APPROXIMATELY HOLOMORPHIC METHODS IN SYMPLECTIC TOPOLOGY

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Lecture notes and bibliography

1. Introduction

A *symplectic structure* on a smooth manifold is a closed non-degenerate 2-form ω (i.e., $d\omega = 0$ and $\omega^n = \mathrm{vol} > 0$). Unlike the Riemannian case where curvature is a local invariant, all symplectic manifolds are locally symplectomorphic to $\mathbb{R}^{2n}$ equipped with the standard form $\omega_0 = \sum dx_i \wedge dy_i$ (Darboux's theorem). The problem of classifying symplectic manifolds is therefore mostly of a topological nature.

Riemann surfaces $(\Sigma, \mathrm{vol}_\Sigma)$ are symplectic manifolds; more generally, any Kähler manifold is symplectic, which includes all complex projective manifolds. However the symplectic category is much larger than that of complex manifolds: for example, Gompf has shown in 1994 that any finitely presented group can be realized as the fundamental group of a compact symplectic 4-manifold [Go1].

Even if it is not complex, every symplectic manifold carries a compatible *almost-complex structure*, i.e. an endomorphism $J \in \mathrm{End}(TX)$ such that $J^2 = -\mathrm{Id}$ and that $g(u,v) := \omega(u, Jv)$ is a Riemannian metric. Pointwise, (X, ω, J) is identical to $(\mathbb{C}^n, \omega_0, i)$, but J is not *integrable*: $\nabla J \neq 0$, $\bar\partial^2 \neq 0$, and the Lie bracket of two vector fields of type $(1,0)$ is not necessarily of type $(1,0)$. As a consequence, there are no holomorphic functions, even locally, and in particular no holomorphic local coordinates.

The problems that symplectic topology aims to solve are questions such as: which smooth manifolds admit a symplectic structure? can one classify the symplectic structures on a given smooth manifold? (Moser's theorem states that, if the cohomology class $[\omega] \in H^2(X, \mathbb{R})$ is fixed, then all deformations of the symplectic structure are trivial). Motivations are both of physical (classical mechanics; string theory; ...) and geometric nature (symplectic manifolds appear as a natural second step once complex manifolds are understood).

Some properties of complex manifolds extend to the symplectic case, but this is far from being a general rule. The situation is best understood in dimension 4, in particular thanks to the work of Taubes on the structure of Seiberg-Witten invariants of symplectic manifolds and their relation with Gromov-Witten invariants [Ta]. However, very little is known when $\dim X \geq 6$: in particular, there is no known non-trivial obstruction to the existence of a symplectic structure on a given compact manifold (besides the existence of $[\omega] \in H^2(X, \mathbb{R})$ such that $[\omega]^{\wedge n} \neq 0$ and that of an almost-complex structure).

2. Approximately holomorphic techniques and symplectic submanifolds

The idea introduced by Donaldson in the mid-90's is the following: in presence of an almost-complex sructure, there are no holomorphic objects (sections, linear systems), but one can work in a similar manner with approximately holomorphic objects.

Let (X, ω) be a compact symplectic manifold of dimension $2n$. We will assume throughout this text that $\frac{1}{2\pi}[\omega] \in H^2(X, \mathbb{Z})$; this integrality condition does not restrict the topological type of X, since any symplectic form can be perturbed to make its cohomology class rational, and then integral by multiplication by a constant factor. Let J be an almost-complex structure compatible with ω, and let $g(.,.) = \omega(., J.)$ be the corresponding Riemannian metric.

We consider a complex line bundle L over X such that $c_1(L) = \frac{1}{2\pi}[\omega]$, endowed with a Hermitian metric and a Hermitian connection ∇^L with curvature 2-form $F(\nabla^L) = -i\omega$. The almost-complex structure induces a splitting of the connection : $\nabla^L = \partial^L + \bar\partial^L$, where $\partial^L s(v) = \frac{1}{2}(\nabla^L s(v) - i\nabla^L s(Jv))$ and $\bar\partial^L s(v) = \frac{1}{2}(\nabla^L s(v) + i\nabla^L s(Jv))$.

If the almost-complex structure J is integrable, i.e. if X is a Kähler complex manifold, then L is an ample holomorphic line bundle, and for large enough values of k the line bundles $L^{\otimes k}$ admit many holomorphic sections. Therefore, the manifold X can be embedded into a projective space (Kodaira); generic hyperplane sections are smooth hypersurfaces in X (Bertini), and more generally the linear system formed by the sections of $L^{\otimes k}$ allows one to construct various structures (Lefschetz pencils, ...).

When the manifold X is only symplectic, the lack of integrability of J prevents the existence of holomorphic sections. Nonetheless, it is possible to find an *approximately holomorphic* local model: a neighborhood of a point $x \in X$, equipped with the symplectic form ω and the almost-complex structure J, can be identified with a neighborhood of the origin in $\mathbb{C}^n$ equipped with the standard symplectic form ω_0 and an almost-complex structure of the form $i+O(|z|)$. In this local model, the line bundle $L^{\otimes k}$ endowed with the connection $\nabla = (\nabla^L)^{\otimes k}$ of curvature $-ik\omega$ can be identified with the trivial line bundle $\underline{\mathbb{C}}$ endowed with the connection $d + \frac{k}{4}\sum(z_j\, d\bar z_j - \bar z_j\, dz_j)$. The section of $L^{\otimes k}$ given in this trivialization by $s_{k,x}(z) = \exp(-\frac{1}{4}k|z|^2)$ is then approximately holomorphic [Do1].

More precisely, a sequence of sections s_k of $L^{\otimes k}$ is said to be approximately holomorphic if, with respect to the rescaled metrics $g_k = kg$, and after normalization of the sections to ensure that $\|s_k\|_{C^r,g_k} \sim C$, an inequality of the form $\|\bar\partial s_k\|_{C^{r-1},g_k} < C'k^{-1/2}$ holds, where C and C' are constants independent of k. The change of metric, which dilates all distances by a factor of $\sqrt{k}$, is required in order to be able to obtain uniform estimates, due to the larger and larger curvature of the line bundle $L^{\otimes k}$. The intuitive idea is that, for large k, the sections of the line bundle $L^{\otimes k}$ with curvature $-ik\omega$ probe the geometry of X at small scale ($\sim 1/\sqrt{k}$), which makes the almost-complex structure J almost integrable and allows one to achieve better and better approximations of the holomorphicity condition $\bar\partial s = 0$.

It is worth noting that, since the above requirement is an open condition, it is not possible to define a "space of approximately holomorphic sections" of $L^{\otimes k}$ in any simple manner (cf. the work of Borthwick and Uribe [BU] or Shiffman and Zelditch for other approaches to this problem).

Once many approximately holomorphic sections have been made available, the aim is to find among them some sections whose geometric behavior is as generic as possible. Donaldson has obtained the following result [Do1]:

Theorem 1 (Donaldson)**.** *For $k \gg 0$, $L^{\otimes k}$ admits approximately holomorphic sections s_k whose zero sets W_k are smooth symplectic hypersurfaces.*

This result starts from the observation that, if the section s_k vanishes transversely and if $|\bar\partial s_k(x)| \ll |\partial s_k(x)|$ at every point of $W_k = s_k^{-1}(0)$, then the submanifold W_k is symplectic (i.e., $\omega_{|W_k}$ is non-degenerate, which implies that $(W_k, \omega_{|W_k})$ is symplectic), and even approximately J-holomorphic (i.e. $J(TW_k)$ is close to TW_k). The crucial point is therefore to obtain a lower bound for ∂s_k at every point of W_k, in order to make up for the lack of holomorphicity.

Sections s_k of $L^{\otimes k}$ are said to be *uniformly transverse to* 0 if there exists a constant $\eta > 0$ (independent of k) such that the inequality $|\partial s_k(x)|_{g_k} > \eta$ holds at any point of X where $|s_k(x)| < \eta$. In order to prove Theorem 1, it is sufficient to achieve this uniform estimate on the tranversality of some approximately holomorphic sections s_k. The idea of the construction of such sections consists of two main steps. The first one is an effective local transversality result for complex-valued functions. Donaldson's argument makes use of a result of Yomdin on the complexity of real semi-algebraic sets; however a somewhat simpler proof has recently been found [Au2]. The second step is a remarkable globalization process, which makes it possible to achieve uniform transversality over larger and larger open subsets by means of successive perturbations of the sections s_k, until transversality holds over the entire manifold X [Do1].

The symplectic submanifolds constructed by Donaldson present several remarkable properties which make them closer to complex submanifolds than to arbitrary symplectic submanifolds. For instance, they satisfy the Lefschetz hyperplane theorem: up to half the dimension of the submanifold, the homology and homotopy groups of W_k are identical to those of X [Do1]. More importantly, these submanifolds are, in a sense, asymptotically unique: for given large enough k, the submanifolds W_k are, up to symplectic isotopy, independent of all the choices made in the construction (including that of the almost-complex structure J) [Au1].

Finally, it is worth noting that an odd-dimensional analogue of Donaldson's construction (for contact manifolds) has been obtained by Ibort, Martinez-Torres and Presas [IMP].

3. Symplectic Lefschetz pencils and fibrations

Considering no longer one, but two sections of $L^{\otimes k}$, Donaldson has shown that any compact symplectic manifold can be equipped with a structure of *symplectic Lefschetz pencil* [Do2, Do3]: a pair of suitably chosen approximately holomorphic sections (s^0_k, s^1_k) of $L^{\otimes k}$ defines a family of symplectic hypersurfaces $\Sigma_{k,\alpha} = \{x \in X,\ s^0_k(x) - \alpha s^1_k(x) = 0\}$, $\alpha \in \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. The submanifolds $\Sigma_{k,\alpha}$ are all smooth except for finitely many of them which present an isolated singularity; they intersect along the *base points* of the pencil, which form a smooth symplectic submanifold $Z_k = \{s^0_k = s^1_k = 0\}$ of codimension 4.

One can also define the projective map $f_k = (s^0_k : s^1_k) : X - Z_k \to \mathbb{CP}^1$, whose critical points correspond to the singularities of the fibers $\Sigma_{k,\alpha}$. The function f_k is a complex Morse function, i.e. near any of its critical points it is given by the local model $f_k(z) = z_1^2 + \cdots + z_n^2$ in approximately holomorphic coordinates.

Donaldson's argument again relies on successive perturbations of the sections s^0_k and s^1_k in order to achieve uniform transversality properties, not only for the sections (s^0_k, s^1_k) but also for the derivative ∂f_k [Do3]. Donaldson also shows that, for given $k \gg 0$, the constructed Lefschetz pencils are all identical up to isotopy, independently of the choices made in the construction.

After blowing up X along Z_k, one obtains a *Lefschetz fibration* $\hat{f}_k : \hat{X} \to \mathbb{CP}^1$, for which one can study the *monodromy* around the singular fibers (corresponding to the critical values of f_k). This monodromy takes values in the symplectic *mapping class group* $\mathrm{Map}^\omega(\Sigma_k, Z_k) = \pi_0(\{\phi \in \mathrm{Symp}(\Sigma_k, \omega),\ \phi_{|V(Z_k)} = \mathrm{Id}\})$, where Σ_k is a generic fiber of $\hat{f}_k$ (corresponding to the choice of a reference point $\alpha \in \mathbb{CP}^1$), and $V(Z_k)$ is a neighborhood of Z_k inside Σ_k. This yields a monodromy homomorphism $\psi_k : \pi_1(\mathbb{C} - \mathrm{crit}\, f_k) \to \mathrm{Map}^\omega(\Sigma_k, Z_k)$. The monodromy around a singular fiber is a (generalized) *Dehn twist* along the *vanishing cycle*, which is an embedded Lagrangian sphere $S^{n-1} \subset \Sigma_k$.

The most studied case is when X is a 4-manifold: the fibers are then Riemann surfaces, and Z_k is a finite set of points. The group $\mathrm{Map}^\omega(\Sigma_k, Z_k)$ therefore identifies with the mapping class group $\mathrm{Map}_{g,N}$ of a Riemann surface of genus $g = g(\Sigma_k)$ with $N = \mathrm{card}\, Z_k$ punctures, and the monodromy around a singular fiber (a Riemann surface with an ordinary double point) is a Dehn twist along an embedded loop.

Donaldson's asymptotic uniqueness result implies that, for large enough k, the monodromy of a Lefschetz pencil constructed from approximately holomorphic sections of $L^{\otimes k}$ is an invariant of the symplectic manifold (X, ω). Conversely, Gompf has shown that the monodromy morphism completely determines the manifold X together with its symplectic structure; in fact, in the 4-dimensional case, the total space of any "topological Lefschetz fibration" over $\mathbb{CP}^1$ (with at least one singular fiber) always carries a symplectic structure [GS].

The geometric and topological properties of 4-dimensional Lefschetz pencils and fibrations have been extensively studied over the past few years; see e.g. [ABKP], [Sm1], [EK]. For example, Smith has shown that (relatively minimal) symplectic Lefschetz fibrations admitting sections of square -1 cannot be decomposed into non-trivial fiber sums [Sm2]; Siebert and Tian have shown that all genus 2 Lefschetz fibrations without reducible singular fibers and with transitive monodromy are in fact holomorphic [ST]. Moreover, Donaldson and Smith have shown that certain Gromov-Witten type invariants counting sections of higher-dimensional fibrations associated to a Lefschetz pencil can be used to reprove various results of Taubes on the existence of symplectic curves realizing certain homology classes without using Seiberg-Witten theory [DS]. In a different direction, the work of Seidel shows that it is possible to associate to a symplectic Lefschetz pencil a "directed Fukaya category", involving the Floer homology of the vanishing cycles of the fibration [Se]; this construction, which in the 4-dimensional case is purely combinatorial, provides information on Floer homology for Lagrangian spheres in the total space of the pencil.

4. Branched covers of $\mathbb{CP}^2$, linear systems and projective maps

We now consider a linear system generated by three approximately holomorphic sections (s_k^0, s_k^1, s_k^2) of $L^{\otimes k}$: for $k \gg 0$, it is again possible to ensure a generic behavior of the ($\mathbb{CP}^2$-valued) projective map $f_k = (s_k^0 : s_k^1 : s_k^2)$ associated to the linear system.

Theorem 2 ([Au3, Au4])**.** *For large enough k, three suitably chosen approximately holomorphic sections of $L^{\otimes k}$ determine a map $f_k : X - \{\text{base points}\} \to \mathbb{CP}^2$ with generic local models, in a canonical way up to isotopy.*

When the manifold X has dimension 4, the linear system has no base points, and the projective map f_k is a *branched covering.* Given any point $x \in X$, there exist approximately holomorphic coordinates over neighborhoods of x and $f_k(x)$ such that f_k can be identified with one of the three generic local models for a holomorphic map from $\mathbb{C}^2$ to itself: $(u, v) \mapsto (u, v)$ (local diffeomorphism), $(u, v) \mapsto (u^2, v)$ (branching of order 2) or $(u, v) \mapsto (u^3 - uv, v)$ (cusp).

When $\dim X > 4$, the base locus $Z_k = \{s_k^0 = s_k^1 = s_k^2 = 0\}$ is a smooth symplectic submanifold, of real codimension 6; near a point of Z_k, a local model for f_k is $(z_1, \dots, z_n) \mapsto (z_1 : z_2 : z_3)$. Away from Z_k, the three generic local models become respectively: $(z_1, \dots, z_n) \mapsto (z_1, z_2)$; $(z_1, \dots, z_n) \mapsto (z_1^2 + \dots + z_{n-1}^2, z_n)$; $(z_1, \dots, z_n) \mapsto (z_1^3 - z_1 z_n + z_2^2 + \dots + z_{n-1}^2, z_n)$.

In all cases, the set of critical points of f_k is a (connected) smooth symplectic curve $R_k \subset X$. However, its image $D_k = f_k(R_k) \subset \mathbb{CP}^2$ (the *branch curve*, or *discriminant curve*) is immersed only outside of the points where the third local model holds; at those points, it presents a *complex cusp* $(27x^2 = 4y^3)$. The other generic singularities of D_k are *nodes* (transverse double points), which do not appear in the local models because they correspond to ramification phenomena at two distinct points in the same fiber of f_k; although D_k is approximately holomorphic, both orientations are a priori possible for these nodes, contrarily to the complex case. For large k, the topology of the curve D_k is therefore a symplectic invariant only up to creation or cancellation of admissible pairs of nodes.

The idea of the proof of Theorem 2 is to enumerate the various possibilities, generic or not, for the behavior of the map f_k at a given point of X; each one corresponds to the vanishing of a certain quantity that can be expressed in terms of the sections s_k^0, s_k^1, s_k^2 and their derivatives. In fact, this determines a *stratification* of a jet bundle by approximately holomorphic submanifolds. The main ingredient is then a uniform transversality result for jets of approximately holomorphic sections with respect to such stratifications [Au5], which can be proved using methods similar to Donaldson's arguments.

The topological data characterizing a branched covering of $\mathbb{CP}^2$ consist of the branch curve $D_k \subset \mathbb{CP}^2$ (up to isotopy and cancellations of pairs of nodes), and a monodromy morphism $\theta_k : \pi_1(\mathbb{CP}^2 - D_k) \to S_N$ describing the configuration of the $N = \deg f_k$ sheets of the covering above $\mathbb{CP}^2 - D_k$. Conversely, given a symplectic curve D_k and a morphism θ_k (satisfying certain elementary compatibility conditions), it is possible to recover not only the smooth 4-manifold X but also its symplectic structure: up to isotopy, the symplectic form $k\omega$ can be expressed as

$f_k^*\omega_{FS} + \epsilon\alpha$, where $\epsilon > 0$ is a small constant and α is an *exact* 2-form which restricts positively to $\operatorname{Ker} df_k$ at every point of the ramification curve.

The topology of a singular curve $D \subset \mathbb{CP}^2$ can be studied by means of the braid group techniques introduced in complex geometry by Moishezon and Teicher [Mo1, Te] : the idea is to choose a linear projection $\pi : \mathbb{CP}^2 - \{\mathrm{pt}\} \to \mathbb{CP}^1$, e.g. $\pi(x:y:z) = (x:y)$, in such a way that the curve D is in general position with respect to the fibers of π. The restriction $\pi_{|D}$ is then a singular branched covering of degree $d = \deg D$; besides the singular points of D (nodes and cusps), one must also consider *tangency points*, where D becomes tangent to the fibers of π.

Except for those which contain a special point of D (cusp, node or tangency), the fibers of π are lines intersecting the curve D in d distinct points. After choosing a reference point in $\mathbb{CP}^1$ (and the corresponding fiber $\ell \simeq \mathbb{C} \subset \mathbb{CP}^2$ of π), and after restriction to an affine open subset in order to trivialize the fibration π, the topology of the branched covering $\pi_{|D}$ can be described by a morphism $\rho : \pi_1(\mathbb{C} - \{\mathrm{pts}\}) \to B_d$, where B_d is the braid group on d strings: the braid $\rho(\gamma)$ describes the the motion of the d points of $\ell \cap D$ inside the fibers of π as one moves along the loop γ. Equivalently, the morphism ρ can be replaced by a *factorization* in the braid group B_d, involving the monodromy around each of the special points of D. The morphism ρ and the corresponding factorization depend on trivialization choices; they are only defined up to simultaneous conjugation (corresponding to a change of trivialization of the reference fiber ℓ) and Hurwitz operations (change of generators for $\pi_1(\mathbb{C} - \{\mathrm{pts}\})$). There is a one-to-one correspondence between morphisms $\rho : \pi_1(\mathbb{C} - \{\mathrm{pts}\}) \to B_d$, up to Hurwitz operations and conjugation, and singular plane curves of degree d compatible with the projection π, up to isotopy [AK].

Unlike the complex case, it is not clear in the symplectic case that the branch curve D_k can satisfy the expected properties with respect to any linear projection π; in fact, this requires an improved version of Theorem 2 [AK, Au4]. Moreover, one must take into account the possibility of admissible creations or cancellations of pairs of nodes in D_k, which affect the morphism $\rho_k : \pi_1(\mathbb{C} - \{\mathrm{pts}\}) \to B_d$ by insertions or deletions of pairs of factors. The uniqueness statement of Theorem 2 then implies the following result:

Theorem 3 ([AK]). *Given a large enough value of k, the combinatorial data (ρ_k, θ_k) are, up to conjugation, Hurwitz operations and insertions or deletions, invariants of the symplectic manifold (X, ω). Moreover, these are* complete *invariants, i.e. from the morphisms ρ_k and θ_k it is possible to reconstruct the symplectic manifold (X, ω) up to symplectomorphism.*

It is worth mentioning that the symplectic Lefschetz pencils constructed by Donaldson can be recovered very easily from the branched coverings f_k, simply by considering the compositions $\pi \circ f_k$ with values in $\mathbb{CP}^1$. Therefore, the fibers $\Sigma_{k,\alpha}$ of the pencil are the preimages by f_k of the fibers of π, and the singular fibers of the pencil correspond to the tangency points of D_k.

In fact, the monodromy morphisms ψ_k describing the Lefschetz pencils can be obtained explicitly from θ_k and ρ_k : the restriction to the line $\bar{\ell} = \ell \cup \{\infty\}$ of the S_N-valued morphism θ_k describes the topology of a fiber of the pencil as an N-fold simple branched covering of $\mathbb{CP}^1$ with d branch points, which makes it possible to define a *lifting homomorphism* $(\theta_k)_*$ from a subgroup of B_d with values in $\mathrm{Map}(\Sigma_k, Z_k) = \mathrm{Map}_{g,N}$. The monodromy of the Lefschetz pencil is then given by $\psi_k = (\theta_k)_* \circ \rho_k$ [AK].

When $\dim X > 4$, the topology of the projective map f_k and of the curve $D_k \subset \mathbb{CP}^2$ can again be described in the same manner, with the only difference that the morphism θ_k describing the monodromy of the fibration above $\mathbb{CP}^2 - D_k$ now takes values in the symplectic mapping class group $\mathrm{Map}^\omega(\Sigma_k, Z_k)$ of the generic fiber of f_k. Theorem 3 remains true in this context [Au4]; however, these invariants are seldom usable in practice, because the structure of the group $\mathrm{Map}^\omega(\Sigma_k, Z_k)$ is essentially unknown when $\dim \Sigma_k \geq 4$.

However, this difficulty can be avoided by means of a dimensional induction process. Indeed, the restriction of f_k to the line $\bar{\ell} \subset \mathbb{CP}^2$ determines a Lefschetz pencil structure on a symplectic hypersurface $W_k \subset X$, with generic fiber Σ_k and monodromy θ_k. This structure can be enriched by introducing an additional section of $L^{\otimes k}$ so as to obtain a map from W_k to $\mathbb{CP}^2$, which can again be described by monodromy invariants, and so on. At the end of the process, given a symplectic manifold (X^{2n}, ω) and an integer $k \gg 0$, one obtains $n-1$ singular curves $D_k^{(n)}, D_k^{(n-1)}, \dots, D_k^{(2)} \subset \mathbb{CP}^2$, described by $n-1$ braid group-valued morphisms, and a morphism $\theta_k^{(2)}$ from $\pi_1(\mathbb{CP}^2 - D_k^{(2)})$ with values in a symmetric group. These invariants determine the manifold (X, ω) up to symplectomorphism [Au4].

In view of the above results, the classification of symplectic manifolds seems to largely reduce to the study of certain singular plane curves, or equivalently of certain words in braid groups. However, even though the invariants defined by Theorem 3 are explicitly computable for various examples ($\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$ [Mo2], some complete intersections [Ro], the Hirzebruch surface $\mathbb{F}_1$, and arbitrary double covers of $\mathbb{CP}^1 \times \mathbb{CP}^1$ [ADKY]; cf. also [AK2]), they cannot be used directly to compare two symplectic manifolds, because no algorithm is available to compare to words in a braid group up to Hurwitz operations. Therefore, one must search for less complete but more usable invariants.

In the case of complex surfaces, Moishezon and Teicher have studied the fundamental group $\pi_1(\mathbb{CP}^2 - D)$ as an effectively computable invariant to study a branch curve $D \subset \mathbb{CP}^2$ [Mo2, Te]; the monodromy morphism ρ provides an explicit presentation of $\pi_1(\mathbb{CP}^2 - D)$ via the Zariski-Van Kampen theorem. In the symplectic case, the possibility of node creations or cancellations makes it necessary to consider a certain quotient $\bar{G}$ (the *stabilized fundamental group*) rather than $\pi_1(\mathbb{CP}^2 - D)$ itself; this quotient is a symplectic invariant [ADKY].

It is worth mentioning that, for all known examples, for large enough k the stabilization operation becomes trivial ($\bar{G}_k = \pi_1(\mathbb{CP}^2 - D_k)$). Moreover, the structure of the group $\bar{G}_k$ for these examples is remarkably simple, and various indications lead to conjecture that, at least when the manifold X is simply connected, the structure of $\bar{G}_k$ can be described entirely in terms of the cohomology of X [ADKY]; therefore, a lot of information seems to be lost when considering only the fundamental group of the complement of the branch curve rather than its braid monodromy. This suggests that other constructions will need to be investigated before monodromy invariants can be used successfully in classification problems.

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