)

Given a pair of triangles T and T' contained in $\mathcal{V}$, realized as the images of holomorphic maps u, u' from the unit disc with three boundary marked points, the intersection number of the product map (u, u') with the diagonal in $\mathrm{Sym}^2(F')$ can be evaluated by considering the rotation number of the boundaries around each other: namely, embedding $\mathcal{V}$ into $\mathbb{R}^2$, the restriction of $u' - u$ to the unit circle defines a loop in $\mathbb{R}^2 \setminus \{0\}$, whose degree is easily seen to equal the intersection number of (u, u') with the diagonal. One then checks that configurations where T and T' are disjoint or intersect in a triangle ("head-to-tail overlap") lead to an intersection number of 0 and are hence allowed; however, all other configurations, e.g. when T and T' are contained inside each other or intersect in a quadrilateral, lead to an intersection number of 1 and are hence forbidden. Similarly for triangles in $\mathcal{V}'$.

We conclude that the Floer product counts k-tuples of embedded triangles in F' which either are disjoint or overlap head-to-tail (compare [6, Lemma 2.6]).

Recall that α_i^-, α_j and α_l^+ intersect pairwise if and only if $i \leq j \leq l$. In that case, these curves bound exactly two embedded triangles T_{ijl} and T'_{ijl}, the former contained in $\mathcal{V}$ and the latter contained in $\mathcal{V}'$, unless $i = j = l$ in which case there is a single triangle $T_{iii} = T'_{iii}$ obtained by deforming the triple intersection at the branch point p_i (see Figure 3). Under the dictionary introduced in the proof of Proposition 3.4, the triangle T_{ijl} corresponds to the concatenation of strands connecting a_i to a_j and a_j to a_l to obtain a strand connecting a_i to a_l, while T'_{ijl} corresponds to the concatenation of strands connecting a_{2g+i} to a_{2g+j} and a_{2g+l} to a_{2g+l} to obtain a strand connecting a_{2g+i} to a_{2g+l}; the special case $i = j = l$ corresponds to the concatenation of pairs of horizontal dotted lines.

Finally, consider two triangles T_{ijl} and $T_{i'j'l'}$ where $i \leq j \leq l$, $i' \leq j' \leq l'$, and $i < i'$: the concatenation of the strands connecting a_i to a_j and a_j to a_l intersects the concatenation of the strands connecting $a_{i'}$ to $a_{j'}$ and $a_{j'}$ to $a_{l'}$ twice if and only if $j > j'$ and $l < l'$, i.e. the forbidden case is $i < i' \leq j' < j \leq l < l'$. A tedious but straightforward enumeration of cases shows that this is precisely the scenario in which the triangles T_{ijl} and $T_{i'j'l'}$ overlap in a forbidden manner (other than head-to-tail). Thus, the rules defining the product operations in $\mathcal{A}_{1/2}(F', k)$ and $\mathcal{F}(f_{2g,k})$ agree with each other. □

The last ingredient is the following:

Proposition 3.6. *The higher compositions involving the thimbles D_s ($s \in \mathcal{S}_k$) in $\mathcal{F}(f_{2g,k})$ are identically zero.*

Proof. The argument is similar to the first part of the proof of Proposition 3.5. Namely, the ℓ-fold composition m_ℓ is determined by picking $\ell + 1$ different perturbations of the thimbles, and identifying them in the relevant portion of $\mathrm{Sym}^k(F')$ with products of arcs obtained by perturbing the α_i. The resulting diagram generalizes in the obvious manner that of Figure 3 (with $\ell + 1$ sets of $2g$ arcs).

Consider the class ϕ of a holomorphic $(\ell + 1)$-pointed disc in $\mathrm{Sym}^k(F')$ that contributes to m_ℓ: then by Theorem 4.2 of [12] we have

$$\mu(\phi) = i(\phi) + 2e(\phi) - (\ell - 1)k/2.$$

We can calculate the Euler measure as in the proof of Proposition 3.5 by setting up a perturbation of the diagram in which all intersections occur at angles that are multiples of $\pi/(\ell + 1)$, and summing local contributions. (The local contribution of a vertex with angle $r\pi$ to the Euler measure is $\frac{1}{4} - \frac{r}{2}$). The same argument as before shows that each of the $(\ell + 1)k$ corners contributes $\frac{1}{4} - \frac{1}{2(\ell+1)} = \frac{\ell-1}{4(\ell+1)}$ to the Euler measure, so that $e(\phi) = (\ell - 1)k/4$ and $\mu(\phi) = i(\phi) \geq 0$.

On the other hand, m_ℓ counts rigid holomorphic discs, i.e. discs of Maslov index $2-\ell$. The above calculation shows that for $\ell \geq 3$ there are no such discs. □

Theorem 1.1 follows from Propositions 3.4, 3.5 and 3.6.

Remark 3.7. Seidel's definition of the Fukaya category of a Lefschetz fibration [15] is slightly more restrictive than the version we gave in Section 3.2 above, in that the only non-compact Lagrangians he allows are thimbles; the difference between the two versions is not expected to be significant when one passes to twisted complexes, but the cautious reader may wish to impose this additional restriction. With this understood, Theorem 18.24 of [15] implies that the Fukaya category of the Lefschetz fibration $f_{2g,k}$ is generated by the exceptional collection of thimbles $\{D_s,\ s \in \mathcal{S}_k\}$, in the sense that, after passing to twisted complexes, the inclusion of the finite directed subcategory $\mathcal{A}_{1/2}(F', k)$ into $\mathcal{F}(f_{2g,k})$ induces a quasi-equivalence $Tw\mathcal{A}_{1/2}(F', k) \to Tw\mathcal{F}(f_{2g,k})$.

Remark 3.8. In the next sections we will consider the slightly larger surface F and the Lefschetz fibration $f_{2g+1,k} : \mathrm{Sym}^k(F) \to \mathbb{C}$. Assume that the points $p_j = i\theta_j$ have been chosen so that $\theta_1 < \cdots < \theta_{2g} < 0 < \theta_{2g+1}$ and $|\theta_{2g+1}| \gg |\theta_1|$: then the double covers $F \to \mathbb{C}$ and $F' \to \mathbb{C}$ can be identified outside of a neighborhood of the positive imaginary axis. Passing to symmetric products, the Lefschetz fibrations $f_{2g+1,k}$ and $f_{2g,k}$ agree over a large convex open subset $\mathcal{U}$ which includes the $\binom{2g}{k}$ critical points of $f_{2g,k}$ and the corresponding thimbles. In this situation, the Fukaya category $\mathcal{F}(f_{2g.k})$ embeds as a full A_∞-subcategory of $\mathcal{F}(f_{2g+1,k})$, namely the subcategory generated by the thimbles $D_s,\ s \in \mathcal{S}_k$ $(= \mathcal{S}_k^{2g} \subsetneq \mathcal{S}_k^{2g+1})$. Indeed, the Lagrangian submanifolds and holomorphic discs considered above all lie within $\mathcal{U}$ and do not see the difference between $f_{2g,k}$ and $f_{2g+1,k}$. This alternative description of $\mathcal{A}_{1/2}(F', k)$ as a subcategory of $\mathcal{F}(f_{2g+1,k})$ amounts to viewing it as the strands algebra associated to a *twice pointed* matched circle, rather than a pair of pointed circles.

4. Partially wrapped Fukaya categories and the algebra $\mathcal{A}(F, k)$

4.1. **Partially wrapped Fukaya categories.**

The Fukaya category of a Lefschetz fibration, as discussed in Section 3.2, is a particular instance of a more general construction, which also encompasses the so-called wrapped Fukaya category (see [2]). In both cases, the idea is to allow noncompact Lagrangian manifolds with appropriate behavior at infinity, and to define their intersection theory by means of suitable Hamiltonian perturbations which achieve a certain geometric behavior at infinity.

Let (M, ω) be an exact symplectic manifold with contact boundary. Let $\hat{M}$ be the completion of M, i.e. the symplectic manifold obtained by attaching to M the positive part $([1,\infty) \times \partial M, d(r\alpha))$ of the symplectization of ∂M. Let $H : \hat{M} \to \mathbb{R}$ be a Hamiltonian function such that $H \geq 0$ everywhere and $H(r, y) = r$ on $[1,\infty) \times \partial M$.

The objects of the wrapped Fukaya category of M (or $\hat{M}$) are exact Lagrangian submanifolds of $\hat{M}$ with cylindrical ends modelled on Legendrian submanifolds of ∂M. The morphisms are defined by $\hom(L_1, L_2) = \lim_{w\to+\infty} CF^*(\phi_{wH}(L_1), L_2)$, where ϕ_{wH} is the Hamiltonian diffeomorphism generated by wH; in the symplectization, this Hamiltonian isotopy "wraps" L_1 by the time w flow of the Reeb vector field. The differential, composition, and higher products are defined in terms of suitably perturbed versions of the holomorphic curve equation; i.e., they can be understood in terms of holomorphic discs with boundary on increasingly perturbed versions of the Lagrangians. The reader is referred to §3 of [2] for details.

We now consider "partially wrapped" Fukaya categories, tentatively defined in the following manner:

"Definition" 4.1. *Given a smooth function $\rho : \partial M \to [0,1]$, let $H_\rho : \hat{M} \to \mathbb{R}$ be a Hamiltonian function such that $H_\rho \geq 0$ everywhere and $H_\rho(r, y) = \rho(y)\, r$ on the positive symplectization $[1,\infty)\times\partial M$. The objects of the "$\rho$-wrapped" Fukaya category $\mathcal{F}(M, \rho)$ are exact Lagrangian submanifolds of $\hat{M}$ with cylindrical ends modelled on Legendrian submanifolds of $\partial M \setminus \rho^{-1}(0)$, and the morphisms and compositions are defined by perturbing the Lagrangians by the long-time flow generated by H_ρ. Namely,*

$$\hom(L_1, L_2) = \lim_{w\to+\infty} CF^*(\phi_{wH_\rho}(L_1), L_2),$$

and the differential, composition, and higher products are defined as in [2] by counting solutions of the Cauchy-Riemann equations perturbed by the Hamiltonian flow of H_ρ.

At the boundary, the flow generated by H_ρ can be viewed as the Reeb flow for the contact form $\rho^{-1}\alpha$ on the non-compact hypersurface $\{r = \rho^{-1}\} \simeq \partial M \setminus \rho^{-1}(0)$. The effect of this modification is to slow down the wrapping so that the long time flow never quite reaches $\rho^{-1}(0)$.

The direct limit in Definition 4.1 relies on the existence of well-defined continuation maps from $CF^*(\phi_{wH_\rho}(L_1), L_2)$ to $CF^*(\phi_{w'H_\rho}(L_1), L_2)$ for $w' > w$. Even though exactness prevents bubbling and the positivity of H_ρ implies an a priori energy bound on perturbed holomorphic discs, it is not entirely clear that the construction is well-defined in full generality.[1] Here, we will only consider settings in which $\phi_{wH_\rho}(L_1)$ and L_2 are transverse to each other for all sufficiently large w, and in particular no intersections appear or disappear for $w \gg 0$. This simplifies things greatly, as the complex stabilizes for large enough w. The continuation maps can then be constructed by the "homotopy method" (see Appendix A), and turn out to be the obvious ones for w, w' large enough. The product maps can also be defined similarly by counting "cascades" of (unperturbed) holomorphic discs, i.e. trees of rigid holomorphic discs with boundaries on the Lagrangian submanifolds $\phi_{w_i H_\rho}(L_i)$ (where the parameter w_i is sometimes

[1] Ongoing work of Mohammed Abouzaid provides a treatment of the important case where ρ is lifted from an open book on ∂M.

fixed, and sometimes allowed to vary); see Appendix A for details. However, in our case the upshot will be that the complexes, differentials and products behave exactly as if one simply considered sufficiently perturbed copies of the Lagrangians.

Remark 4.2. In many situations (exact Lefschetz fibrations over the disc with convex fibers, symmetric products of Riemann surfaces with boundary, etc.), one is naturally given an exact symplectic manifold with corners; one then needs to "round the corners" to obtain a contact boundary. Concretely, in the case of a product of Stein domains $M_1 \times M_2$, we consider the completed Stein manifolds $(\hat{M}_i, dd^c\varphi_i)$ and equip their product with the plurisubharmonic function $\pi_1^*\varphi_1 + \pi_2^*\varphi_2$, then restrict to a sublevel set to obtain a Stein domain again. More importantly for our purposes, a similar procedure can be used to round the corners of the symmetric product of a Riemann surface with boundary.

The Fukaya category of a Lefschetz fibration over the disc can now be understood as a partially wrapped Fukaya category for a suitably chosen ρ, which vanishes in the direction of the fiberwise boundary (recall that one only considers Lagrangians on which the projection is proper) and also in the fiber above one point of the boundary (or a subinterval of the boundary).

Another property that we expect of partially wrapped Fukaya categories is the existence of "acceleration" A_∞-functors $\mathcal{F}(M,\rho) \to \mathcal{F}(M,\rho')$ whenever $\rho \leq \rho'$ (i.e., from a "less wrapped" Fukaya category to one that is "more wrapped"). Specifically, because $H_\rho \leq H_{\rho'}$ one should have well-defined continuation maps from $CF^*(\phi_{wH_\rho}(L_1), L_2)$ to $CF^*(\phi_{wH_{\rho'}}(L_1), L_2)$, which (taking direct limits) define the linear term of the functor. However, the construction in the general case is well beyond the scope of this paper. In our case, we will consider a very specific setting in which the "less wrapped" Floer complex turns out to be a *subcomplex* of the "more wrapped" one, and the acceleration functor is simply given by the inclusion map.

4.2. **Partially wrapped categories for symmetric products.** Let S be a Riemann surface with boundary, equipped with an exact area form, and fix a point $z \in \partial S$. Then $M = \mathrm{Sym}^k(S)$ is an exact symplectic manifold with corners, and $V = \{z\} \times \mathrm{Sym}^{k-1}(S) \subset \partial M$. As in Remark 4.2, we can complete M to $\hat{M} = \mathrm{Sym}^k(\hat{S})$ where $\hat{S}$ is a punctured Riemann surface obtained by attaching cylindrical ends to S, and use a plurisubharmonic function on $\hat{M}$ to round the corners of M.

Consider a Lagrangian submanifold of $\hat{M}$ of the form $\hat{L} = \hat{\lambda}_1 \times \cdots \times \hat{\lambda}_k$, where $\hat{\lambda}_i$ are disjoint properly embedded arcs in $\hat{S}$ obtained by extending arcs $\lambda_i \subset S$ into the cylindrical ends. We assume that the end points of λ_i lie away from z, so that $\hat{L}$ is tentatively an object of the partially wrapped Fukaya category.

Away from the diagonal strata, the exact symplectic structure on $\hat{M}$ is the product one, and the Hamiltonian H that defines wrapped Floer homology in $\hat{M}$ is just a sum

$H([z_1,\dots,z_k]) = \sum_i h(z_i)$, where h is a Hamiltonian on $\hat{S}$. Thus, wrapping preserves the product structure away from the diagonal: wrapping the product Lagrangian $\hat{L}$ inside the symmetric product $\hat{M}$ is equivalent to wrapping each factor $\hat{\lambda}_i$ inside $\hat{S}$.

Due to the manner in which the smooth structure on the symmetric product $\hat{M} = \mathrm{Sym}^k(\hat{S})$ is defined near the diagonal, it is impossible for a nontrivial smooth Hamiltonian on $\hat{M}$ to preserve the product structure everywhere. Thus, if we wish to preserve the interpretation of holomorphic discs in $\hat{M}$ in terms of holomorphic curves in $\hat{S}$, we cannot perturb the holomorphic curve equations by an inhomogeneous Hamiltonian term. This is one of the key reasons why we choose to set up wrapped Floer theory in the language of cascades: then we consider genuine holomorphic discs (for the product complex structure) with boundary on product Lagrangian submanifolds (recall that H preserves the product structure away from a small neighborhood of the diagonal, and in particular near the Lagrangian submanifolds that we consider).

When we work relatively to $V = \{z\} \times \mathrm{Sym}^{k-1}(S)$, we are "slowing down" the wrapping whenever one of the k components approaches z or, in the completion, the ray $\hat{Z} = \{z\} \times [1,\infty)$ generated by z in the cylindrical end. Observe that $\{0\} \times \mathrm{Sym}^{k-1}(\mathbb{C}) \subset \mathrm{Sym}^k(\mathbb{C})$ is the (transverse) zero set of the k-th elementary symmetric function $\sigma_k(x_1,\dots,x_k) = \prod x_i$. Hence, a natural way to associate a partially wrapped Fukaya category to the pair (M, V) is to use a function ρ which decomposes as a product: $\rho([z_1,\dots,z_k]) = \prod \rho_{S,z}(z_i)$, where $\rho_{S,z} : S \to [0,1]$ is a smooth function that vanishes to order 2 at $z \in \partial S$.

In this situation the wrapping flow no longer preserves the product structure as soon as one of the points z_i gets too close to $\hat{Z}$, even away from the diagonal. So, if we consider two product Lagrangians $\hat{L}$, $\hat{L}'$ which are disjoint from the support of $1-\rho$, the wrapping perturbation applied to $\hat{L}$ only preserves the product structure until $\phi_{wH_\rho}(\hat{L})$ enters the neighborhood of $\hat{Z} \times \mathrm{Sym}^{k-1}(\hat{S})$ where $\rho \neq 1$. While it can be checked that this is not an issue when it comes to the definition of the Floer complexes and differentials, it is not entirely clear at this point that the product operations are well-defined and reduce to calculations in the surface S. Thus, to avoid technical difficulties, we will use a different choice of ρ to construct the A_∞-category $\mathcal{F}_z$.

Let us specialize right away to the case at hand, and consider again the situation where $\hat{S} = \hat{F}$ is a punctured genus g surface, equipped with a double covering map $\pi : \hat{F} \to \mathbb{C}$ with branch points $p_1,\dots,p_{2g+1} \in \mathbb{C}$ (with $\operatorname{Im} p_1 < \dots < \operatorname{Im} p_{2g+1}$), the subsurface $F \subset \hat{F}$ is the preimage of some large disc, say of radius a, and $z \in \partial F$ is one of the two points in $\pi^{-1}(-a)$.

First version. We first equip $\hat{F}$ with a Hamiltonian constructed as follows. Let $a' > 0$ be such that $\max |p_j| \ll a' \ll a$, define $U = \pi^{-1}(D^2(a')) \subset \hat{F}$, and let $\hat{Z} \subset \hat{F}$ be the component of $\pi^{-1}((-\infty, -a'])$ which passes through z. We define

$h_\rho(w) = \chi(w)\,|\pi(w)|^2$, where $\chi : \hat{F} \to [0,1]$ is a smooth function which vanishes on $\hat{Z} \cup U$ and equals 1 everywhere away from $\hat{Z} \cup U$. Note that h_ρ has no critical points outside of $\hat{Z} \cup U$, and it has the right growth rate at infinity for the purposes of constructing a partially wrapped Fukaya category for the pair $(F, \{z\})$.

The long-time flow of X_{h_ρ} acts on properly embedded arcs in $\hat{F}$ in a straightforward manner: the flow is identity inside the subset U, while in the cylindrical end the flow wraps in the positive direction and accumulates onto the ray $\hat{Z}$ (if χ is chosen suitably). To be more specific, we identify $\hat{F} \backslash U$ with a cylinder, with radial coordinate $|\pi(\cdot)|^2$ and angular coordinate $\vartheta = \frac{1}{2} \arg \pi(\cdot)$ (with, say, $\vartheta = \pi/2$ at z to fix things). The level sets of h_ρ are asymptotic (from both sides) to the ray $\hat{Z}$, where $\vartheta = \pi/2$; thus the wrapping by the positive (resp. negative) time flow generated by h_ρ moves any point outside $\hat{Z} \cup U$ towards infinity, with ϑ increasing (resp. decreasing) towards $\pi/2$.

In particular, the positive (resp. negative) time flow of h_ρ maps the arcs $\alpha_j = \pi^{-1}(p_j + \mathbb{R}_{\geq 0}) \subset \hat{F}$ to arcs which, after a compactly supported isotopy, look like the arcs $\tilde{\alpha}_j^-$ (resp. $\tilde{\alpha}_j^+$) pictured in Figure 4 below (the last arc α_{2g+1} is not pictured, but behaves in a similar manner). Note however that, due to the degeneracy of h_ρ inside U, the arcs $\phi_{\pm w h_\rho}(\alpha_i)$ are never transverse to each other: without further perturbation, the flow of h_ρ only yields an A_∞-precategory, i.e. morphisms and compositions are only defined for objects which are mutually transverse within U (and in particular, endomorphisms are not well-defined). In order to construct an honest A_∞-category one needs to choose further (compactly supported) Hamiltonian perturbations in a consistent manner; see below.

With h_ρ at hand, we equip $\hat{M} = \mathrm{Sym}^k(\hat{F})$ with a Hamiltonian H_ρ such that, outside of a small neighborhood of the diagonal strata, $H_\rho([z_1, \dots, z_k]) = \sum_i h_\rho(z_i)$. In particular, the Hamiltonian flow generated by H_ρ preserves the product structure away from a small neighborhood of the diagonal. Thus, given k disjoint embedded arcs $\hat{\lambda}_1, \dots, \hat{\lambda}_k \subset \hat{F}$, for suitable values of w the flow maps $\hat{L} = \hat{\lambda}_1 \times \cdots \times \hat{\lambda}_k$ to

$$\phi_{wH_\rho}(\hat{L}) = \phi_{wh_\rho}(\hat{\lambda}_1) \times \cdots \times \phi_{wh_\rho}(\hat{\lambda}_k). \tag{4.1}$$

Remark 4.3. Due to the specifics of the construction, for large w the image under ϕ_{wH_ρ} of a product of disjoint arcs does approach the diagonal, where the product structure is not preserved by the flow; we will want to correct this and ensure that (4.1) holds for all w. There are several ways to proceed. A first option would be to modify the definition of h_ρ appropriately in order to control the manner in which things can accumulate towards the diagonal; this comes at the expense of making h_ρ non-constant over U, which complicates the geometric behavior of the flow. A second possibility, suggested by the referee, is to let the Hamiltonian H_ρ be singular along the diagonal. This is valid because in our technical setup the Hamiltonian is never used to perturb the Cauchy-Riemann equation (see Appendix A); instead, we consider honest holomorphic curves with boundary on the images of the Lagrangians under the flow, and these remain smooth for Lagrangians which do not intersect the diagonal. One would also need to make the Kähler form singular along the diagonal, which is actually not a problem in our case. A third approach, strictly equivalent to the previous one and which we will use instead, is to allow the choice of H_ρ near the diagonal to depend on the product Lagrangian $\hat{L}$ under consideration; it is then not hard to ensure that (4.1) holds for all w.

Hamiltonian perturbations. One way to address the degeneracy of h_ρ would be to replace it by a non-degenerate Hamiltonian; however, this affects the long-term dynamics inside U in a counter-intuitive manner. Another approach is to keep using a degenerate Hamiltonian, but further add small compactly supported Hamiltonian perturbations in order to achieve transversality. This is conceptually similar to the approach taken by Seidel in [15], except we again consider cascades of honest holomorphic curves with boundaries on perturbed Lagrangian submanifolds, rather than perturbing the holomorphic curve equation.

Concretely, for each pair of Lagrangians (L_1, L_2), we choose a family of Hamiltonians $\{H'_{L_1,L_2,\tau}\}_{\tau\geq 0}$, uniformly bounded, depending smoothly on τ, and with $H'_{L_1,L_2,0} = 0$, with the property that $\phi_{wH_\rho+H'_{L_1,L_2,w}}(L_1)$ is transverse to L_2 for all sufficiently large w. We then define

$$\hom(L_1, L_2) = \lim_{w\to+\infty} CF^*(\phi_{wH_\rho+H'_{L_1,L_2,w}}(L_1), L_2).$$

The definition of product structures requires additional transversality properties, and the choice of suitable homotopies between the Hamiltonian perturbations; these are incorporated into the definition of the A_∞-operations via cascades. The details can be found in §A.3 where, for simplicity, we only describe the construction in the case where the perturbation $H'_{L_1,L_2,w} = H'_{L_1,w}$ is chosen to depend only on L_1 and w, not on L_2. This assumption makes the construction much simpler, but prevents us from achieving transversality for arbitrary pairs of Lagrangians.

In our case, we will essentially be able to use small multiples of a same Hamiltonian perturbation H' for all the thimbles D_s. Namely, we pick a Hamiltonian $h' : \hat{F} \to \mathbb{R}$ with the following properties:

- the branch points $q_1, \dots, q_{2g+1}$ of the projection π are nondegenerate critical points of h';
- h' is bounded, and constant on the level sets of h_ρ in the cylindrical end of $\hat{F}$;
- $h'_{|\alpha_j}$ is a Morse function with a single minimum at q_j.

The second property ensures that the flow of h' commutes with that of h_ρ (which makes perturbed cascades more intuitive) and does not affect the behavior at infinity; the third one ensures that the images of the arcs α_j under the flow generated by $wh_\rho + \epsilon h'$ (for $\epsilon > 0$) behave exactly like the arcs $\tilde{\alpha}_j^-$ pictured in Figure 4.

As above, we define a Hamiltonian H' on $\hat{M} = \mathrm{Sym}^k(\hat{F})$ such that, outside of a small neighborhood of the diagonal strata, $H'([z_1,\dots,z_k]) = \sum_i h'(z_i)$. We can in particular arrange for its Hamiltonian flow to preserve the product structure away from the diagonal and commute with that of H_ρ. Thus, given k sufficiently disjoint embedded arcs $\hat{\lambda}_1,\dots,\hat{\lambda}_k \subset \hat{F}$, the flow generated by $wH_\rho + \epsilon H'$ maps the product $\hat{L} = \hat{\lambda}_1 \times \dots \times \hat{\lambda}_k$ to $\phi_{wH_\rho+\epsilon H'}(\hat{L}) = \phi_{wh_\rho+\epsilon h'}(\hat{\lambda}_1) \times \dots \times \phi_{wh_\rho+\epsilon h'}(\hat{\lambda}_k)$, at least away from the diagonal. As explained in Remark 4.3, we can ensure that this identity remains true for all large w and small ϵ by letting the choices of H_ρ and H' near the diagonal depend on the Lagrangian $\hat{L}$; we denote these choices by $H_{\rho,\hat{L}}$ and $H'_{\hat{L}}$, though we will often drop the subscript from the notation. (Here again, another option would have been to let H' be singular along the diagonal).

For $s \in \mathcal{S}_k^{2g+1}$ and $\tau \geq 0$, we set $H'_{D_s,\tau} = \epsilon(\tau) H'_{D_s}$, where ϵ is a monotonically increasing smooth function with $\epsilon(0) = 0$ and bounded by a small positive constant. By construction, the image of D_s under the flow generated by $wH_{\rho,D_s} + H'_{D_s,w}$ is transverse to D_t for all large enough w, without any intersections being created or cancelled; moreover, the construction of H' is flexible enough to ensure that the appropriate moduli spaces of holomorphic discs are generically regular (see below). Thus, the necessary technical conditions (Definition A.1, as modified in §A.3 to include the perturbations) are satisfied.

Definition 4.4. *We denote by $\mathcal{F}_z$ the A_∞-(pre)category whose objects are*

1. *closed exact Lagrangian submanifolds contained in* $\mathrm{Sym}^k(U) \subset \mathrm{Sym}^k(\hat{F})$*, and*
2. *exact Lagrangian submanifolds of the form $\hat{\lambda}_1 \times \dots \times \hat{\lambda}_k$, where the $\hat{\lambda}_i$ are disjoint properly embedded arcs in $\hat{F}$ such that $\hat{\lambda}_i \cap (\hat{F} \setminus U)$ consists of two components which project via π to straight lines contained in the right half-plane* $\mathrm{Re}\,\pi > 0$*,*

with morphisms and compositions defined by partially wrapped Floer theory (in the sense of Appendix A) with respect to the product complex structure J, the Hamiltonian H_ρ, and suitably chosen small bounded Hamiltonian perturbations.

We leave the Hamiltonian perturbations $H'_{L,\tau}$ unspecified except for the thimbles D_s. Indeed, the actual choice is immaterial, and the Fukaya categories constructed for different choices of perturbations are quasi-equivalent (the argument is essentially the same as in [15]). The only key requirement is that we need the perturbations to be small and bounded so as to not significantly affect the behavior at infinity of the long-time flow (for non-compact objects as in Definition 4.4(2), the properness of h_ρ away from the ray $\hat{Z}$ ensures that a small bounded Hamiltonian perturbation pulled back from $\hat{F}$ does not modify the large-scale behavior).

We also note that, since the compact objects in Definition 4.4(1) are required to lie in $\mathrm{Sym}^k(U)$, over which H_ρ vanishes, they are not affected by the wrapping.

In general, due to our simplifying assumption on the Hamiltonian perturbations we cannot expect transversality in the sense of §A.3 to hold for arbitrary Lagrangian submanifolds, so that $\mathcal{F}_z$ is only an A_∞-precategory, i.e. morphisms and compositions are only defined for objects which satisfy the transversality conditions. The issue is fairly mild, and can be ignored for all practical purposes, since any ordered sequence of thimbles D_s is transverse. Nonetheless, the cautious reader may wish to restrict the set of objects of $\mathcal{F}_z$ to some fixed countable collection of Lagrangians (such that every isotopy class is represented, and including the thimbles D_s) for which transversality can be achieved.

Remark 4.5. If we modify the construction of h_ρ to make the cut-off function χ vanish on *both* components of $\pi^{-1}((-\infty,-a'])$, then we obtain a "less wrapped" category which is fairly closely related to the Fukaya category of the Lefschetz fibration $f_{2g+1,k}$, at least as far as the thimbles $D_s=\prod_{i\in s}\alpha_i$ are concerned. Indeed, the flow still preserves the product structure, but since the Hamiltonian now vanishes over the entire preimage of an arc connecting p_{2g+1} to $-\infty$, the wrapping now accumulates on the *two* infinite rays $\vartheta=\pm\pi/2$ in the cylindrical end and never crosses the preimage of the negative real axis. Thus the flow now maps the arcs α_i to a configuration which, for all practical purposes, behaves interchangeably with the arcs α_i^- previously introduced. It is an exercise left to the reader to adapt the argument below and show that, in this "less wrapped" Fukaya category, the A_∞-algebra associated to the thimbles D_s, $s\in\mathcal{S}_k^{2g}$ is again $\mathcal{A}_{1/2}(F',k)$, just as in $\mathcal{F}(f_{2g+1,k})$ (cf. Remark 3.8).

In the rest of this section, we will be considering the thimbles $D_s=\prod_{j\in s}\alpha_j$, where $s\in\mathcal{S}_k^{2g}$ ranges over all k-element subsets of $\{1,\dots,2g\}$, viewed as objects of the partially wrapped Fukaya category $\mathcal{F}_z$. The following lemma says that we can ignore the technicalities of the construction of the partially wrapped Fukaya category, and simply perturb D_s to $\tilde{D}_s^\pm=\prod_{j\in s}\tilde\alpha_j^\pm$, where the $\tilde\alpha_j^\pm$ are the arcs pictured in Figure 4.

Lemma 4.6. *The full subcategory of $\mathcal{F}_z$ with objects D_s, $s\in\mathcal{S}_k^{2g}$ is quasi-isomorphic to the A_∞-category with the same objects,* $\hom(D_s,D_t)=CF^*(\tilde D_s^-,\tilde D_t^+)$*, and product operations given by counting holomorphic discs bounded by suitably perturbed versions of the D_s (using the long-time flow of H_ρ and the Hamiltonian perturbation H').*

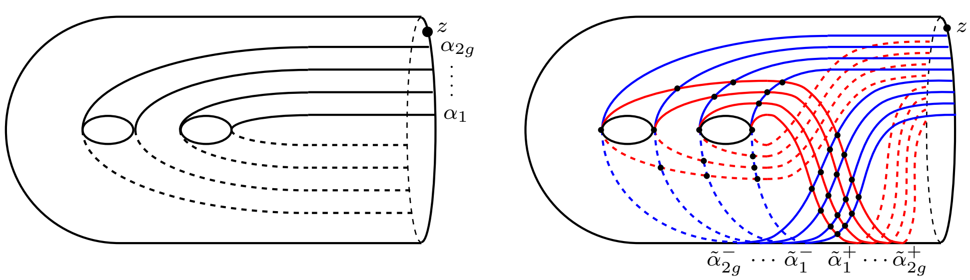


FIGURE 4. The arcs α_j and $\tilde\alpha_j^\pm$ on $\hat F$

Proof. Lemma A.12 gives a criterion under which the infinitely generated complex used to define $\hom(D_s, D_t)$ in the partially wrapped Fukaya category $\mathcal{F}_z$ can be replaced by the ordinary Floer complex $CF^*(\phi_{wH_\rho+\epsilon(w)H'}(D_s), D_t)$ (which is naturally isomorphic to $CF^*(\tilde{D}_s^-, \tilde{D}_t^+)$), and the cascades used to define A_∞-operations are simply rigid holomorphic discs with boundaries on the images of the given Lagrangians under $\phi_{\tau H_\rho+\epsilon(\tau)H'}$ (for sufficiently different values of τ).

The first assumption of the lemma, i.e. the transversality of $\phi_{(\tau+w)H_\rho+\epsilon(\tau+w)H'}(D_s)$ to $\phi_{\tau H_\rho+\epsilon(\tau)H'}(D_t)$ for all $s, t \in \mathcal{S}_k^{2g}$, $\tau \geq 0$ and large enough w, follows from the construction of H' (using the fact that the function ϵ is monotonically increasing). Thus we only need to check that, for $s_0, \dots, s_\ell \in \mathcal{S}_k^{2g}$ and $\tau_0 \gg \tau_1 \gg \dots \gg \tau_\ell \geq 0$, the Lagrangian submanifolds $\phi_{\tau_i H_\rho+\epsilon(\tau_i)H'}(D_{s_i})$ never bound any holomorphic discs of Maslov index less than $2 - \ell$.

We claim that the diagram formed by the arcs α_j and their images under the flow generated by $\tau_i h_\rho + \epsilon(\tau_i)h'$ has the same nice properties as the diagram considered in Section 3. Namely, one can draw the bounded regions of the diagram formed by $\ell+1$ different increasingly wrapped perturbations of the arcs α_j ($1 \leq j \leq 2g$) in such a way that all intersections occur at angles that are multiples of $\pi/(\ell+1)$, and find as in the proof of Proposition 3.6 that the Maslov index of any holomorphic disc is equal to its intersection number with the diagonal strata, $\mu(\phi) = i(\phi) \geq 0$. (See the argument below and Figures 5 and 6.) This immediately implies the absence of discs of index less than $2 - \ell$ except in the case $\ell = 1$.

Next, we observe that a Maslov index 0 holomorphic strip would have to be disjoint from the diagonal strata in $\mathrm{Sym}^k(\hat{F})$ (since $\mu(\phi) = i(\phi) = 0$). Thus, such a strip can be viewed as a k-tuple of holomorphic strips in $\hat{F}$; however, $\phi_{wh_\rho+\epsilon(w)h'}(\alpha_i)$ and α_j (or equivalently, $\tilde{\alpha}_i^-$ and $\tilde{\alpha}_j^+$) do not bound any non-trivial discs in $\hat{F}$. Hence there are no nonconstant Maslov index 0 holomorphic strips, which completes the verification of the assumptions of Lemma A.12. The result follows. □

4.3. **Proof of Theorem 1.2.** The proof of Theorem 1.2 goes along the same lines as that of Theorem 1.1, but using the arcs $\tilde{\alpha}_j^\pm$ instead of $\alpha_j^\pm$. The theorem follows from Lemma 4.6 and the following three propositions.

Proposition 4.7. *The chain complexes* $\hom_{\mathcal{F}_z}(D_s, D_t)$ *and* $\hom_{\mathcal{A}(F,k)}(s,t)$ *are isomorphic for all* $s, t \in \mathcal{S}_k^{2g}$.

Proof. The intersections of $\tilde{D}_s^-$ with $\tilde{D}_t^+$ consist of k-tuples of intersections between the arcs $\tilde{\alpha}_i^-$, $i \in s$ and $\tilde{\alpha}_j^+$, $j \in t$. These can be determined by looking at Figure 4. Namely, the "left half" of $\hat{F}$ looks similar to the configuration of Section 3, while in the "right half" the wrapping creates one new intersection between each $\tilde{\alpha}_i^-$ and each $\tilde{\alpha}_j^+$. Intersections of the first type are again interpreted as strands which do not cross the interval $[2g, 2g+1]$ on the pointed matched circle, while the new intersection point between $\tilde{\alpha}_i^-$ and $\tilde{\alpha}_j^+$ is interpreted as a strand connecting a_i to a_{2g+j}. The dictionary between intersection points and strands is now as follows:

- For $i < j$, $\tilde{\alpha}_i^- \cap \tilde{\alpha}_j^+$ consists of three points; the point at the upper-left on the front part of Figure 4 is interpreted as $\left[\begin{smallmatrix} 2g+i \\ 2g+j \end{smallmatrix}\right]$, while the point at the lower-left on the back part of the figure corresponds to $\left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right]$, and the point in the lower-right part of the figure corresponds to $\left[\begin{smallmatrix} i \\ 2g+j \end{smallmatrix}\right]$;
- For $i = j$, $\tilde{\alpha}_i^- \cap \tilde{\alpha}_j^+$ consists of two points; the branch point of π in the left part of the figure corresponds to the double dotted line $[\,^i\,]$, while the point in the lower-right part of the figure corresponds to $\left[\begin{smallmatrix} i \\ 2g+i \end{smallmatrix}\right]$;
- For $i > j$, $\tilde{\alpha}_i^- \cap \tilde{\alpha}_j^+$ consists of a single point, interpreted as $\left[\begin{smallmatrix} i \\ 2g+j \end{smallmatrix}\right]$.

As before, by considering the set of k-tuples for which the labels in s and t each appear exactly once we obtain a bijection between the generators of $\hom_{\mathcal{F}_z}(D_s, D_t)$ and $\hom_{\mathcal{A}(F,k)}(s,t)$.

Next we consider the Floer differential. One easily checks that the bounded regions of $\hat{F}$ delimited by the arcs $\tilde{\alpha}_1^-, \dots, \tilde{\alpha}_{2g}^-$ and $\tilde{\alpha}_1^+, \dots, \tilde{\alpha}_{2g}^+$ are all rectangles; see Figure 5 for a picture of the relevant portion of the diagram (Figure 5 is obtained from Figure 4 by cutting open $\hat{F}$ at the back in a manner that splits each arc $\tilde{\alpha}_i^\pm$ at the branch point q_i; thus, pairs of rectangles which touch by a corner at q_i are now separated).

Let us mention in passing that our dictionary between intersections and strands is easy to understand in terms of Figure 5: the columns of the diagram, from right to left, can be viewed as the $4g$ starting positions for strands, while the rows, from bottom to top, correspond to the ending positions. The intersection at column i and row j is then the strand $\left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right]$; however the intersection at the branch point q_i appears in two places in the diagram, namely at (i,i) and at $(2g+i, 2g+i)$.

Since the diagram $(\hat{F}, \{\tilde{\alpha}_i^-\}, \{\tilde{\alpha}_i^+\})$ is nice, the Floer differential on $CF^*(\tilde{D}_s^-, \tilde{D}_t^+)$ counts empty embedded rectangles. As in the proof of Proposition 3.4, rectangles

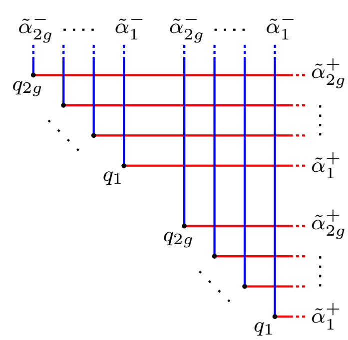


FIGURE 5. The bounded regions of the diagram $(\hat{F}, \{\tilde{\alpha}_i^-\}, \{\tilde{\alpha}_i^+\})$

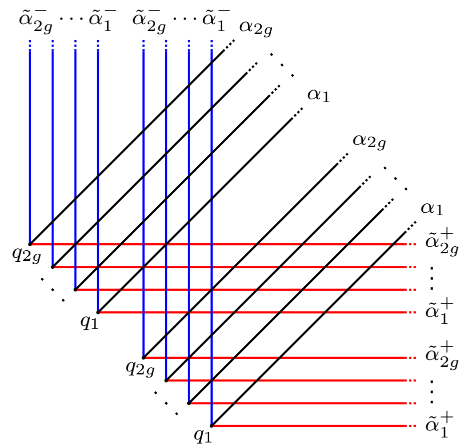


FIGURE 6. The bounded regions of the diagram $(\hat{F}, \{\tilde{\alpha}_i^-\}, \{\alpha_i\}, \{\tilde{\alpha}_i^+\})$

correspond to resolutions of crossings in the strand diagram, and the emptiness condition amounts to the requirement that the resolution does not create any double crossing. Thus the differentials agree. □

Next we compare the products in $\mathcal{F}_z$ and $\mathcal{A}(F,k)$. Given $s,t,u \in \mathcal{S}_k^{2g}$, the composition $\hom(D_s,D_t)\otimes\hom(D_t,D_u) \to \hom(D_s,D_u)$ in $\mathcal{F}_z$ can be computed by wrapping the thimbles in such a way that each pair lies in the correct relative position at infinity. Concretely, we can consider $\tilde{D}_s^-$, D_t, and $\tilde{D}_u^+$, which are products of arcs as in Figure 4 (with the understanding that the end points of the α_i all lie on the portion of ∂F in between the end points of the $\tilde{\alpha}_i^+$ and those of the $\tilde{\alpha}_i^-$).

Proposition 4.8. *The isomorphism of Proposition 4.7 intertwines the product structures of $\mathcal{F}_z$ and $\mathcal{A}(F,k)$.*

Proof. The argument is similar to the proof of Proposition 3.5. Namely, the arcs $\tilde{\alpha}_i^-$, α_i and $\tilde{\alpha}_i^+$ can be drawn on $\hat{F}$ so as to form a diagram with non-generic triple intersections; after cutting $\hat{F}$ open at the q_i, the relevant portion of the diagram is shown on Figure 6. The triple intersections can be perturbed as in Figure 3 right.

By the same argument as in the proof of Proposition 3.5, the Euler measure of any 2-chain ϕ that contributes to the Floer product is equal to $k/4$, and the condition $\mu(\phi)=0$ then implies that ϕ is disjoint from the diagonal strata in $\mathrm{Sym}^k(\hat{F})$. Hence the product counts k-tuples of embedded triangles in $\hat{F}$ which either are disjoint or overlap head-to-tail. Finally, the same argument as before shows that embedded triangles correspond to strand concatenations, and that the forbidden overlaps correspond to concatenations that create double crossings. □

Proposition 4.9. *The higher compositions involving the thimbles D_s ($s \in \mathcal{S}_k^{2g}$) in $\mathcal{F}_z$ are identically zero.*

The proof is identical to that of Proposition 3.6 and simply relies on a Maslov index calculation to show that there are no rigid discs.

4.4. **Other matchings.** In [7], Lipshitz, Ozsváth and Thurston construct the algebra $\mathcal{A}(F,k)$ for an arbitrary pointed matched circle, i.e. the $2g$ pairs of labels assigned to the $4g$ points on the circle need not be in the configuration $1,\dots,2g,1,\dots,2g$ that we have used throughout. The only requirement is that the surface obtained by attaching bands connecting the pairs of identically labelled points and filling in a disc should have genus g and a single boundary component.

We claim that Theorem 1.2 admits a natural extension to this more general setting. Namely, take the configuration of arcs depicted in Figure 6 and view it as lying in a disc D, with the $4g$ end points (previously labelled $q_1,\dots,q_{2g},q_1,\dots,q_{2g}$) lying on the boundary. (So there are now $4g$ marked points on the boundary of D, and $4g$ α-arcs emanating from them). Next, attach $2g$ bands to the disc, in such a way that the two ends of each band are attached to small arcs in ∂D containing end points which carry the same label; and push the end points into the bands until they come together in pairs. In this manner one obtains a configuration of $2g$ properly embedded arcs $\eta_1,\dots,\eta_{2g}$ in a genus g surface with boundary $\mathbb{S}$, as well as their perturbed versions $\tilde{\eta}_i^{\pm}$ which enter in the construction of the partially wrapped Fukaya category.

The $\binom{2g}{k}$ objects of the partially wrapped Fukaya category of the k-fold symmetric product which correspond to the primitive idempotents of $\mathcal{A}(F,k)$ are again products $\Delta_s = \prod_{j\in s}\eta_j$; morphisms, differentials and products can be understood by cutting $\mathbb{S}$ open in each band, to obtain diagrams identical to those of Figures 5 and 6 except for a change in labels. The proof of Theorem 1.2 then extends without modification.

5. Generating the partially wrapped category $\mathcal{F}_z$

The goal of this section is to outline a strategy of proof of "Theorem" 1.3. The argument is based on a careful analysis of the relation between the Fukaya category $\mathcal{F}(f_{2g+1,k})$ of the Lefschetz fibration $f_{2g+1,k}$ and the partially wrapped category $\mathcal{F}_z$.

In the definition of $\mathcal{F}_z$, we restricted ourselves to a specific set of noncompact objects with two useful properties. First, the restriction of Re $f_{2g+1,k}$ to these objects is proper and bounded below, and the imaginary part is bounded by a multiple of the real part. This allows us to view them as objects of $\mathcal{F}(f_{2g+1,k})$ (after generalizing the notion of admissible Lagrangian to allow objects to project to a convex angular sector rather than just to a straight line; this does not significantly affect the construction). Second, we only consider products of disjoint properly embedded arcs, for which the behavior of the flow of H_ρ near infinity is easy to understand: namely, in the cylindrical end the flow preserves the product structure and rotates each arc towards the ray $\vartheta = \pi/2$.

Step 1: The acceleration functor. The first ingredient is the existence of a natural A_∞-functor from $\mathcal{F}(f_{2g+1,k})$ (or rather from a full subcategory whose objects are also objects of $\mathcal{F}_z$) to $\mathcal{F}_z$; this is a special case of more general "acceleration" functors between partially wrapped Fukaya categories, from a less wrapped category to a more wrapped one. This functor is identity on objects, and in the simplest cases (e.g. for the thimbles D_s) it is simply given by an inclusion of morphism spaces.

Closed exact Lagrangians contained in $\mathrm{Sym}^k(U)$ (as in Definition 4.4 (1)) are not affected by the flow of X_{H_ρ}, and neither are their intersections with other Lagrangians. Hence, assuming the two categories $\mathcal{F}(f_{2g+1,k})$ and $\mathcal{F}_z$ are built using the same auxiliary Hamiltonian perturbations, as far as morphisms to/from compact objects are concerned the acceleration functor is simply given by the identity map on Floer complexes. Thus we can restrict our attention to noncompact objects.

Let $L = \lambda_1 \times \cdots \times \lambda_k$ and $L' = \lambda'_1 \times \cdots \times \lambda'_k$, where $\lambda_1, \dots, \lambda_k$ and $\lambda'_1, \dots, \lambda'_k$ are mutually transverse k-tuples of disjoint properly embedded arcs as in Definition 4.4 (2). When we view L and L' as objects of $\mathcal{F}(f_{2g+1,k})$, morphisms from L to L' are defined by perturbing L near infinity (in the complement of U) until its slope becomes larger than that of L', i.e. by perturbing each λ_i in the positive direction to obtain a new arc λ_i^- whose image under π lies closer to the positive imaginary axis than the images of $\lambda'_1, \dots, \lambda'_k$. (If needed we also choose a small auxiliary Hamiltonian perturbation to achieve transversality inside U). In other terms, we wrap the arcs $\lambda_1, \dots, \lambda_k$ by a flow that accumulates on the two infinite rays $\vartheta = \pi/4$ and $\vartheta = 5\pi/4$ (recall $\vartheta = \frac{1}{2}\arg \pi(\cdot)$). The complex $\hom_{\mathcal{F}(f_{2g+1,k})}(L, L')$ is then generated by the intersections of $L^- = \lambda_1^- \times \cdots \times \lambda_k^-$ with L'. (One could also keep perturbing the arcs λ_i until they approach the rays $\vartheta = \pm\pi/2$; this further perturbation does not affect things in any significant manner, see Remark 4.5.)

The construction of $\hom_{\mathcal{F}_z}(L, L')$ involves the complexes $CF^*(\phi_{wH_\rho + H'_{L,w}}(L), L')$ for $w \gg 1$. The long-time flow generated by H_ρ wraps each arc λ_i in the positive direction until it approaches the ray $\vartheta = \pi/2$. Assuming the auxiliary Hamiltonian perturbations are chosen in the same manner in both categories, the resulting arc $\tilde{\lambda}_i^-$ can be viewed as a perturbation of λ_i^- in the cylindrical end $\hat{F} \setminus U$, further wrapping the arc in the positive direction to approach $\vartheta = \pi/2$. We set $\tilde{L}^- = \tilde{\lambda}_1^- \times \cdots \times \tilde{\lambda}_k^-$. The key point is that the isotopy from λ_i^- to $\tilde{\lambda}_i^-$ only *creates* intersections with the arcs $\lambda'_1, \dots, \lambda'_k$. Hence, we can keep track of the intersection points under the isotopy, which allows us to identify $L^- \cap L'$ with a subset of $\tilde{L}^- \cap L'$.

Lemma 5.1. *No intersection point created in the isotopy from L^- to $\tilde{L}^-$ can be the outgoing end of a J-holomorphic strip in $\mathrm{Sym}^k(\hat{F})$ with boundary in $\tilde{L}^- \cup L'$ whose incoming end is a previously existing intersection point (i.e., one that arises by deforming a point of $L^- \cap L'$).*

Proof. By contradiction, assume such a J-holomorphic strip exists. Lifting to a branched cover and projecting to $\hat{F}$, we can view it as a holomorphic map from a bordered Riemann surface to $\hat{F}$ (with the boundary mapping to the arcs $\tilde{\lambda}_i^-$ and λ'_j). The argument is then purely combinatorial, but is best understood in terms of the maximum principle applied to the radial coordinate $r = |\pi|^2$. Namely, after a compactly supported isotopy that does not affect intersections, we can assume that, among the points of $\tilde{\lambda}_i^- \cap \lambda'_j$, those which come from $\lambda_i^- \cap \lambda'_j$ have smaller r than the others (i.e., they lie less far in the cylindrical end; see e.g. Figure 4 right). Moreover, in the cylindrical end the various arcs at hand are all graphs (i.e., the angular coordinate ϑ can be expressed as a function of the radial coordinate r), with the property that at a point of $\tilde{\lambda}_i^- \cap \lambda'_j$ the slope of $\tilde{\lambda}_i^-$ is always greater than that of λ'_j. Thus, if an outgoing strip-like end converges to such an intersection point (i.e., the boundary of the holomorphic curve jumps from λ'_j to $\tilde{\lambda}_i^-$), then the radial coordinate r does not have a local maximum. The maximum of r is then necessarily achieved at an incoming strip-like end converging to an intersection point that lies further in the cylindrical end of $\hat{F}$, i.e. one of the intersections created by the isotopy from L^- to $\tilde{L}^-$. This contradicts the assumption about the incoming end of the strip. □

In other terms, the portion of $CF^*(\tilde{L}^-, L')$ generated by the intersection points that come from $L^- \cap L'$ is a subcomplex. However the naive map from $CF^*(L^-, L')$ to $CF^*(\tilde{L}^-, L')$ obtained by "following" the existing generators through the isotopy is not necessarily a chain map; rather, one should construct a continuation map using linear cascades as in Appendix A.

More generally, the same argument applies to the J-holomorphic discs bounded by $\ell + 1$ Lagrangians obtained by partial wrapping of (mutually transverse) products of disjoint properly embedded arcs. Namely, using appropriate isotopies, we can again view the intersection points which define morphisms in $\mathcal{F}(f_{2g+1,k})$ as a subset of those which define morphisms in $\mathcal{F}_z$; the maximum principle applied to the radial coordinate then implies that a J-holomorphic disc whose incoming ends all map to previously existing intersection points must have its outgoing end also mapping to a previously existing intersection point. In other terms, the wrapping isotopy from L^- to $\tilde{L}^-$ satisfies a property similar to condition (2) in Definition A.1. For collections of product Lagrangians which satisfy appropriate transversality properties, this allows us to use cascades of J-holomorphic discs to build an A_∞-functor whose linear term is given by the above-mentioned continuation maps.

The behavior of the acceleration functor is significantly simpler if we consider the thimbles D_s, $s \in \mathcal{S}_k^{2g}$: namely, in that case an argument similar to that of Lemma 4.6 implies that the wrapping isotopy does not produce any exceptional holomorphic discs (of Maslov index less than $2 - \ell$), and hence there are no non-trivial cascades. The acceleration functor is then simply given by the naive embedding of one Floer complex into the other, obtained by following the intersection points through the isotopy. Or, to state things more explicitly via Theorems 1.1 and 1.2, the acceleration functor simply corresponds to the obvious embedding of $\mathcal{A}_{1/2}(F',k)$ as a subalgebra of $\mathcal{A}(F,k)$.

One last property we need to know about the acceleration functor is that it is cohomologically unital (i.e., the induced functor on cohomology is unital). When the auxiliary Hamiltonian perturbations are chosen suitably and identically in both theories, this essentially follows from the fact that the cohomological unit is given by the "same" generator of the Floer complex in $\mathcal{F}(f_{2g+1,k})$ and $\mathcal{F}_z$. (The general case is not much harder). For compact objects contained in $\mathrm{Sym}^k(U)$ this is clear. For products of properly embedded arcs, the small-time flow of H_ρ pushes each arc slightly off itself in the positive direction at infinity, and choosing the perturbation suitably we can arrange for each arc to intersect its pushoff exactly once; the Floer complex then has a single generator, whose image under the relevant continuation maps (or, in the case at hand, inclusion of the Floer complex) is a cohomological unit. (For instance, in the case of the thimbles D_s, this singles out the generator of $\hom(D_s,D_s)$ which consists only of branch points of π; that generator turns out to be a strict unit.) The behavior of the continuation maps which make up the acceleration functor then ensures unitality of the induced functor on cohomology.

Step 2: Generation by thimbles. The next ingredient is Seidel's result which states that the Fukaya category $\mathcal{F}(f_{2g+1,k})$ is generated by a collection of Lefschetz thimbles, e.g. the $\binom{2g+1}{k}$ product thimbles D_s, $s\in\mathcal{S}_k^{2g+1}$ (Theorem 18.24 of [15]). To be more precise, the only non-compact Lagrangians allowed by Seidel are Lefschetz thimbles, so while his result implies that any compact exact Lagrangian is quasi-isomorphic to a twisted complex built out of the thimbles D_s, the argument in [15] does not apply to the products of disjoint properly embedded arcs that we also wish to allow as objects. On the other hand, those objects can be shown "by hand" to be generated by the D_s, by interpreting arc slides as mapping cones.

Consider $k+1$ disjoint properly embedded arcs $\lambda_1,\dots,\lambda_k,\lambda_1'$ in $\hat{F}$, all satisfying the conditions in Definition 4.4 (2), and such that one extremity of λ_1' lies immediately next to one extremity of λ_1 in the cylindrical end $\hat{F}\setminus U$, say in the positive direction from it. Let λ_1'' be the arc obtained by sliding λ_1 along λ_1'. Finally, denote by $\lambda_1^-,\dots,\lambda_k^-$ a collection of arcs obtained by slightly perturbing $\lambda_1,\dots,\lambda_k$ in the positive direction in the cylindrical end, with each λ_i^- intersecting λ_i in a single point $x_i\in U$, and λ_1^- intersecting λ_1' in a single point x_1' which lies near the cylindrical end; see Figure 7. Let $L=\lambda_1\times\dots\times\lambda_k$, $L'=\lambda_1'\times\lambda_2\times\dots\times\lambda_k$, and $L''=\lambda_1''\times\lambda_2\times\dots\times\lambda_k$. Then the point $(x_1',x_2,\dots,x_k)\in(\lambda_1^-\times\dots\times\lambda_k^-)\cap(\lambda_1'\times\lambda_2\times\dots\times\lambda_k)$ determines (via the appropriate continuation map between Floer complexes, to account for the need to further perturb L) an element of $\hom(L,L')$, which we call u. We claim:

Lemma 5.2. *In $Tw\,\mathcal{F}(f_{2g+1,k})$, L'' is quasi-isomorphic to the mapping cone of u.*

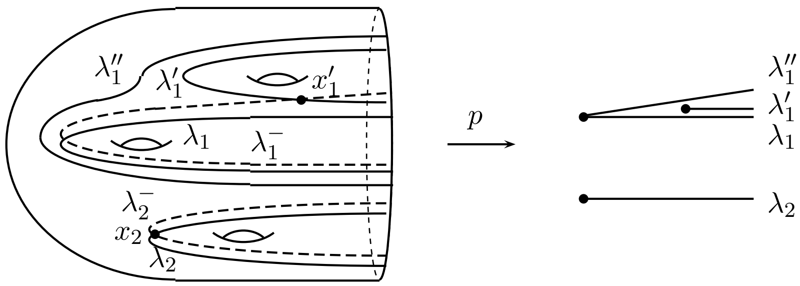


FIGURE 7. Sliding λ_1 along λ_1', and the covering p

Proof. The surface $\hat{F}$ admits a simple branched covering map $p : \hat{F} \to \mathbb{C}$ (i.e., a Lefschetz fibration) with the following properties: (1) the arcs $\lambda_1, \lambda_1', \lambda_2, \dots, \lambda_k$ are thimbles for $k+1$ of the critical points of p (i.e., lifts of half-lines parallel to the real axis and connecting critical values $y_1, y_1', \dots, y_k$ to infinity), with the critical value for λ_1' lying immediately above and very close to the vanishing path for λ_1; (2) the monodromies around the critical points of p corresponding to λ_1 and λ_1' are two transpositions with one common index, and sliding the vanishing arc that lifts to λ_1 around that which lifts to λ_1' yields a new vanishing arc, whose Lefschetz thimble is isotopic to λ_1''. See Figure 7 right. (The covering p, whose degree may be very large, can be built by first projecting a neighborhood of $\lambda_1 \cup \lambda_1'$ to $\mathbb{C}$ by a 3:1 map with two branch points, and a neighborhood of every other λ_i by a 2:1 map with a single branch point, and then extending the map over the rest of $\hat{F}$). Note that p is not holomorphic with respect to the given complex structure on $\hat{F}$, but we can arrange for it to be holomorphic near the branch points.

As in Section 2, we use p to build a symplectic Lefschetz fibration $P : \mathrm{Sym}^k(\hat{F}) \to \mathbb{C}$, defined by $P([z_1, \dots, z_k]) = \sum p(z_i)$ (at least away from the diagonal strata; smoothness requires a slight modification of P near the diagonal, which is irrelevant for our purposes). As before, the critical points of P are tuples of distinct critical points of P, and the thimbles associated to straight line vanishing arcs are just products of the corresponding thimbles for p. In particular, the thimbles associated to the two critical points $[y_1, \dots, y_k]$ and $[y_1', y_2, \dots, y_k]$ of F are respectively L and L', and sliding one vanishing arc over the other one turns L into a product Lagrangian isotopic to L''. (The thimble obtained is not strictly speaking L'', because the sliding operation forces us to consider vanishing arcs with a small positive slope, so the factors $\lambda_2, \dots, \lambda_k$ need to be perturbed accordingly.)

It is then a result of Seidel [15, Proposition 18.23] that L'' is quasi-isomorphic to the mapping cone of the unique generator of $\hom(L, L')$ in the Fukaya category of the Lefschetz fibration P. (Or, in other terms, the objects L, L' and L'' sit in an exact triangle). In order to return to the Fukaya category of $f_{2g+1,k}$, we observe that the construction of homomorphisms in the Fukaya category of the Lefschetz fibration

P requires less wrapping in the positive direction than when we consider $f_{2g+1,k}$. (In fact, the perturbation needed to bring admissible Lagrangians into positive position with respect to P can be made arbitrarily small by choosing p of sufficiently high degree). Thus, there is again an "acceleration" A_∞-functor from the Fukaya category of P to that of $f_{2g+1,k}$. Taking the image of the exact triangle involving L, L', L'' by this functor (and recalling that A_∞-functors are exact) yields the result. □

The other useful fact is that sliding one factor of L over another factor of L only affects L by a Hamiltonian isotopy. For instance, if we denote by $\tilde{\lambda}_1$ the arc obtained by sliding λ_1 along λ_2, then $\tilde{L} = \tilde{\lambda}_1 \times \lambda_2 \times \cdots \times \lambda_k$ is Hamiltonian isotopic to L. This follows immediately from the main result in [11]. (More precisely, the result in [11] is for product tori in symmetric products of closed surfaces; one can reduce to that case by doubling F along its boundary to obtain a closed surface and reflecting the arcs $\lambda_1, \ldots, \lambda_k$ to obtain disjoint closed curves; the arc slide operation then becomes a handle slide and the result of [11] applies.)

With these two results about arc slides in hand, it is fairly easy to show that any product of disjoint properly embedded arcs in $\hat{F}$ (satisfying the conditions in Definition 4.4 (2)) is quasi-isomorphic in $Tw\,\mathcal{F}(f_{2g+1,k})$ to a complex built out of copies of the thimbles D_s, $s \in \mathcal{S}_k^{2g+1}$.

Using the exactness of the acceleration A_∞-functor constructed in Step 1, it now follows that every object of $\mathcal{F}_z$ is quasi-isomorphic in $Tw\,\mathcal{F}_z$ to a complex built out of the thimbles D_s, $s \in \mathcal{S}_k^{2g+1}$.

Step 3: Eliminating α_{2g+1}. We now show that, even though all $\binom{2g+1}{k}$ thimbles are needed to generate $\mathcal{F}(f_{2g+1,k})$, in the case of $\mathcal{F}_z$ it is enough to consider the $\binom{2g}{k}$ thimbles D_s for $s \subseteq \{1, \ldots, 2g\}$. For simplicity, let us assume as in Remark 3.8 that, of the $2g+1$ critical values $p_j = i\theta_j$ of $\pi : \hat{F} \to \mathbb{C}$, $p_1, \ldots, p_{2g}$ lie close to the origin along the negative imaginary axis, while p_{2g+1} lies further away along the positive imaginary axis; for instance, let's say that $|\theta_j| < \frac{1}{k}$ for $j \leq 2g$, whereas $\theta_{2g+1} > 1$.

The key observation is that α_{2g+1} can be isotoped, without crossing the ray $\vartheta = \pi/2$ nor any of the arcs $\alpha_1, \ldots, \alpha_{2g}$, to a properly embedded arc η contained within the open subset $\pi^{-1}(\{\operatorname{Im} w < 0\})$ (which can be identified with the subsurface F' considered in Section 3); see Figure 8. Hence, for $s = \{i_1, \ldots, i_{k-1}, 2g+1\} \in \mathcal{S}_k^{2g+1}$ the thimble $D_s = \alpha_{i_1} \times \cdots \times \alpha_{i_{k-1}} \times \alpha_{2g+1}$ can be isotoped without crossing the diagonal nor $\hat{Z} \times \operatorname{Sym}^{k-1}(\hat{F})$ to the product $\Delta = \alpha_{i_1} \times \cdots \times \alpha_{i_{k-1}} \times \eta$. By construction, Δ lies within the preimage by $f_{2g+1,k}$ of the lower half-plane $\{\operatorname{Im} w < 0\}$, which can be identified with an open subset of the Lefschetz fibration $f_{2g,k}$ (see Remark 3.8). The generation argument we have outlined in Step 2 above then implies that, in $Tw\,\mathcal{F}(f_{2g,k})$, Δ is quasi-isomorphic to a cone built out of the $\binom{2g}{k}$ thimbles corresponding to the elements of $\mathcal{S}_k^{2g}$ (we will make this more explicit below). Recalling that $\mathcal{F}(f_{2g,k})$ embeds as a

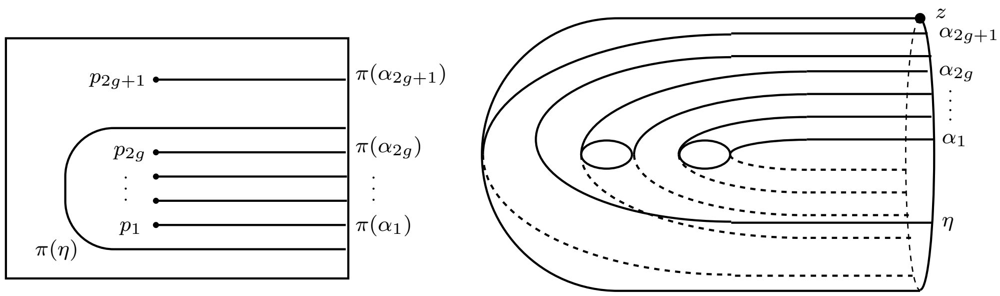


FIGURE 8. Isotoping α_{2g+1} into $\pi^{-1}(\{\mathrm{Im}\, w < 0\})$

full subcategory into $\mathcal{F}(f_{2g+1,k})$, the same result holds in $Tw\,\mathcal{F}(f_{2g+1,k})$; and hence in $Tw\,\mathcal{F}_z$ as well, via the acceleration functor of Step 1.

However, the isotopy from α_{2g+1} to η does not cross the ray $\vartheta = \pi/2$. Hence D_s is isotopic to Δ among product Lagrangians for which partially wrapped Floer theory (with respect to the Hamiltonian H_ρ) is well-defined, and the continuation map induced by the isotopy (defined using cascades as in Appendix A) yields a quasi-isomorphism between these two objects in $\mathcal{F}_z$. (Note here that one could have allowed more general objects in the category $\mathcal{F}_z$, since the construction of partially wrapped Floer theory does not require the arcs to project to the right half-plane, as long as they stay away from the ray $\vartheta = \pi/2$.) Hence D_s is quasi-isomorphic in $Tw\,\mathcal{F}_z$ to a complex built out of the thimbles D_t, $t \subseteq \{1, \dots, 2g\}$. This completes the proof.

It is not hard to write down explicitly a complex to which D_s is quasi-isomorphic. Observe that η can be obtained by first sliding α_1 along α_2 (at the end which lies at the back on Figure 8 right), then sliding the resulting arc successively along $\alpha_3, \dots, \alpha_{2g}$ (at the front of the picture when sliding over odd α_i's, and at the back when sliding over even α_i's). For instance, in the case $k = 1$, this sequence of arc slides tells us that α_{2g+1} is quasi-isomorphic to the complex

$$\alpha_1 \xrightarrow{\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]} \alpha_2 \xrightarrow{\left[\begin{smallmatrix}2g+2\\2g+3\end{smallmatrix}\right]} \alpha_3 \xrightarrow{\left[\begin{smallmatrix}3\\4\end{smallmatrix}\right]} \alpha_4 \xrightarrow{\left[\begin{smallmatrix}2g+4\\2g+5\end{smallmatrix}\right]} \cdots \xrightarrow{\left[\begin{smallmatrix}2g-1\\2g\end{smallmatrix}\right]} \alpha_{2g}$$

(using the notations from $\mathcal{A}(F, k = 1)$ to describe the morphisms). For $k > 1$, we can similarly express $\alpha_{i_1} \times \cdots \times \alpha_{i_{k-1}} \times \alpha_{2g+1}$ in terms of the generators by using the same sequence of arc slides; however, some of the moves now amount to Hamiltonian isotopies while the others are mapping cones.

6. $\widehat{CFA}$ AND THE PAIRING THEOREM

6.1. **Lagrangian correspondences and partially wrapped Fukaya categories.** As explained in §1.1, work in progress of Lekili and Perutz [5] shows that Heegaard-Floer homology can be understood in terms of quilted Floer homology (cf. [16, 17])

for Lagrangian correspondences between symmetric products. The relevant correspondences were introduced by Perutz in his thesis [10]; the construction requires a non-exact perturbation of the Kähler form the symmetric product.

Given a Riemann surface Σ, Perutz equips $\mathrm{Sym}^k(\Sigma)$ with a Kähler form in a class of the form $s\eta_\Sigma + t\theta_\Sigma$, where $s, t \in \mathbb{R}_+$, η_Σ is Poincaré dual to $\{pt\} \times \mathrm{Sym}^{k-1}(\Sigma)$, and $\theta_\Sigma - g\eta_\Sigma$ is Poincaré dual to $\sum_1^g a_i \times b_i \times \mathrm{Sym}^{k-2}(\Sigma)$ where $\{a_i, b_i\}$ is a symplectic basis of $H_1(\Sigma)$ (see [10]). In our case Σ is a punctured Riemann surface, so η_Σ is trivial, and we choose $[\omega]$ to be a positive multiple of θ_Σ, or equivalently, a negative multiple of the first Chern class $c_1(T\mathrm{Sym}^k(\Sigma)) = (n+1-g)\eta_\Sigma - \theta_\Sigma$. Moreover, we arrange for ω to coincide with the product Kähler form on Σ^k away from the diagonal; this ensures that the Hamiltonian flow used in the construction of the partially wrapped Fukaya category still preserves the product structure away from the diagonal.

With this understood, let γ be a non-separating simple closed curve on Σ, and Σ_γ the surface obtained from Σ by deleting a tubular neighborhood of γ and gluing in two discs. Equip Σ_γ with a complex structure which agrees with that of Σ away from γ, and equip $\mathrm{Sym}^k(\Sigma)$ and $\mathrm{Sym}^{k-1}(\Sigma_\gamma)$ with Kähler forms ω and ω_γ chosen as above.

Theorem 6.1 (Perutz [10])**.** *The simple closed curve γ determines a Lagrangian correspondence T_γ in the product $(\mathrm{Sym}^{k-1}(\Sigma_\gamma) \times \mathrm{Sym}^k(\Sigma), -\omega_\gamma \oplus \omega)$, canonically up to Hamiltonian isotopy.*

Given r disjoint simple closed curves $\gamma_1, \dots, \gamma_r$, linearly independent in $H_1(\Sigma)$, we can consider the sequence of correspondences that arise from successive surgeries along $\gamma_1, \dots, \gamma_r$. The main properties of these correspondences (see Theorem A in [10]) imply immediately that their composition defines an embedded Lagrangian correspondence $T_{\gamma_1,\dots,\gamma_r}$ in $\mathrm{Sym}^{k-r}(\Sigma_{\gamma_1,\dots,\gamma_r}) \times \mathrm{Sym}^k(\Sigma)$.

When $r = k = g(\Sigma)$, this construction yields a Lagrangian torus in $\mathrm{Sym}^k(\Sigma)$, which by [10, Lemma 3.20] is smoothly isotopic to the product torus $\gamma_1 \times \dots \times \gamma_k$; Lekili and Perutz show that these two tori are in fact Hamiltonian isotopic [5].

Now, consider as in the introduction a 3-manifold Y with connected boundary $\partial Y \simeq F \cup_{S^1} D^2$ of genus g. Viewing Y as a succession of elementary cobordisms from D^2 to F (e.g. by considering a Morse function $f : Y \to [0,1]$ with index 1 and 2 critical points only, with $f^{-1}(1) = D^2$ and $f^{-1}(0) = F$), Y can be described by a Heegaard diagram consisting of a once punctured surface Σ of genus $\bar{g}$ carrying $\bar{g}$ simple closed curves $\beta_1, \dots, \beta_{\bar{g}}$ (corresponding to the index 2 critical points) and $\bar{g} - g$ simple closed curves $\alpha_1^c, \dots, \alpha_{\bar{g}-g}^c$ (determined by the index 1 critical points). These determine respectively the product torus $T_\beta = \beta_1 \times \dots \times \beta_{\bar{g}} \subset \mathrm{Sym}^{\bar{g}}(\Sigma)$ and a correspondence $\bar{T}_\alpha$ from $\mathrm{Sym}^{\bar{g}}(\Sigma)$ to $\mathrm{Sym}^g(F)$. The formal composition of T_β and $\bar{T}_\alpha$ then defines an object $\mathbb{T}_Y$ of the extended Fukaya category $\mathcal{F}^\sharp(\mathrm{Sym}^g(F))$ (in the sense of Ma'u, Wehrheim and Woodward [9]).

Theorem 6.2 (Lekili-Perutz [5]). *Up to quasi-isomorphism the object $\mathbb{T}_Y$ is independent of the choice of Heegaard diagram for Y.*

Even though we are no longer in the exact setting, technical difficulties in the definition of Floer homology can be avoided by considering *balanced* (also known as *Bohr-Sommerfeld monotone*) Lagrangians. Namely, equip the anticanonical bundle $K^{-1} = \det TM^{1,0}$ of $M = \mathrm{Sym}^g(F)$ (resp. $\mathrm{Sym}^{\bar{g}}(\Sigma)$) with a connection ∇ whose curvature is a constant multiple of the Kähler form. We say that an orientable Lagrangian submanifold L is balanced with respect to ∇ if the restriction of ∇ to L (which is automatically flat) has trivial holonomy, and if moreover the trivialization of $K^{-1}_{|L}$ induced by a ∇-parallel section is homotopic to the natural trivialization given by projecting a basis of TL to $TM^{1,0}$.

In the context of Heegaard-Floer theory, the balancing condition is closely related to admissibility of the Heegaard diagram, and can be similarly ensured by a suitable perturbation of the diagram. Its usefulness is due to the following observation: if L and L' are balanced, then the symplectic area of a pseudo-holomorphic strip with boundary on L, L' connecting two given intersection points is determined *a priori* by its Maslov index (cf. [17, Lemma 4.1.5]). Moreover, the Lagrangians that we consider do not bound any holomorphic discs, because the inclusion of L into M is injective on fundamental groups and hence $\pi_2(M, L) = \pi_2(M) = 0$ (recall that we are considering symmetric products of punctured surfaces); this prevents bubbling and makes Floer homology well-defined.

These properties allow us to extend the construction of the partially wrapped Fukaya category $\mathcal{F}_z$ to this setting, essentially without modification (considering balanced Lagrangians with $\pi_2(M, L) = 0$ instead of exact ones). Moreover, we can enlarge $\mathcal{F}_z$ to allow sufficiently well-behaved generalized Lagrangians. Namely, denote by $\mathcal{F}_z^\sharp$ the A_∞-(pre)category whose objects are

(1) closed balanced Lagrangian tori constructed as products of disjoint, homologically linearly independent simple closed curves, and generalized Lagrangians obtained as images of such balanced product tori under balanced correspondences between symmetric products arising from Perutz's construction;
(2) products of disjoint properly embedded arcs as in Definition 4.4(2);

with morphisms and compositions defined by partially wrapped Floer theory using the Hamiltonian H_ρ on $\mathrm{Sym}^g(\hat{F})$ and suitably chosen small Hamiltonian perturbations. As in §4.2, we require the closed objects to be contained inside the bounded subset $\mathrm{Sym}^g(U)$, where H_ρ vanishes; thus these objects and their intersections with other Lagrangians are not affected by the wrapping.

Proposition 6.3. *The statement of "Theorem" 1.3 remains valid if $\mathcal{F}_z$ is replaced by $\mathcal{F}_z^\sharp$.*

Sketch of proof. The general strategy of proof is the same as in §5. However, we now associate to the Lefschetz fibration $f_{2g+1,k}$ an extended Fukaya category $\mathcal{F}^\sharp(f_{2g+1,k})$, whose compact closed objects are the same balanced generalized Lagrangian submanifolds as in (1) above (whereas the non-compact objects remain the same as in $\mathcal{F}(f_{2g+1,k})$). The key point is that Seidel's generation result still holds in this setting, namely $\mathcal{F}^\sharp(f_{2g+1,k})$ is generated by the thimbles D_s, $s \in \mathcal{S}_k^{2g+1}$.

Seidel's argument relies on viewing the Fukaya category of a Lefschetz fibration as a piece of the $\mathbb{Z}/2$-equivariant Fukaya category of a branched double cover ramified along a smooth reference fiber (i.e., the pullback by a 2:1 base change). In our case, we choose the reference fiber to be disjoint from $\mathrm{Sym}^k(U)$, e.g. we take $f_{2g+1,k}^{-1}(c)$ for $c \in \mathbb{R}_+$ sufficiently large. The thimbles D_s, viewed as Lagrangian discs with boundary in the reference fiber, lift to Lagrangian spheres $\tilde{D}_s$ in the double cover $\tilde{M}$, while a compact object L lifts to the disjoint union of its two preimages $\tilde{L} = \tilde{L}_+ \cup \tilde{L}_-$. (All these lifts have to be equipped with suitable $\mathbb{Z}/2$-equivariant structures.)

Compact generalized Lagrangian submanifolds contained in $\mathrm{Sym}^k(U)$ also lift naturally to the disjoint union of two compact generalized Lagrangians in $\tilde{M}$. These behave in the same manner as ordinary Lagrangians. In particular, the product of the Dehn twists about the Lagrangian spheres $\tilde{D}_s$ interchanges the two preimages $\tilde{L}_\pm$ of a compact object L of $\mathcal{F}^\sharp(f_{2g+1,k})$ (cf. §18 of [15]). Moreover, Seidel's long exact sequence for Dehn twists generalizes to the quilted setting: namely, Wehrheim and Woodward show that the graph of the Dehn twist about $\tilde{D}_s$ fits into an exact triangle in the extended Fukaya category of $\tilde{M} \times \tilde{M}$, from which the long exact sequence follows (see §7 of [18]). This in turn implies by the same argument as in [15, Lemma 18.15 and Proposition 18.17] that, in $Tw\,\mathcal{F}^\sharp(f_{2g+1,k})$, compact objects of $\mathcal{F}^\sharp(f_{2g+1,k})$ are quasi-isomorphic to twisted complexes built out of the thimbles D_s.

With this understood, the rest of the argument works as in §5. Namely, using the acceleration A_∞-functor from $\mathcal{F}^\sharp(f_{2g+1,k})$ to $\mathcal{F}_z^\sharp$ we conclude that $\mathcal{F}_z^\sharp$ is also generated by the thimbles D_s, and the final step (reducing from $\mathcal{S}_k^{2g+1}$ to $\mathcal{S}_k^{2g}$) is unchanged. □

6.2. $\widehat{CFA}$ **via Lagrangian correspondences.** We now give a brief outline of the proof of "Theorem" 1.4. As before, we represent a 3-manifold Y with parameterized boundary $\partial Y \simeq F \cup_{S^1} D^2$ by a Heegaard diagram consisting of a surface Σ of genus $\bar{g} \geq g$ with one boundary component, carrying a base point $z \in \partial\Sigma$ and:

- $\bar{g} - g$ simple closed curves $\alpha_1^c, \dots, \alpha_{\bar{g}-g}^c$, which determine a Lagrangian correspondence T_α from $\mathrm{Sym}^g(F)$ to $\mathrm{Sym}^{\bar{g}}(\Sigma)$ and the opposite correspondence $\bar{T}_\alpha$ from $\mathrm{Sym}^{\bar{g}}(\Sigma)$ to $\mathrm{Sym}^g(F)$;
- $2g$ arcs $\alpha_1^a, \dots, \alpha_{2g}^a$, which after surgery along $\alpha_1^c, \dots, \alpha_{\bar{g}-g}^c$ are assumed to correspond exactly to the arcs $\alpha_1, \dots, \alpha_{2g} \subset F$ considered in previous sections;
- $\bar{g}$ simple closed curves $\beta_1, \dots, \beta_{\bar{g}}$, which determine a product torus T_β in $\mathrm{Sym}^{\bar{g}}(\Sigma)$.

As in the case of F, we complete Σ to a punctured Riemann surface $\hat{\Sigma}$, whose cylindrical end can be identified naturally with that of $\hat{F}$, and consider partially wrapped Floer theory for balanced product Lagrangians in the symmetric product $\mathrm{Sym}^{\bar{g}}(\hat{\Sigma})$.

Namely, we associate to $\mathrm{Sym}^{\bar{g}}(\hat{\Sigma})$ a partially wrapped category $\bar{\mathcal{F}}^\sharp = \mathcal{F}_z^\sharp(\mathrm{Sym}^{\bar{g}}(\hat{\Sigma}))$, defined similarly to $\mathcal{F}_z^\sharp$ except we allow noncompact objects which are balanced products of mutually disjoint simple closed curves and properly embedded arcs in $\hat{\Sigma}$. As before, the simple closed curves are constrained to lie within a bounded subset U' (corresponding to $U \subset \hat{F}$ after surgery along the curves α_i^c, and assumed to contain all the closed curves of the Heegaard diagram), while the properly embedded arcs are constrained to go to infinity in the same manner as in Definition 4.4(2).

The Hamiltonian $\bar{H}_\rho$ used to define wrapped Floer homology is constructed exactly as in §4.2. Namely, away from the diagonal strata it is pulled back from a Hamiltonian $\bar{h}_\rho : \hat{\Sigma} \to \mathbb{R}$ which vanishes over U', so that the flow of $\bar{H}_\rho$ preserves the product structure away from the diagonal and is trivial inside $\mathrm{Sym}^{\bar{g}}(U')$. Moreover, we pick $\bar{h}_\rho$ to agree with h_ρ over $\hat{\Sigma} \setminus U' \simeq \hat{F} \setminus U$, so that the wrapping flow acts similarly on a noncompact object of $\mathcal{F}_z^\sharp$ and on its image under the Lagrangian correspondence T_α.

For $s \in \mathcal{S}_g^{2g}$, we consider the object $\Delta_{\alpha,s} = \prod_{i \in s} \alpha_i^a \times \prod_{j=1}^{\bar{g}-g} \alpha_j^c$ of $\bar{\mathcal{F}}^\sharp$.

Lemma 6.4. *$\Delta_{\alpha,s}$ is Hamiltonian isotopic to the image $T_\alpha(D_s)$ of $D_s \subset \mathrm{Sym}^g(\hat{F})$ under the correspondence T_α.*

This follows directly from the results in [5] (since after doubling F and Σ along their boundaries we can reduce to the case of product tori).

As in §4.2, we choose Hamiltonian perturbations for $\Delta_{\alpha,s}$ in such a way that they preserve the product structure and commute with the flow of $\bar{H}_\rho$. More specifically, we choose a Hamiltonian $\bar{h}' : \hat{\Sigma} \to \mathbb{R}$ which agrees with $h' : \hat{F} \to \mathbb{R}$ away from the α_i^c, and whose restriction to each α_i^c is a Morse function with only two critical points, and we use it to construct a Hamiltonian $\bar{H}'$ on $\mathrm{Sym}^{\bar{g}}(\hat{\Sigma})$. This choice of perturbation ensures that $\hom_{\bar{\mathcal{F}}^\sharp}(\Delta_{\alpha,s}, \Delta_{\alpha,t}) \simeq \hom_{\mathcal{F}_z^\sharp}(D_s, D_t) \otimes H^*(T^{\bar{g}-g}, \mathbb{Z}_2)$.

By the work of Ma'u-Wehrheim-Woodward [9], the Lagrangian correspondences T_α and $\bar{T}_\alpha$ induce A_∞-functors $\Phi_\alpha : \mathcal{F}_z^\sharp \to \bar{\mathcal{F}}^\sharp$ and $\bar{\Phi}_\alpha : \bar{\mathcal{F}}^\sharp \to \mathcal{F}_z^\sharp$. (More precisely, we only have A_∞-functors between suitable full subcategories, due to the slightly different restrictions we placed on objects in $\mathcal{F}_z^\sharp$ and $\bar{\mathcal{F}}^\sharp$.) The presence of wrapping Hamiltonians does not create any significant technical difficulties, since H_ρ and $\bar{H}_\rho$ were chosen compatibly near infinity, and the α_i^c are contained inside U' where $\bar{h}_\rho$ vanishes identically.

The functor Φ_α induces an A_∞-homomorphism from $\mathcal{A}(F,g) = \bigoplus_{s,t} \hom_{\mathcal{F}_z^\sharp}(D_s, D_t)$ to $\bar{\mathcal{A}} = \bigoplus_{s,t} \hom_{\bar{\mathcal{F}}^\sharp}(\Delta_{\alpha,s}, \Delta_{\alpha,t})$. In fact, with the choices of perturbations given above, this map is simply the embedding of $\mathcal{A}(F,g)$ into $\bar{\mathcal{A}} \simeq \mathcal{A}(F,g) \otimes H^*(T^{\bar{g}-g}, \mathbb{Z}_2)$ given by $x \mapsto x \otimes 1$. This makes any A_∞-module over $\bar{\mathcal{A}}$ into a module over $\mathcal{A}(F, g)$. With this understood, we have:

Proposition 6.5. *$\widehat{CFA}(Y)$ is quasi-isomorphic to $\bigoplus_{s\in\mathcal{S}_g^{2g}} \hom_{\bar{\mathcal{F}}^\sharp}(T_\beta, \Delta_{\alpha,s})$.*

Sketch of proof. Recall from [7] that $\widehat{CFA}(Y)$ is generated as a $\mathbb{Z}_2$-vector spaces by $\bar{g}$-tuples of intersections between the closed loops β_i and the loops and arcs α_i^c, α_i^a such that each of $\beta_1, \dots, \beta_{\bar{g}}$ is used exactly once, each α_i^c is used exactly once, and each α_i^a is used at most once. Denoting by s the set of α_i^a which are involved in the intersection, these tuples correspond exactly to points of $T_\beta \cap \Delta_{\alpha,s}$. Thus the two sides can be identified as $\mathbb{Z}_2$-vector spaces.

The A_∞-module structure on $\widehat{CFA}(Y)$ comes from considering holomorphic curves in $[0,1]\times\mathbb{R}\times\hat{\Sigma}$ with additional strip-like ends mapping to Reeb chords between the α_i^a. Meanwhile, the A_∞-module structure on $\bigoplus_s \hom(T_\beta, \Delta_{\alpha,s})$ comes from perturbing the arcs α_i^a by the flow of $\bar{h}_\rho$, which turns all Reeb chords avoiding the base point z into intersection points (as seen in §4.3), and counting holomorphic discs in $\mathrm{Sym}^{\bar{g}}(\hat{\Sigma})$. There are two steps involved in relating these two holomorphic curve counts.

The first step is to view holomorphic discs in $\mathrm{Sym}^{\bar{g}}(\hat{\Sigma})$ as curves in $[0,1] \times \mathbb{R} \times \hat{\Sigma}$. This is essentially identical to Lipshitz's "cylindrical" reformulation of Heegaard-Floer homology. Namely, consider a holomorphic map u from the disc to $\mathrm{Sym}^{\bar{g}}(\hat{\Sigma})$, with boundary mapping to T_β and to suitably wrapped copies of objects Δ_{α,s_i} ($i = 1, \dots, k$) in that order (where the wrapping times τ_i satisfy $\tau_1 \gg \tau_2 \gg \dots \gg \tau_k$). There exists a unique biholomorphism $\varphi : D^2 \to (0,1) \times \mathbb{R}$ such that the boundary marked points corresponding to the intersections involving T_β are sent to the strip-like ends at $\pm\infty$ while the boundary marked points corresponding to the intersections between the perturbed Δ_{α,s_i}'s are sent to points $t_1, \dots, t_{k-1}$ of $\{1\} \times \mathbb{R}$. Denoting by $\pi : S \to D^2$ a suitable ramified $\bar{g}$:1 covering, we can turn u into a holomorphic map $\hat{u} : S \to [0,1] \times \mathbb{R} \times \hat{\Sigma}$, whose first component is $\varphi \circ \pi$ and whose second component maps the $\bar{g}$ preimages of a point $x \in D^2$ to the $\bar{g}$ elements of $u(x)$. The boundary components of S lying above $\{0\}\times\mathbb{R}$ map to the closed curves β_i, while the boundary components lying above $\{1\}\times\mathbb{R}$ map to perturbed copies of the α arcs and curves (switching from one to another above each $t_i \in \{1\} \times \mathbb{R}$).

The second step is to get rid of Hamiltonian perturbations and replace the intersection points occurring at the punctures above each t_i by Reeb chords. The main idea is to "stretch the neck" near $\partial\bar{U}$, i.e. deform the complex structure on $\hat{\Sigma}$ so that the compact subsurface Σ is separated from the region where the wrapping Hamiltonian $\bar{h}_\rho$ is nonzero by a cylinder of arbitrarily large modulus. (Equivalently, we do not modify $\hat{\Sigma}$ but change the choice of $\bar{h}_\rho$ so that its support lies further and further out at infinity.) Simultaneously, we turn off the auxiliary perturbation $\bar{h}'$, so that the k different versions of the α-arcs and curves converge towards each other in arbitrarily large subsets of $\hat{\Sigma}$. Under this deformation, holomorphic curves in $[0,1]\times\mathbb{R}\times\hat{\Sigma}$ converge to multi-stage curves (in the sense of symplectic field theory).

The "bottom" stage of the limit curve is again a holomorphic curve in $[0,1]\times\mathbb{R}\times\hat{\Sigma}$, but the k portions of the boundary over $\{1\}\times\mathbb{R}$ now all map to unperturbed α-arcs and curves. The strip-like ends which used to converge to intersection points lying outside of $\bar{U}$ in the wrapped setting now map to "Reeb chords", i.e. unbounded strips with boundary on α-arcs in the cylindrical end of $\hat{\Sigma}$, as expected in bordered Heegaard-Floer theory. Meanwhile, wherever in the wrapped setting one had a strip-like end converging to an intersection point lying inside $\bar{U}$ (hence, an intersection between copies of a *same* α-arc or curve), the limit curve has a smooth boundary point, together with a gradient flow trajectory for the restriction of $\bar{h}'$ to the appropriate arc or loop. Since we are considering rigid curves, the limit curve has no intermediate stages, and the top stage is constant in the $[0,1]\times\mathbb{R}$ factor and consists of strips in the infinite cylinder $\mathbb{R}\times S^1$ each connecting a Reeb chord (at the negative end of the cylinder) to the corresponding intersection point between the wrapped α-arcs.

We claim that the two-stage limit configurations we have just described are in one-to-one correspondence with the curves used to define the module structure on $\widehat{CFA}(Y)$. This follows from two observations.

First, the upper stage of the limit curve is uniquely determined by the bottom stage, since each Reeb chord between two of the arcs α_i^a (not passing over the base point) is connected to the corresponding intersection point between appropriately wrapped versions of the arcs by a unique rigid holomorphic strip in the cylinder $\mathbb{R}\times S^1$.

Second, whenever an intersection point between an arc or loop $\eta\in\{\alpha_i^c,\alpha_i^a\}$ and its image under the perturbation $\bar{h}'$ lies inside $\bar{U}$ and occurs in a generator of $\Phi_\alpha(\mathcal{A}(F,g))\subset\bar{\mathcal{A}}$, it is necessarily the *minimum* of the restriction of $\bar{h}'$ to η. Indeed, in the case of α_i^a the only intersection inside $\bar{U}$ (corresponding to the pair of horizontal dotted lines $\left[\,{}^{i}\,\right]$ in the notation of [7]) is by construction the minimum of $\bar{h}'$ on α_i^a; and in the case of the closed loop α_i^c, the claim follows from the description of the embedding of $\mathcal{A}(F,g)$ into $\bar{\mathcal{A}}$ given just before the statement of the proposition.

Thus, at each of the boundary marked point which does not degenerate to a Reeb chord, the limit curve instead has a smooth boundary on some arc η, together with a Morse gradient flow line of $\bar{h}'_{|\eta}$ from the marked point on the boundary of the limit curve to the minimum. Since every generic point of η is connected to the minimum by a *unique* gradient flow line of $\bar{h}'_{|\eta}$, we conclude that turning the Hamiltonian perturbation $\bar{h}'$ on or off does not affect the count of holomorphic curves. □

Proposition 6.6. *The $\mathcal{A}(F,g)$-modules $\bigoplus_s \hom_{\bar{\mathcal{F}}^\sharp}(T_\beta,\Delta_{\alpha,s})$ and $\bigoplus_s \hom_{\mathcal{F}^\sharp_z}(\mathbb{T}_Y,D_s)$ are quasi-isomorphic.*

Remark 6.7. Recalling that $\Delta_{\alpha,s} \simeq \Phi_\alpha(D_s)$ and $\mathbb{T}_Y = \bar{\Phi}_\alpha(T_\beta)$, this proposition is a special case of a more general statement, namely that the A_∞-functors $\Phi_\alpha : \mathcal{F}_z^\sharp \to \bar{\mathcal{F}}^\sharp$ and $\bar{\Phi}_\alpha : \bar{\mathcal{F}}^\sharp \to \mathcal{F}_z^\sharp$ induced by the Lagrangian correspondence T_α are mutually *adjoint*. As evident from the proof, this is a general feature of functors induced by Lagrangian correspondences and in no way specific to the specific example at hand.

Sketch of proof. The existence of an isomorphism between $\hom_{\bar{\mathcal{F}}^\sharp}(T_\beta, \Phi_\alpha(D_s))$ and $\hom_{\mathcal{F}_z^\sharp}(\bar{\Phi}_\alpha(T_\beta), D_s)$ as vector spaces follows directly from the definition of extended Fukaya categories, since both are given by the quilted Floer complex $CF^*(T_\beta, T_\alpha, D_s)$.

In order to compare the module structures, we describe the relevant operations graphically in terms of quilted holomorphic curves. In $\mathcal{F}_z^\sharp$, the k-fold product

$$\hom(\bar{\Phi}_\alpha(T_\beta), D_{s_1}) \otimes \hom(D_{s_1}, D_{s_2}) \otimes \cdots \otimes \hom(D_{s_{k-1}}, D_{s_k}) \to \hom(\bar{\Phi}_\alpha(T_\beta), D_{s_k})$$

is given by a count of quilted holomorphic discs with boundaries on T_β and on $D_{s_1}, \dots, D_{s_k}$, with a seam mapping to the correspondence T_α, as depicted in the left half of Figure 9. On the other hand, the right half of Figure 9 represents the quilted discs which contribute to the product operation

$$\hom(T_\beta), \Phi_\alpha(D_{s_1})) \otimes \hom(\Phi_\alpha(D_{s_1}), \Phi_\alpha(D_{s_{j_1}})) \otimes \cdots \to \hom(T_\beta, \Phi_\alpha(D_{s_k}))$$

in $\bar{\mathcal{F}}^\sharp$, together with the quilted discs which govern the A_∞-homomorphism from $\mathcal{A}(F,k)$ to $\bar{\mathcal{A}}$ induced by Φ_α. (Actually, in our case the higher order terms of this A_∞-homomorphism vanish, so the latter quilted discs have only one input and look

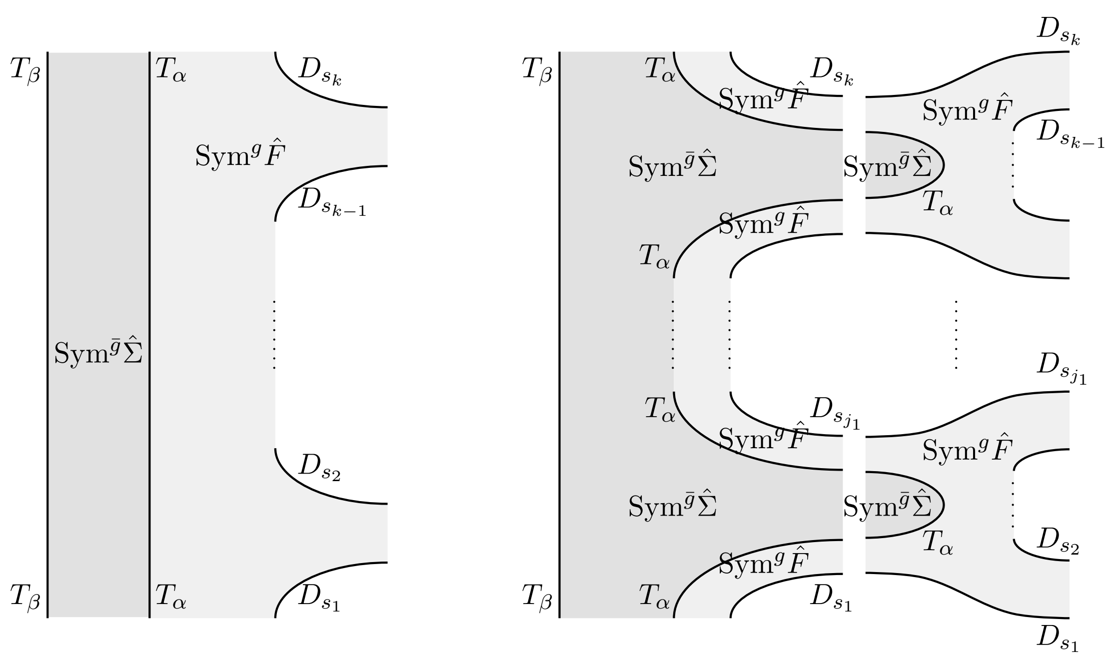


FIGURE 9. The $\mathcal{A}(F,g)$-module structure on $\bigoplus CF^*(\bar{\Phi}_\alpha(T_\beta), D_s)$ (left) and the $\Phi_\alpha(\mathcal{A}(F,g))$-module structure on $\bigoplus CF^*(T_\beta, \Phi_\alpha(D_s))$ (right)

like those in [16, Figure 10].) The right-hand side picture can be deformed to that on the left-hand side by gluing the various components together and moving the seam across; thus the two module maps agree up to a chain homotopy. □

Finally, "Theorem" 1.4 is a direct corollary of Propositions 6.5 and 6.6.

6.3. **The pairing theorem.** We now sketch the proof of "Theorem" 1.5. Consider a closed 3-manifold Y which decomposes as the union of two 3-manifolds Y_1 and Y_2 with $\partial Y_1 = -\partial Y_2 = F \cup_{S^1} D^2$. As in the previous section, Heegaard diagrams for Y_1 and for $-Y_2$ allow us to associate to these manifolds two objects $\mathbb{T}_{Y_1}$ and $\mathbb{T}_{-Y_2}$ of $\mathcal{F}_z^\sharp$. These generalized Lagrangian submanifolds of $\mathrm{Sym}^g(F)$ can also be constructed by viewing Y_1 and $-Y_2$ as successions of elementary cobordisms between Riemann surfaces, starting from D^2 and ending with F. From this perspective, Y_2 is obtained by considering the same sequence of elementary cobordisms as for $-Y_2$ but in reverse order, starting from F and ending with D^2; thus Y_2 defines the opposite correspondence $\mathbb{T}_{Y_2} = \bar{\mathbb{T}}_{-Y_2}$ from $\mathrm{Sym}^g(F)$ to $\mathrm{Sym}^0(D^2) = \mathrm{pt}$.

By the work of Lekili and Perutz [5], these Lagrangian correspondences allow us to compute the Heegaard-Floer homology of Y, namely $\widehat{CF}(Y)$ is quasi-isomorphic to the quilted Floer complex of the sequence of correspondences $(\mathbb{T}_{Y_1}, \mathbb{T}_{Y_2})$. (Indeed, this sequence arises from a particular way of representing the complement of a ball in Y as a cobordism from D^2 to D^2; the claim then follows from Theorem 6.2, which we now apply in the context of the manifold $Y \setminus B^3$ with boundary $S^2 = D^2 \cup_{S^1} D^2$.) Thus, we have

$$\widehat{CF}(Y) \simeq \hom_{\mathcal{F}_z^\sharp}(\mathbb{T}_{Y_1}, \mathbb{T}_{-Y_2}).$$

Next, recall that we have a contravariant Yoneda functor $\mathcal{Y} : \mathcal{F}_z^\sharp \to \mathcal{A}(F, g)$-mod, given on objects by

$$\mathbb{L} \mapsto \mathcal{Y}(\mathbb{L}) = \textstyle\bigoplus_s \hom_{\mathcal{F}_z^\sharp}(\mathbb{L}, D_s),$$

and that by "Theorem" 1.4 we have $\widehat{CFA}(Y_1) \simeq \mathcal{Y}(\mathbb{T}_{Y_1})$ and $\widehat{CFA}(-Y_2) \simeq \mathcal{Y}(\mathbb{T}_{-Y_2})$.

Proposition 6.8. *$\mathcal{Y}$ is a cohomologically full and faithful (contravariant) embedding.*

Proof. The usual Yoneda embedding of $\mathcal{F}_z^\sharp$ into $\mathcal{F}_z^\sharp$-mod is cohomologically full and faithful (cf. e.g. [15, Corollary 2.13]). Moreover, by Proposition 6.3 (the analogue of "Theorem" 1.3 for the extended category $\mathcal{F}_z^\sharp$), the natural functor from $\mathcal{F}_z^\sharp$-mod to $\mathcal{A}(F, g)$-mod is an equivalence. The result follows. □

"Theorem" 1.5 follows, since we now have

$$\begin{aligned} \hom_{\mathcal{A}(F,g)\text{-mod}}(\widehat{CFA}(-Y_2), \widehat{CFA}(Y_1)) &\simeq \hom_{\mathcal{A}(F,g)\text{-mod}}(\mathcal{Y}(\mathbb{T}_{-Y_2}), \mathcal{Y}(\mathbb{T}_{Y_1})) \\ &\simeq \hom_{\mathcal{F}_z^\sharp}(\mathbb{T}_{Y_1}, \mathbb{T}_{-Y_2}) \\ &\simeq \widehat{CF}(Y). \end{aligned}$$

Appendix A. Cascades and partially wrapped Floer theory

In this appendix, we outline the construction of the partially wrapped Floer complexes and their A_∞-operations. Generally speaking, things are very similar to the wrapped case defined by Abouzaid and Seidel in [2]. However, instead of considering solutions of inhomogeneous Cauchy-Riemann equations with Hamiltonian perturbations, we study trees of genuine J-holomorphic curves with boundaries on perturbed Lagrangian submanifolds. This construction, which was pointed out to us by Mohammed Abouzaid and is similar to that in Section 10e of [15], allows us both to avoid compactness issues, and to relate the outcome more directly to Heegaard-Floer theory. On the other hand, we need to make some assumptions about the behavior of Lagrangian intersections upon wrapping.

A.1. **Linear cascades and the partially wrapped Floer complex.** Let (M,ω) be an exact symplectic manifold with convex contact boundary $(\partial M,\alpha)$, and let $\hat{M}$ be the exact symplectic manifold obtained by attaching the positive symplectization $([1,\infty)\times\partial M, d(r\alpha))$ along the boundary of M. We consider a Hamiltonian function $H_\rho:\hat{M}\to\mathbb{R}$ such that $H_\rho\geq 0$ everywhere and $H_\rho(r,y)=\rho(y)\,r$ on the positive symplectization, where $\rho:\partial M\to[0,1]$ is a smooth function on the contact boundary. To a pair of exact Lagrangians $L_1,L_2\subset\hat{M}$ with cylindrical ends modelled on Legendrian submanifolds of $\partial M\setminus\rho^{-1}(0)$, we wish to associate a chain complex $\hom(L_1,L_2)$ which behaves as the direct limit for $w\to\infty$ of the Floer complexes $CF^*(\phi_{wH_\rho}(L_1),L_2)$. Following Abouzaid and Seidel [2], we actually define $\hom(L_1,L_2)$ to be the infinitely generated complex $\bigoplus_{w=1}^\infty CF^*(\phi_{wH_\rho}(L_1),L_2)[q]$, or rather the quasi-isomorphic truncation $\bigoplus_{w=m}^\infty CF^*(\phi_{wH_\rho}(L_1),L_2)[q]$ for some $m\geq 1$ (see Definition A.1 below), where the formal variable q has degree -1 and satisfies $q^2=0$, equipped with a differential of the form

$$\begin{array}{ccccc}
CF^*(\phi_{H_\rho}(L_1),L_2) \circlearrowleft_{\delta} & & CF^*(\phi_{2H_\rho}(L_1),L_2) \circlearrowleft_{\delta} & & CF^*(\phi_{3H_\rho}(L_1),L_2) \circlearrowleft_{\delta} \\
\uparrow{\scriptstyle id} & \nearrow^{\kappa} & \uparrow{\scriptstyle id} & \nearrow^{\kappa} & \\
q\,CF^*(\phi_{H_\rho}(L_1),L_2) \circlearrowright_{\delta} & & q\,CF^*(\phi_{2H_\rho}(L_1),L_2) \circlearrowright_{\delta} & & \dots
\end{array} \tag{A.1}$$

Here δ is the usual Floer differential, counting index 1 J-holomorphic strips with boundary on $\phi_{wH_\rho}(L_1)$ and L_2, while κ is a continuation map. Before we give its definition, let us list the technical assumptions that will enable our construction to be well-defined.

Definition A.1. *We say that a collection* $\{L_i,\ i \in I\}$ *of exact Lagrangian submanifolds of* $\hat{M}$ *is transverse with respect to the Hamiltonian* H_ρ *and the almost-complex structure* J *if the following properties hold.*

(1) $\phi_{wH_\rho}(L_i)$ *is transverse to* L_j *for all* $i, j \in I$ *and for all integer values of* w *greater or equal to some lower bound* $m = m_{i,j}$.

(2) *For* $w \geq m$, *each point of* $\phi_{wH_\rho}(L_i) \cap L_j$ *lies on a unique maximal smooth arc* $t \mapsto \gamma(t)$ *parametrized by either the whole interval* $[m, \infty)$ *or a subinterval of the form* (t_0, ∞), *such that* $\gamma(t)$ *is a transverse intersection of* $\phi_{tH_\rho}(L_i)$ *and* L_j *for all* t. *In the second case* ($t_0 > m$), $\gamma(t)$ *tends to infinity as* $t \to t_0$, *and there exists* $\epsilon > 0$ *such that for* $t \in (t_0, t_0 + \epsilon)$ *no* J*-holomorphic disc can have an outgoing strip-like end converging to* $\gamma(t) \in \phi_{tH_\rho}(L_i) \cap L_j$.

(3) *Given any* $i_0, \dots, i_\ell \in I$, *and any integers* $m_{i_{j-1},i_j} \leq w_j^- \leq w_j^+$, $j = 1, \dots, \ell$ *and* $0 = w_{\ell+1}^- \leq w_{\ell+1}^+$, *consider all* J*-holomorphic discs in* $\hat{M}$ *with boundary on the Lagrangian submanifolds* $\phi_{\tau_j H_\rho}(L_{i_j})$ $(0 \leq j \leq \ell)$, *where* $\tau_j = \sum_{k=j+1}^{\ell+1} w_k$ *and* $w_j \in [w_j^-, w_j^+]$, *with* $\ell + 1$ *marked points mapping to given intersections (in the sense of condition (2) above) and representing a given relative class* φ. *Then the moduli space of such discs is smooth and of the expected dimension* $\mu(\varphi) + \ell - 2 + \#\{j \,|\, w_j^- < w_j^+\}$, *and all these discs are regular (as elements of the parametrized moduli space). Moreover, all nontrivial projections to the real parameters* τ_j *and* w_j *are generic and transverse to each other with respect to gluing operations (whenever the outgoing marked point in one moduli space matches with an incoming marked point in another moduli space).*

Condition (2) can be stated more informally as follows: as w increases continuously from m to ∞, existing intersections between $\phi_{wH_\rho}(L_i)$ and L_j persist and remain transverse, whereas new intersections may be created "at infinity" but only provided that, each time this happens, the newly created intersection cannot be the outgoing end of any J-holomorphic disc. In particular, given $p \in \phi_{wH_\rho}(L_i) \cap L_j$ and $w' \geq w$, we can associate to p a unique point of $\phi_{w'H_\rho}(L_i) \cap L_j$, which we denote by $\vartheta_w^{w'}(p)$.

Finally, condition (3) states that all the moduli spaces of holomorphic discs we will consider are regular, and behave in the expected manner with respect to gluing; the precise meaning of the transversality requirement will become clear in the subsequent discussion. As usual, we only need this property to hold for 0- and 1-dimensional moduli spaces in order for the construction to be well-defined (while invariance properties also involve 2-dimensional moduli spaces).

Remark A.2. One should keep in mind the following subtlety: when defining higher products, one sometimes needs to consider cascades in which two of the components are given by the same data, in which case it is impossible to make the projections to the time and width parameters transverse, so that condition (3) does not hold.

When it arises, this issue can be addressed by picking perturbations which depend on the full boundary data of the cascade (see Definition A.3), and not just on the component under consideration; see Sections 4.7 and 4.9 of [2] for details.

It is worth mentioning that condition (2) is the key limiting technical assumption in the approach we adopt. Conditions (1) and (3) can often be achieved by introducing suitable perturbations into the story below (see Remark A.2; see also §A.3). On the other hand it is not clear as of this writing how to construct continuation maps via cascades if (2) does not hold. When constructing (partially) wrapped Fukaya categories, condition (2) usually follows from a finiteness property or from an appropriate version of the maximum principle.

To keep the notations under control, in the discussion below we will ignore ignore perturbations; we will also assume that it is always possible to choose $m = 1$ and define

$$\hom(L_i, L_j) = \bigoplus_{w=1}^{\infty} CF^*(\phi_{wH_\rho}(L_i), L_j)[q].$$

In the general case, we will leave it up to the reader to replace these complexes by their quasi-isomorphic truncations (restricting to $w \geq m$, or replacing H_ρ by a multiple).

Given two transverse exact Lagrangians L_1, L_2 and a positive integer w, we can now define the continuation map $\kappa : CF^*(\phi_{wH_\rho}(L_1), L_2) \to CF^*(\phi_{(w+1)H_\rho}(L_1), L_2)$ as follows. Given $p \in \phi_{wH_\rho}(L_1) \cap L_2$ and $q \in \phi_{(w+1)H_\rho}(L_1) \cap L_2$, a *$k$-step linear cascade* from p to q is a sequence of k finite energy J-holomorphic strips $u_1, \dots, u_k : \mathbb{R} \times [0,1] \to \hat{M}$ such that:

- $u_i(\mathbb{R} \times 0) \subset \phi_{w_i H_\rho}(L_1)$ and $u_i(\mathbb{R} \times 1) \subset L_2$, for some $w_1 \leq \cdots \leq w_k$ in the interval $[w, w+1]$;
- denoting by $p_i^\pm \in \phi_{w_i H_\rho}(L_1) \cap L_2$ the intersection points to which the strips u_i converge at $\pm\infty$, and setting $p_0^+ = p$ and $p_{k+1}^- = q$, the points p_i^+ and p_{i+1}^- match up in the sense of property A.1(2), i.e. $p_{i+1}^- = \vartheta_{w_i}^{w_{i+1}}(p_i^+)$ $\forall\, 0 \leq i \leq k$.

As a special case we allow $k = 0$, i.e. the empty sequence of strips, provided that $q = \vartheta_w^{w+1}(p)$.

We denote by $\mathcal{M}_1^{\{1\}}(L_1, L_2; w; p, q; \varphi)$ the moduli space of all linear cascades from p to q which represent a given total relative homotopy class φ (the precise definition of the homotopy class involves completing the broken trajectory to a continuous arc in the path space using the Hamiltonian isotopy; the details are left to the reader). The coefficient of q in $\kappa(p)$ is then defined as a count of rigid linear cascades from p to q, i.e. those which represent classes φ for which the Maslov index $\mu(\varphi)$ is zero. By the regularity assumption, these are cascades in which each component is a Maslov index 0 holomorphic strip at which the linearized $\bar\partial$ operator has a one-dimensional cokernel ("exceptional" holomorphic strips).

Linear cascades are a special case of the more general cascades we will introduce below. Informally, these objects can be understood by considering the perturbed holomorphic strips normally used to define Floer continuation maps, with a Hamiltonian perturbation term of the form $\beta(t)X_{H_\rho}$ where the smooth function $\beta:\mathbb{R}\to\mathbb{R}$ tends to w at $+\infty$ and $w+1$ at $-\infty$, and taking the limit where the derivative of β tends to zero; it is then reasonable to expect that perturbed holomorphic strips converge (in the sense of Gromov compactness) to linear cascades.

The algebraic properties of κ are determined by the behavior of one-dimensional moduli spaces of linear cascades. These moduli spaces are obtained by gluing together various pieces, corresponding to different numbers of steps and/or individual homotopy classes of the components. Namely, the part of the boundary of the moduli space of k-step cascades where one of the k components breaks into two J-holomorphic strips is glued with the part of the boundary of the moduli space of $k+1$-step cascades where two values w_i and w_{i+1} become equal. The only remaining boundaries correspond to the cases $w_1=w$ and $w_k=w+1$, which amounts to breaking off of a J-holomorphic strip contributing to the usual Floer differential δ. Thus $\kappa\delta=\delta\kappa$ (up to sign), i.e. κ is a chain map between the Floer complexes $CF^*(\phi_{wH_\rho}(L_1),L_2)$ and $CF^*(\phi_{(w+1)H_\rho}(L_1),L_2)$, and the differential on the complex (A.1) squares to zero.

A.2. **Cascades and A_∞ operations.** The construction of the partially wrapped Fukaya A_∞-category $\mathcal{F}(M,\rho)$ relies on that of a series of maps

(A.2)
$$m_\ell^F : CF^*(\phi_{w_\ell H_\rho}(L_{\ell-1}),L_\ell)\otimes\cdots\otimes CF^*(\phi_{w_1H_\rho}(L_0),L_1)\to CF^*(\phi_{w_{out}H_\rho}(L_0),L_\ell),$$

where $L_0,\dots,L_\ell$ are a transverse collection of exact Lagrangians ($\ell\geq 1$), F is a subset of $\{1,\dots,\ell\}$, $w_1,\dots,w_\ell$ are positive integers, and $w_{out}=w_1+\cdots+w_\ell+|F|$.

The maps m_ℓ^F generalize both the usual Floer-theoretic product operations, which correspond to $F=\emptyset$, and the continuation map κ defined above, which corresponds to $\ell=1$ and $F=\{1\}$. Up to sign, m_ℓ^F is precisely the part of the ℓ-fold product operation which maps $q^{\epsilon_\ell}CF^*(\phi_{w_\ell H_\rho}(L_{\ell-1}),L_\ell)\otimes\cdots\otimes q^{\epsilon_1}CF^*(\phi_{w_1H_\rho}(L_0),L_1)$ to $CF^*(\phi_{w_{out}H_\rho}(L_0),L_\ell)$, where $\epsilon_i=1$ if $i\in F$ and 0 otherwise; see Section 3.8 of [2].

We will define the map m_ℓ^F differently from the construction in Section 3 of [2], which involves counts of perturbed holomorphic curves called "popsicles". We will instead use cascades of (unperturbed) holomorphic discs.

Definition A.3. *We call* boundary data *a tuple* $(\underline{L};\underline{w},F;\underline{p},q)$ *where:*

- $\underline{L}=(L_0,\dots,L_\ell)$ *is a transverse collection of exact Lagrangian submanifolds;*
- $\underline{w}=(w_1,\dots,w_\ell)\in\mathbb{R}_+^\ell$ *are positive real numbers;*
- F *is a (possibly empty) subset of* $\{1,\dots,\ell\}$*; set* $w_i'=w_i+1$ *if* $i\in F$ *and* $w_i'=w_i$ *otherwise, and* $w_{out}=\sum_{i=1}^\ell w_i+|F|=\sum_{i=1}^\ell w_i'$*;*
- $\underline{p}=(p_1,\dots,p_\ell)$*,* $p_i\in\phi_{w_iH_\rho}(L_{i-1})\cap L_i$*, and* $q\in\phi_{w_{out}H_\rho}(L_0)\cap L_\ell$ *are transverse intersection points.*

A labelled planar tree *for the boundary data* $(\underline{L};\underline{w},F;\underline{p},q)$ *consists of:*

1. *a planar tree* Γ *with* $\ell+1$ *leaves (properly embedded in* D^2*, with the leaves mapping to the boundary and the other vertices mapping to the interior), together with a distinguished leaf called* output*; all the edges of* Γ *are oriented so they point towards the output, and the components of* $D^2\setminus\Gamma$ *are numbered by integers* $0,\dots,\ell$ *and labelled by the Lagrangians* $L_0,\dots,L_\ell$ *in counterclockwise order starting from the output;*
2. *for each vertex* v *of* Γ *and for each region* i *adjacent to* v*, a "time"* $\tau_{i,v}\in\mathbb{R}$*. These are required to satisfy the following conditions:*
   (a) *at the output leaf* v_{out}*,* $\tau_{0,v_{out}}=w_{out}$ *and* $\tau_{\ell,v_{out}}=0$*;*
   (b) *at the* i*-th input leaf* $v_{in,i}$*,* $\tau_{i-1,v_{in,i}}=w_i$ *and* $\tau_{i,v_{in,i}}=0$*;*
   (c) *for every directed edge* $e=(v^-,v^+)$ *separating regions* i *and* j $(i<j)$*,* $\tau_{j,v^-}\leq\tau_{j,v^+}$*, and* $w_{e,v^-}:=\tau_{i,v^-}-\tau_{j,v^-}\leq w_{e,v^+}=\tau_{i,v^+}-\tau_{j,v^+}\leq\sum_{i<k\leq j}w'_k$*;*
3. *for each vertex* v *of* Γ *and each edge* e *adjacent to* v*, separating two regions* i *and* j*, a point* $p_{e,v}\in\phi_{\tau_{i,v}H_\rho}(L_i)\cap\phi_{\tau_{j,v}H_\rho}(L_j)$*. These are required to satisfy the following conditions:*
   (a) *at the output leaf,* $p_{e,v_{out}}=q$*;*
   (b) *at the input leaves,* $p_{e,v_{in,i}}=p_i$*;*
   (c) *for every directed edge* $e=(v^-,v^+)$ *separating regions* i *and* j $(i<j)$*, the points* p_{e,v^-} *and* p_{e,v^+} *match up in the sense of property A.1(2), i.e.* $p_{e,v^+}=\phi_{\tau_{j,v^+}H_\rho}\circ\vartheta_{w_{e,v^-}}^{w_{e,v^+}}\circ\phi^{-1}_{\tau_{j,v^-}H_\rho}(p_{e,v^-})$*.*

We denote by ℓ_v*,* $\underline{L}_v$*,* $\underline{\tau}_v$*,* $\underline{p}_v$ *and* q_v *the number of inputs, Lagrangian submanifolds, times, incoming and outgoing intersection points associated to the vertex* v*.*

The elementary building blocks of cascades are J-holomorphic discs with boundaries on the images of given Lagrangian submanifolds by the Hamiltonian flow generated by H_ρ. Given a transverse collection $\underline{L}=(L_0,\dots,L_\ell)$ of exact Lagrangians, a tuple of real numbers $\underline{\tau}=(\tau_0,\dots,\tau_\ell)\in\mathbb{R}^{\ell+1}$, a tuple of intersection points $\underline{p}=(p_1,\dots,p_\ell)$, $p_i\in\phi_{\tau_{i-1}H_\rho}(L_{i-1})\cap\phi_{\tau_iH_\rho}(L_i)$, $q\in\phi_{\tau_0H_\rho}(L_0)\cap\phi_{\tau_\ell H_\rho}(L_\ell)$, and a relative homotopy class φ, we denote by $\mathcal{M}^{\mathrm{hol}}_\ell(\underline{L};\underline{\tau};\underline{p},q;\varphi)$ the moduli space of J-holomorphic maps from the disc with $\ell+1$ (ordered) boundary marked points to $\hat{M}$, with the boundary arcs mapping to the Lagrangian submanifolds $\phi_{\tau_iH_\rho}(L_i)$ and the marked points mapping to $p_1,\dots,p_\ell,q$, representing the class φ.

The Floer product operation

$$m_\ell=m_\ell^{\emptyset}:CF^*(\phi_{w_\ell H_\rho}(L_{\ell-1}),L_\ell)\otimes\cdots\otimes CF^*(\phi_{w_1H_\rho}(L_0),L_1)\to CF^*(\phi_{w_{out}H_\rho}(L_0),L_\ell)$$

(where $w_{out}=\sum w_i$) corresponding to the case $F=\emptyset$ differs from a mere count of J-holomorphic discs in that one needs to apply to all inputs the A_∞-functors which intertwine Lagrangian intersection theory for the pairs $(\phi_{w_iH_\rho}(L_{i-1}),L_i)$ and

$(\phi_{\tau_{i-1}H_\rho}(L_{i-1}), \phi_{\tau_i H_\rho}(L_i))$, where $\tau_i = \sum_{j>i} w_j$. The standard way of doing this relies on a Hamiltonian perturbation of the holomorphic curve equation; instead, the homotopy method leads us to consider cascades of holomorphic discs. To distinguish the cascades for $F = \emptyset$ from the more general case (for arbitrary F), we will sometimes call them "plain cascades".

Definition A.4. *A* (plain) cascade *of J-holomorphic discs for the boundary data $(\underline{L}; \underline{w}, \emptyset; \underline{p}, q)$ consists of:*

- *a labelled planar tree $(\Gamma, \{\tau_{i,v}\}, \{p_{e,v}\})$ for the boundary data (in the sense of Definition A.3);*
- *for each interior vertex v of Γ, a holomorphic disc $u_v \in \mathcal{M}^{\text{hol}}_{\ell_v}(\underline{L}_v; \underline{\tau}_v; \underline{p}_v, q_v; \varphi_v)$ representing some homotopy class $[u_v] = \varphi_v$.*

We denote by $\mathcal{M}^{\emptyset}_{\ell}(\underline{L}; \underline{w}; \underline{p}, q; \varphi)$ the moduli space of such cascades representing a total homotopy class $\sum [u_v] = \varphi$.

Note that, since $w_{out} = \sum w_i$, it must be the case that in condition A.3(2)(c) the equality $w_{e,v^-} = w_{e,v^+} = \sum_{i<k\leq j} w_k$ holds for every directed edge $e = (v^-, v^+)$ of Γ separating regions i and j.

The transversality condition A.1(3) implies that, when the w_i are positive integers, the moduli space $\mathcal{M}^{\emptyset}_{\ell}(\underline{L}; \underline{w}; \underline{p}, q; \varphi)$ is smooth and of the expected dimension, i.e. $\mu(\varphi) + \ell - 2$. The coefficient of q in $m^{\emptyset}_{\ell}(p_\ell, \dots, p_1)$ is then defined as a count of cascades in the moduli spaces $\mathcal{M}^{\emptyset}_{\ell}(\underline{L}; \underline{w}; \underline{p}, q; \varphi)$ for which $\mu(\varphi) = 2 - \ell$.

The simplest case is when the graph Γ has a single interior vertex, and the cascade consists of a single holomorphic disc in $\mathcal{M}^{\text{hol}}_{\ell}(\underline{L}; \underline{\tau}; \underline{p}^+, q; \varphi)$, where $\tau_i = \sum_{j>i} w_j$ and $p_i^+ = \phi_{\tau_i H_\rho}(p_i)$. More generally, the cascades which contribute to $m^{\emptyset}_{\ell}$ consist of a "root component" which is a rigid holomorphic disc carrying the output marked point, and other components which are exceptional holomorphic discs of index $1 - \ell_v$ for a component with ℓ_v inputs (indeed, the time parameters $\tau_{i,v}$ for the root component are completely fixed, while for the other components they are only determined up to a simultaneous translation).

Example A.5. By the above discussion, the cascades which contribute to $m^{\emptyset}_1$ consist of a single index 1 holomorphic disc, so $m^{\emptyset}_1$ equals the usual Floer differential δ. The situation is more interesting for $\ell \geq 2$; for instance, Figure 10 depicts a rigid plain cascade that contributes to $m^{\emptyset}_3$.

Lemma A.6. *The operations $m^{\emptyset}_{\ell}$ satisfy the A_∞-relations, i.e.*

$$\sum_{i,k} (-1)^* \, m^{\emptyset}_{\ell-k+1}(p_\ell, \dots, p_{i+k+1}, m^{\emptyset}_k(p_{i+k}, \dots, p_{i+1}), p_i, \dots, p_1) = 0.$$

(Since we work with $\mathbb{Z}/2$ coefficients, we will not worry about orientations or signs.)

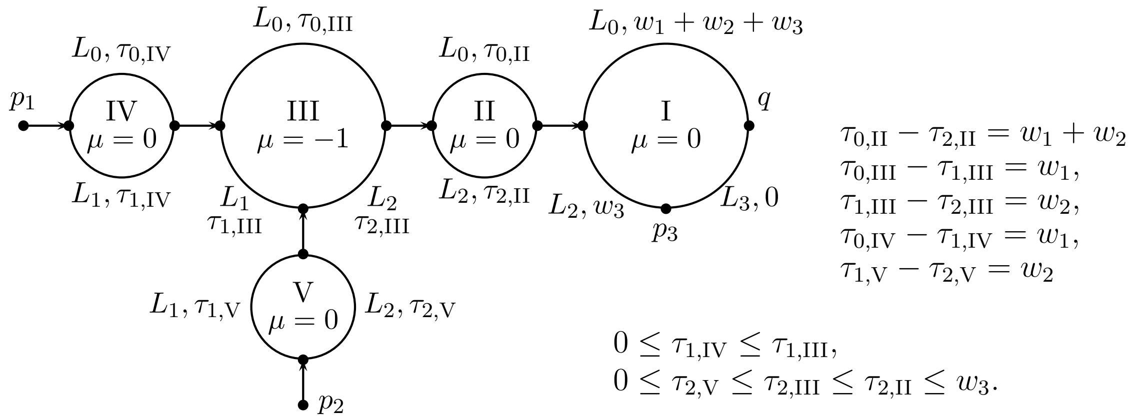


FIGURE 10. A rigid cascade contributing to $m_3^{\emptyset}$. The arrows indicate intersections that match via the flow ϕ_{H_ρ}.

Sketch of proof. The argument relies as usual on an analysis of 1-dimensional moduli spaces of cascades. These moduli spaces are composed of various pieces, depending on the combinatorial type of the tree Γ and the classes represented by the individual components. At interior points, exactly one of the components admits a one-parameter family of deformations, while the others are rigid.

With one exception, the portions of the boundary where the non-rigid component breaks into a pair of J-holomorphic disks match exactly with those where the inequality $\tau_{j,v^-} \leq \tau_{j,v^+}$ in condition A.3(2)(c) becomes an equality for some directed edge $e = (v^-, v^+)$ connecting two interior vertices of Γ (one of them carrying the non-rigid component) and separating regions $i < j$. Accordingly, we glue the various moduli spaces together along these common boundary strata.

The exceptional case is when the root component breaks into a pair of rigid discs, one carrying the ℓ-th input and the other carrying the output. In that case we create an edge $e = (v^-, v^+)$ in Γ to record the combinatorics of the breaking, and then split Γ along e to obtain a pair of planar graphs Γ', whose root vertex v^- carries the ℓ-th input, and Γ'', whose root vertex v^+ carries the original output (this case has to be treated separately because $\tau_{\ell,v^-} = \tau_{\ell,v^+} = 0$). One easily checks that the cascade now decomposes into the union of two cascades with underlying graphs Γ' and Γ''.

The remaining portions of the boundary correspond to the cases where the inequality $\tau_{j,v^-} \leq \tau_{j,v^+}$ becomes an equality for a directed edge $e = (v^-, v^+)$ that connects an input leaf to an interior vertex of Γ. In that case, we have $\tau_{j,v^+} = 0$, and j is necessarily the largest index among all the regions of $D^2 \setminus \Gamma$ adjacent to the vertex v^+. Splitting Γ along the outgoing edge from the vertex v^+ (and creating a pair of leaves) yields a pair of planar graphs Γ' (with root vertex v^+) and Γ'' (with the same root vertex as Γ); it is then easy to check that the cascade decomposes into the union of two cascades with underlying graphs Γ' and Γ''.

Conversely, two cascades such that the outgoing intersection point of one matches with one of the inputs of the other can be glued to obtain one of the boundary configurations described above. Thus, the boundary of the moduli space of cascades can be identified with a union of fibered products of smaller moduli spaces of cascades, and the A_∞-relations follow. □

We are now ready to define the more general cascades which determine the operation m_ℓ^F for an arbitrary subset F of $\{1,\dots,\ell\}$.

Definition A.7. *A* cascade *of holomorphic discs for the boundary data* $(\underline{L};\underline{w},F;\underline{p},q)$ *consists of:*

- *a labelled planar tree* $(\Gamma,\{\tau_{i,v}\},\{p_{e,v}\})$ *for the boundary data, such that for every vertex v of Γ, the region of greatest index j among those adjacent to v satisfies $\tau_{j,v}=0$;*
- *for each interior vertex v of Γ, an element of $\mathcal{M}^{\emptyset}_{\ell_v}(\underline{L}_v;\underline{w}_v;\underline{p}_v^-,q_v;\varphi_v)$, i.e. a plain cascade representing some homotopy class φ_v, where $\underline{w}_v$ is the collection of widths $w_{e,v}$ for the incoming edges at the vertex v, and $p^-_{e,v}=\phi^{-1}_{\tau_{j,v}H_\rho}(p_{e,v})$ for an incoming edge separating regions i and j, $i<j$.*

We denote by $\mathcal{M}^F_\ell(\underline{L};\underline{w};\underline{p},q;\varphi)$ the moduli space of cascades representing a total homotopy class $\sum\varphi_v=\varphi$.

The transversality condition A.1(3) implies that, when the w_i are positive integers, the moduli space $\mathcal{M}^F_\ell(\underline{L};\underline{w};\underline{p},q;\varphi)$ is smooth and of the expected dimension, i.e. $\mu(\varphi)+\ell-2+|F|$. The coefficient of q in $m_\ell^F(p_\ell,\dots,p_1)$ is then defined as a count of cascades in the moduli spaces $\mathcal{M}^F_\ell(\underline{L};\underline{w};\underline{p},q;\varphi)$ for which $\mu(\varphi)=2-\ell-|F|$. Note that the operation $m_1^{\{1\}}$ includes the empty cascade (where Γ has no interior vertices).

Given an interior vertex v of Γ, the width parameter $w_{e,v}$ associated to an incoming edge e is *a priori* free to vary if and only if e can be reached by a directed path that starts at some input leaf $v_{in,i}$, $i\in F$. Thus, denoting by f_v the number of such incoming edges at v and by ℓ_v the total number of incoming edges, the dimension of the parametrized moduli space attached to the vertex v is $\mu(\varphi_v)+\ell_v-2+f_v$. Hence, the rigid cascades which contribute to m_ℓ^F consist of trees such that the equality

$$\mu(\varphi_v)=2-\ell_v-f_v \tag{A.3}$$

holds for each interior vertex v of Γ.

When $F\neq\emptyset$, generic cascades have the property that $f_v\geq 1$ for every interior vertex v, i.e. each vertex can be reached by a directed path from some input leaf $v_{in,i}$, $i\in F$; for otherwise the sum of the individual dimensions $\mu(\varphi_v)+\ell_v-2+f_v$ turns out to be strictly less than $\mu(\varphi)+\ell-2+|F|$. (In the case $F=\emptyset$ the same argument implies that for generic cascades Γ has a single interior vertex, i.e. we are reduced to Definition A.4).

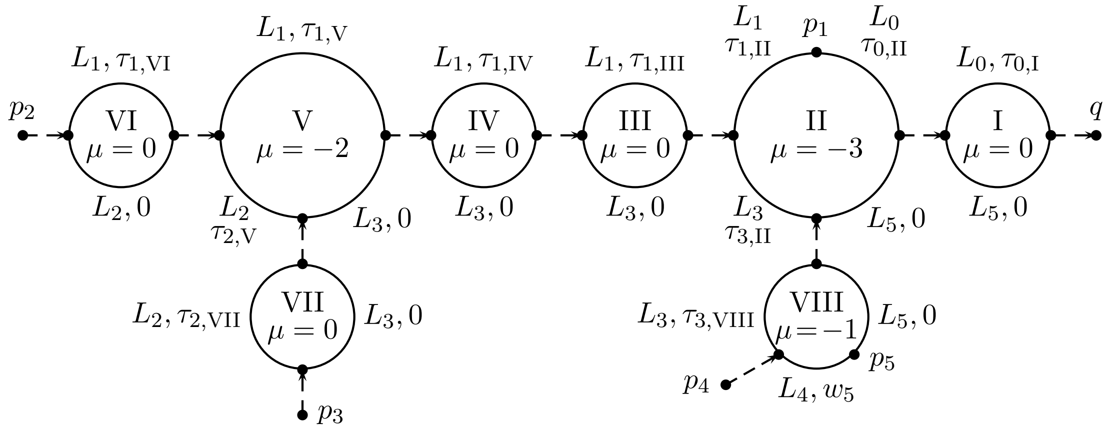


FIGURE 11. A rigid cascade contributing to $m_5^{\{2,3,4\}}$. The dotted lines indicate intersections that match via the maps $\vartheta_w^{w'}$ from A.1(2).

Example A.8. Figure 11 depicts a rigid cascade that contributes to $m_5^{\{2,3,4\}}$. Each circle represents either a single holomorphic disc, or more generally a plain cascade as in Definition A.4. The times $\tau_{i,v}$ satisfy:

- $\tau_{0,\mathrm{II}} - \tau_{1,\mathrm{II}} = w_1$;
- $w_2 \leq \tau_{1,\mathrm{VI}} \leq \tau_{1,\mathrm{V}} - \tau_{2,\mathrm{V}} \leq w_2 + 1$;
- $w_3 \leq \tau_{2,\mathrm{VII}} \leq \tau_{2,\mathrm{V}} \leq w_3 + 1$;
- $\tau_{1,\mathrm{V}} \leq \tau_{1,\mathrm{IV}} \leq \tau_{1,\mathrm{III}} \leq \tau_{1,\mathrm{II}} - \tau_{3,\mathrm{II}} \leq w_2 + w_3 + 2$;
- $w_4 + w_5 \leq \tau_{3,\mathrm{VIII}} \leq \tau_{3,\mathrm{II}} \leq w_4 + w_5 + 1$;
- $\tau_{0,\mathrm{II}} \leq \tau_{0,\mathrm{I}} \leq w_{out}$.

In order to state the algebraic relation satisfied by the m_ℓ^F's, we first recall the notion of "admissible cut" introduced by Abouzaid and Seidel (cf. Section 3.6 of [2]).

Definition A.9 ([2], Definition 3.8)**.** *An* admissible cut *of $F \subseteq \{1, \dots, \ell\}$ consists of $\ell_+, \ell_- \geq 1$ such that $\ell_- + \ell_+ = \ell + 1$, a number $i \in \{1, \dots, \ell_+\}$, and subsets $F_\pm \subseteq \{1, \dots, \ell_\pm\}$ satisfying $|F_-| + |F_+| = |F|$, and with the following property: F contains all $k \in F_+$ satisfying $k < i$, the numbers $k + \ell_- - 1$ for all $k \in F_+$ with $k > i$, and all the numbers $k + i - 1$ for $k \in F_-$. If $i \notin F_+$ those are all the elements of F, otherwise F has one more element, which lies in the range $\{i, \dots, i + \ell_- - 1\}$.*

An admissible cut arises when a cascade decomposes into a pair of cascades by splitting the graph Γ along some edge to obtain a pair of planar graphs Γ^- (carrying input vertices i to $i + \ell_- - 1$) and Γ^+ (carrying input vertices 1 to $i - 1$ and $i + \ell_-$ to ℓ, plus a new input vertex arising from the edge that was cut). The elements of F then decompose in the obvious manner; however we allow ourselves to delete one element from $F \cap \{i, \dots, i + \ell_- - 1\}$ when forming F_-, in which case i becomes an element of F_+. The width associated to the cut (i.e., to the output leaf of Γ^- and to the i-th input leaf of Γ^+) is $w_{cut} = \sum_{k=i}^{i+\ell_- -1} w_k + |F_-|$; naturally, the cut is legal only the required inequalities A.3(2)(c) hold on either side of the cut, i.e. the edge $e = (v^-, v^+)$ along which Γ is split should satisfy $w_{e,v^-} \leq w_{cut} \leq w_{e,v^+}$.

The same relation as in [2, equation (61)] then holds:

Proposition A.10. $\sum(-1)^* m_{\ell_+}^{F_+}(p_\ell, \dots, p_{i+\ell_-}, m_{\ell_-}^{F_-}(p_{i+\ell_- -1}, \dots, p_i), p_{i-1}, \dots, p_1) = 0$*, where the sum ranges over all admissible cuts.*

Sketch of proof. The argument again relies on the study of 1-dimensional moduli spaces of cascades. As before, these are composed of various pieces (according to the combinatorial type of the tree Γ) glued together along part of their boundaries. For a cascade in a 1-dimensional moduli space, all but one of the interior vertices of Γ satisfy the equality $\mu(\varphi_v) = 2 - \ell_v - f_v$ (i.e., the corresponding plain cascade is rigid); the remaining interior vertex v_0 is associated to a one-dimensional parametrized moduli space of plain cascades. There are various boundary strata, corresponding to the following possibilities:

(1) the plain cascade at the vertex v_0 breaks up into a pair of plain cascades, as in the proof of Lemma A.6; the limiting cascade is described by a tree with one more vertex;
(2) the inequality $w_{e,v^-} \leq w_{e,v^+}$ becomes an equality for some directed edge $e = (v^-, v^+)$ connecting two interior vertices (one of which is v_0);
(3) the inequality $w_{e,v^-} \leq w_{e,v^+}$ becomes an equality for some directed edge $e = (v^-, v^+)$ connecting an input leaf $v^- = v_{in,i}$ ($i \in F$) to an interior vertex (necessarily $v^+ = v_0$);
(4) the inequality $w_{e,v^+} \leq \sum_{i<k\leq j} w'_k$ becomes an equality for some directed edge $e = (v^-, v^+)$ separating regions i and j (necessarily $v^+ = v_0$).

We do not consider the case where $w_{e,v^-} \leq w_{e,v^+}$ becomes an equality for $v^- = v_0$ and $v^+ = v_{out}$ the outgoing leaf, since it is a subcase of (4). Moreover, no boundary strata arise from the inequality $\tau_{j,v^-} \leq \tau_{j,v^+}$ (where $e = (v^-, v^+)$ is a directed edge separating regions i and j) becoming an equality: indeed, Definition A.7 implies that $\tau_{j,v^-} = 0$, whereas τ_{j,v^+} is always zero if j is the greatest index among all regions adjacent to v^+, and always positive and bounded from below otherwise (due to the positivity of the input width w_{j+1}).

Next, we observe that cases (1) and (2) match up exactly, i.e. they correspond to strata along which different moduli spaces are glued together. (Here it is worth noting that the plain cascades at vertices v^- and v^+ can be glued together to form a single plain cascade precisely when the widths w_{e,v^-} and w_{e,v^+} are equal, regardless of whether the times τ_{j,v^-} and τ_{j,v^+} match or not.) We are left with cases (3) and (4), which correspond precisely to the two types of admissible cuts.

In case (4), we split the tree Γ along the edge e to obtain two trees, Γ^- with root vertex v^- and a new output leaf with width $w_{cut} = \sum_{i<k\leq j} w'_k$, and Γ^+ with a new input leaf with width $w_{cut}\,(= w_{e,v^+})$. We obtain a pair of rigid cascades subordinate

to an admissible cut (with $i \notin F_+$). (Note: since $i \notin F_+$, the cut decreases f_{v^+} by one, which makes the plain cascade at the vertex v^+ rigid after splitting.)

In case (3), namely when w_{e,v^+} becomes equal to w_i for a directed edge e connecting the i-th input leaf to the vertex $v^+ = v_0$, we will find an admissible cut such that the i-th input leaf and the vertex v_0 lie within the tree Γ^-, and the element i is deleted from F_-. Namely, denote by $\hat{e} = (\hat{v}^-, \hat{v}^+)$ the first directed edge encountered along the path from v_0 to the output leaf with the property that

$$w_{\hat{e},\hat{v}^-} \leq \sum_{\hat{\imath}<k\leq\hat{\jmath}} w'_k - 1 \leq w_{\hat{e},\hat{v}^+}$$

where $\hat{\imath}$ and $\hat{\jmath}$ are the labels of the regions separated by $\hat{e}$. Then we split Γ along the edge $\hat{e}$ to obtain two trees: Γ^-, with root vertex $\hat{v}^-$ and a new output leaf with width $w_{cut} = \sum_{\hat{\imath}<k\leq\hat{\jmath}} w'_k - 1$, and Γ^+ with a new input leaf with the same width w_{cut}. The cascade then splits into a pair of rigid cascades subordinate to the relevant cut, where the label associated to the input leaf $v_{in,i}$ is deleted from F_- (i.e., $i - \hat{\imath} + 1 \notin F_-$ after relabelling), whereas the new input is added to F_+ (i.e., $\hat{\imath} \in F_+$).

Finally, each pair of cascades which contributes to the sum in the statement of the proposition arises precisely once from the splitting of some configuration at the boundary of a 1-dimensional moduli space in the manner we have described; the result follows. ☐

Proposition A.10 allows us to construct partially wrapped Fukaya categories (under the assumptions of Definition A.1) in the same manner as Abouzaid and Seidel [2], except we substitute cascades for popsicles.

We end with the following useful observation:

Lemma A.11. *Let $\{L_i,\ i \in I\}$ be a transverse collection of exact Lagrangian submanifolds of $\hat{M}$, with the following additional properties:*

1. *for all $i, j \in I$, $\phi_{wH_\rho}(L_i)$ is transverse to L_j for all large enough w ($w \geq m$ for some integer $m = m_{i,j}$), without any intersections being created or cancelled;*
2. *given any boundary data $(\underline{L}; \underline{w}, F; \underline{p}, q)$ where $\underline{L} = (L_{i_0}, \dots, L_{i_\ell})$ is a sequence of exact Lagrangians chosen among the L_i, and the widths w_k are large enough ($w_k \geq m_{i_k,i_{k+1}}$), and given $\underline{\tau} = (\tau_0, \dots, \tau_\ell) \in \mathbb{R}^{\ell+1}_{\geq 0}$ with $\tau_k - \tau_{k+1} = w_k$ and a nontrivial relative class φ such that $\mu(\varphi) < 2-\ell$, the Lagrangian submanifolds $\phi_{\tau_k H_\rho}(L_{i_k})$ do not bound any holomorphic disc in the relative class φ, i.e. $\mathcal{M}^{\mathrm{hol}}_\ell(\underline{L}; \underline{\tau}; \underline{p}, q; \varphi) = \emptyset$.*

Then the operations m^F_ℓ are identically zero for $F \neq \emptyset$, except $m^{\{1\}}_1 = \kappa$ which is the natural isomorphism between Floer complexes induced by identifying intersection points via the map ϑ^{w+1}_w. Thus, up to quasi-isomorphism we can replace the infinitely generated complex $\hom(L_i, L_j)$ by $CF^*(\phi_{wH_\rho}(L_i), L_j)$ (for any $w \geq m_{i,j}$). Moreover, *the ℓ-fold product operation*

$$m_\ell^\emptyset : CF^*(\phi_{w_\ell H_\rho}(L_{\ell-1}), L_\ell) \otimes \cdots \otimes CF^*(\phi_{w_1 H_\rho}(L_0), L_1) \to CF^*(\phi_{w_{out} H_\rho}(L_0), L_\ell)$$

simply counts rigid J-holomorphic discs in the moduli spaces $\mathcal{M}_\ell^{\mathrm{hol}}(\underline{L};\underline{\tau};\underline{p},q;\varphi)$, where $\tau_i = \sum_{j>i} w_j$ and we identify the generators of $CF^*(\phi_{w_i H_\rho}(L_{i-1}), L_i)$ with those of $CF^*(\phi_{\tau_{i-1} H_\rho}(L_{i-1}), \phi_{\tau_i H_\rho}(L_i))$ in the obvious manner.

Proof. Recall from the discussion after Definition A.4 that rigid plain cascades consist of trees of holomorphic discs in which the root component is rigid and the other components have index $1-\ell_v$ where ℓ_v is the number of inputs. However the assuptions give a lower bound by $2-\ell_v$ on the Maslov index of any nontrivial holomorphic disc. Thus, rigid plain cascades (those of index $2-\ell$) consist of a single holomorphic disc, and there are no "exceptional" plain cascades (of index less than $2-\ell$).

Likewise, consider a rigid cascade contributing to m_ℓ^F and modelled after a planar tree Γ. Recall from the discussion after Definition A.7 that for each interior vertex v we have a plain cascade of Maslov index $2-\ell_v-f_v$, where ℓ_v is the number of incoming edges at v and f_v is the number of incoming edges which can be reached from an input leaf tagged by an element of F. Thus the non-existence of plain cascades of index less than $2-\ell_v$ implies that either $F=\emptyset$ or Γ has no interior vertices (the latter case corresponds to the empty cascade, which contributes to $m_1^{\{1\}} = \kappa$). ☐

A.3. **Hamiltonian perturbations.** We now modify the above setup by introducing auxiliary Hamiltonian perturbations in order to make it easier to achieve transversality even with a degenerate Hamiltonian H_ρ. Given two exact Lagrangians L_1, L_2, we fix a family of Hamiltonians $H'_{L_1,L_2,w}$, with the property that $\phi_{wH_\rho+H'_{L_1,L_2,w}}(L_1)$ is transverse to L_2 for large enough w, and we now define

$$\hom(L_1,L_2) = \bigoplus_{w=1}^{\infty} CF^*(\phi_{wH_\rho+H'_{L_1,L_2,w}}(L_1), L_2)[q].$$

The differential is defined in terms of linear cascades, exactly as in the unperturbed case. In order to define products and higher-order operations on these complexes, we need to fix homotopies between the relevant Hamiltonian perturbations, and incorporate them into the definition of plain cascades (general cascades are then built out of plain cascades as in the unperturbed case).

To avoid a lengthy discussion of consistent homotopies between Hamiltonians, we will restrict ourselves to the case where the perturbation can be chosen independent of the second Lagrangian, i.e. $H'_{L_1,L_2,w} = H'_{L_1,w}$. Thus, we pick for every Lagrangian L a family of Hamiltonians $\{H'_{L,\tau}\}_{\tau\geq 0}$, depending smoothly on τ, and with $H'_{L,0}=0$. We then replace $\phi_{\tau H_\rho}(L)$ by $\phi_{\tau H_\rho + H'_{L,\tau}}(L)$ in the construction of plain cascades.

To be more precise, the changes are the following. To start with, we modify Definition A.3 in the obvious manner, so that boundary data now consists of:

- a collection of exact Lagrangians $\underline{L} = (L_0, \dots, L_\ell)$;
- positive real numbers $\underline{w} = (w_1, \dots, w_\ell) \in \mathbb{R}^\ell_+$;
- a subset F of $\{1, \dots, \ell\}$;
- transverse intersection points $\underline{p} = (p_1, \dots, p_\ell)$ and q, where

$$p_i \in \phi_{w_i H_\rho + H'_{L_{i-1}, w_i}}(L_{i-1}) \cap L_i \quad \text{and} \quad q \in \phi_{w_{out} H_\rho + H'_{L_0, w_{out}}}(L_0) \cap L_\ell.$$

The notion of transversality (Definition A.1) is modified as follows:

- In condition (1), we now require $\phi_{(\tau+w)H_\rho + H'_{L_i,\tau+w}}(L_i)$ and $\phi_{\tau H_\rho + H'_{L_j,\tau}}(L_j)$ to intersect transversely for all large enough integer values of w and for all $\tau \geq 0$.
- Condition (2) again says that, as w increases, new intersections may be created "at infinity", but may not be the outgoing ends of J-holomorphic discs.
- Condition (3) now requires all relevant moduli spaces of holomorphic discs with boundaries on the Lagrangians $\phi_{\tau_j H_\rho + H'_{L_j,\tau_j}}(L_j)$ to be regular.

Plain cascades are again built out of J-holomorphic discs, taking the additional Hamiltonian perturbations H'_{L_j,τ_j} into account. Given a transverse collection $\underline{L} = (L_0, \dots, L_\ell)$ of exact Lagrangians, a tuple of real numbers $\underline{\tau} = (\tau_0, \dots, \tau_\ell) \in \mathbb{R}^{\ell+1}$, intersection points $\underline{p} = (p_1, \dots, p_\ell)$, $p_i \in \phi_{\tau_{i-1} H_\rho + H'_{L_{i-1},\tau_{i-1}}}(L_{i-1}) \cap \phi_{\tau_i H_\rho + H'_{L_i,\tau_i}}(L_i)$, $q \in \phi_{\tau_0 H_\rho + H'_{L_0,\tau_0}}(L_0) \cap \phi_{\tau_\ell H_\rho + H'_{L_\ell,\tau_\ell}}(L_\ell)$, and a relative homotopy class φ, we now denote by $\mathcal{M}^{\mathrm{hol}}_\ell(\underline{L}; \underline{\tau}; \underline{p}, q; \varphi)$ the moduli space of J-holomorphic maps from the disc with $\ell + 1$ (ordered) boundary marked points to $\hat{M}$, with the boundary arcs mapping to the Lagrangian submanifolds $\phi_{\tau_i H_\rho + H'_{L_i,\tau_i}}(L_i)$ and the marked points mapping to $p_1, \dots, p_\ell, q$, representing the class φ.

With this change of notation understood, plain cascades are built out of holomorphic discs exactly as in Definition A.4, and general cascades are defined in terms of plain cascades as in Definition A.7. With the obvious adaptations, Lemma A.6, Proposition A.10 and Lemma A.11 still hold in this setting. In particular, we now restate Lemma A.11 in the form needed for our purposes:

Lemma A.12. *Let $\{L_i,\ i \in I\}$ be a transverse collection of exact Lagrangian submanifolds of $\hat{M}$, with the following additional properties:*

(1) *for all $i, j \in I$, $\phi_{(\tau+w)H_\rho + H'_{L_i,\tau+w}}(L_i)$ is transverse to $\phi_{\tau H_\rho + H'_{L_j,\tau}}(L_j)$ for all large enough w ($w \geq m = m_{i,j}$) and all $\tau \geq 0$, without any intersections being created or cancelled;*

(2) *given any boundary data $(\underline{L}; \underline{w}, F; \underline{p}, q)$ where $\underline{L} = (L_{i_0}, \dots, L_{i_\ell})$ is a sequence of exact Lagrangians chosen among the L_i, and the widths w_k are large enough ($w_k \geq m_{i_k, i_{k+1}}$), and given $\underline{\tau} = (\tau_0, \dots, \tau_\ell) \in \mathbb{R}^{\ell+1}_{\geq 0}$ with $\tau_k - \tau_{k+1} = w_k$ and a nontrivial relative class φ such that $\mu(\varphi) < 2 - \ell$, the Lagrangian submanifolds*

$\phi_{\tau_k H_\rho + H'_{L_{i_k},\tau_k}}(L_{i_k})$ *do not bound any holomorphic disc in the relative class* φ, *i.e.* $\mathcal{M}_\ell^{\mathrm{hol}}(\underline{L};\underline{\tau};\underline{p},q;\varphi)=\emptyset$.

Then the operations m_ℓ^F *are identically zero for* $F\neq\emptyset$, *except* $m_1^{\{1\}}=\kappa$ *which is the natural isomorphism between Floer complexes induced by the isotopy. Thus, up to quasi-isomorphism we can replace the infinitely generated complex* $\hom(L_i,L_j)$ *by* $CF^*(\phi_{wH_\rho+H'_{L_i,w}}(L_i),L_j)$ *(for any* $w\geq m_{i,j}$*). Moreover, the* ℓ*-fold product operation*

$$m_\ell^\emptyset : CF^*(\phi_{w_\ell H_\rho + H'_{L_{\ell-1},w_\ell}}(L_{\ell-1}),L_\ell)\otimes\cdots\otimes CF^*(\phi_{w_1 H_\rho+H'_{L_0,w_1}}(L_0),L_1)\to \\ \to CF^*(\phi_{w_{out}H_\rho+H'_{L_0,w_{out}}}(L_0),L_\ell)$$

simply counts rigid J*-holomorphic discs in the moduli spaces* $\mathcal{M}_\ell^{\mathrm{hol}}(\underline{L};\underline{\tau};\underline{p},q;\varphi)$, *where* $\tau_i=\sum_{j>i}w_j$ *and we identify the generators of* $CF^*(\phi_{w_iH_\rho+H'_{L_{i-1},w_i}}(L_{i-1}),L_i)$ *with those of* $CF^*(\phi_{\tau_{i-1}H_\rho+H'_{L_{i-1},\tau_{i-1}}}(L_{i-1}),\phi_{\tau_iH_\rho+H'_{L_i,\tau_i}}(L_i))$ *in the natural manner.*

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