MIRROR SYMMETRY: LECTURE 3

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Last time, we say that a deformation of (X, J) is given by

$$\{s \in \Omega^{0,1}(X, TX) | \bar{\partial} s + \frac{1}{2}[s, s] = 0\} / \mathrm{Diff}(X) \tag{1}$$

To first order, these are determined by $\mathrm{Def}_1(X, J) = H^1(X, TX)$, but extending these to higher order is obstructed by elements of $H^2(X, TX)$. In the Calabi-Yau case, recall that:

Theorem 1 (Bogomolov-Tian-Todorov). *For X a compact Calabi-Yau ($\Omega_X^{n,0} \cong \mathcal{O}_X$) with $H^0(X, TX) = 0$ (automorphisms are discrete), deformations of X are unobstructed.*

Note that, if X is a Calabi-Yau manifold, we have a natural isomorphism $TX \cong \Omega_X^{n-1}, v \mapsto i_v \Omega$, so

$$H^0(X, TX) = H^{n-1,0}(X) \cong H^{0,1} \tag{2}$$

and similarly

$$H^1(X, TX) = H^{n-1,1}, H^2(X, TX) = H^{n-1,2} \tag{3}$$

1. Hodge theory

Given a Kähler metric, we have a Hodge $*$ operator and L^2-adjoints

$$d^* = - * d *, \bar{\partial}^* = - * \partial * \tag{4}$$

and Laplacians

$$\Delta = dd^* + d^* d, \bar{\Box} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^* \bar{\partial} \tag{5}$$

Every $(d/\bar{\partial})$-cohomology class contains a unique harmonic form, and one can show that $\bar{\Box} = \frac{1}{2}\Delta$. We obtain

$$\begin{aligned} H^k_{dR}(X, \mathbb{C}) &\cong \mathrm{Ker}\ (\Delta : \Omega^k(X, \mathbb{C}) \circlearrowleft) = \mathrm{Ker}\ (\bar{\Box} : \Omega^k \circlearrowleft) \\ &\cong \bigoplus_{p+q=k} \mathrm{Ker}\ (\bar{\Box} : \Omega^{p,q} \circlearrowleft) \cong \bigoplus_{p+q=k} H^{p,q}_{\bar{\partial}}(X) \end{aligned} \tag{6}$$

The Hodge $*$ operator gives an isomorphism $H^{p,q} \cong H^{n-p,n-q}$. Complex conjugation gives $H^{p,q} \cong \overline{H^{q,p}}$, giving us a *Hodge diamond*

$$\begin{array}{ccccc}
h^{n,n} & h^{n-1,n} & \cdots & \cdots & h^{0,n} \\
h^{n,n-1} & h^{n-1,n-1} & \cdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \cdots & h^{1,1} & h^{0,1} \\
h^{n,0} & \cdots & \cdots & h^{1,0} & h^{0,0}
\end{array} \tag{7}$$

For a Calabi-Yau, we have

$$H^{p,0} \cong H^{n,n-p} = H^{n-p}_{\overline{\partial}}(X, \Omega^n_X) \cong H^{n-p}_{\overline{\partial}}(X, \mathcal{O}_X) = H^{0,n-p} \cong \overline{H^{n-p,0}} \tag{8}$$

Specifically, for a Calabi-Yau 3-fold with $h^{1,0} = 0$, we have a reduced Hodge diamond

$$\begin{array}{cccc}
1 & 0 & 0 & 1 \\
0 & h^{1,1} & h^{2,1} & 0 \\
0 & h^{2,1} & h^{1,1} & 0 \\
1 & 0 & 0 & 1
\end{array} \tag{9}$$

Mirror symmetry says that there is another Calabi-Yau manifold whose Hodge diamond is the mirror image (or 90 degree rotation) of this one.

There is another interpretation of the Kodaira-Spencer map $H^1(X, TX) \cong H^{n-1,1}$. For $\mathcal{X} = (X, J_t)_{t \in S}$ a family of complex deformations of (X, J), $c_1(K_X) = -c_1(TX) = 0$ implies that $\Omega^n_{(X,J_t)} \cong \mathcal{O}_X$ under the assumption $H^1(X) = 0$, so we don't have to worry about deforming outside the Calabi-Yau case. Then $\exists [\Omega_t] \in H^{n,0}_{J_t}(X) \subset H^n(X, \mathbb{C})$. How does this depend on t? Given $\frac{\partial}{\partial t} \in T_0 S$, $\frac{\partial t}{\partial \Omega t} \in \Omega^{n,0} \oplus \Omega^{n-1,1}$ by Griffiths transversality:

$$\alpha_t \in \Omega^{p,q}_{J_t} \implies \frac{\partial}{\partial t} \alpha_t \in \Omega^{p,q} + \Omega^{p-1,q+1} + \Omega^{p+1,q-1} \tag{10}$$

Since $\frac{\partial \Omega_t}{\partial t}|_{t=0}$ is d-closed ($d\Omega_t = 0$), $(\frac{\partial \Omega_t}{\partial t}|_{t=0})^{(n-1,1)}$ is $\overline{\partial}$-closed, while

$$\overline{\partial}(\frac{\partial \Omega_t}{\partial t}|_{t=0})^{(n-1,1)} + \overline{\partial}(\frac{\partial \Omega_t}{\partial t}|_{t=0})^{(n-1,1)} = 0 \tag{11}$$

Thus, $\exists[(\frac{\partial \Omega_t}{\partial t}|_{t=0})^{(n-1,1)}] \in H^{n-1,1}(X)$.

For fixed Ω_0, this is independent of the choice of Ω_t. If we rescale $f(t)\Omega_t$,

$$\frac{\partial}{\partial t}(f(t)\Omega_t) = \frac{\partial f}{\partial t}\Omega_t + f(t)\frac{\partial \Omega_t}{\partial t} \tag{12}$$

Taking $t \to 0$, the former term is $(n, 0)$, while for the latter, $f(0)$ scales linearly with Ω^0.

$$H^{n-1,1}(X) = H^1(X, \Omega_X^{n-1}) \cong H^1(X, TX) \tag{13}$$

and the two maps $T_0 S \to H^{n-1,1}(X), H^1(X, TX)$ agree. Hence, for $\theta \in H^1(X, TX)$ a first-order deformation of complex structure, $\theta \cdot \Omega \in H^1(X, \Omega_X^n \otimes TX) = H^{n-1,1}(X)$ and (the Gauss-Manin connection) $[\nabla_\theta \Omega]^{(n-1,1)} \in H^{n-1,1}(X)$ are the same. We can iterate this to the third-order derivative: on a Calabi-Yau three-fold, we have

$$\langle \theta_1, \theta_2, \theta_3 \rangle = \int_X \Omega \wedge (\theta_1 \cdot \theta_2 \cdot \theta_3 \cdot \Omega) = \int_X \Omega \wedge (\nabla_{\theta_1} \nabla_{\theta_2} \nabla_{\theta_3} \Omega) \tag{14}$$

where the latter wedge is of a $(3,0)$ and a $(0,3)$ form.

2. Pseudoholomorphic curves

(reference: McDuff-Salamon) Let (X^{2n}, ω) be a symplectic manifold, J a compatible almost-complex structure, $\omega(\cdot, J\cdot)$ the associated Riemannian metric. Furthermore, let (Σ, j) be a Riemann surface of genus g, $z_1, \dots, z_k \in \Sigma$ market points. There is a well-defined moduli space $\mathcal{M}_{g,k} = \{(\Sigma, j, z_1, \dots, z_k)\}$ modulo biholomorphisms of complex dimension $3g - 3 + k$ (note that $\mathcal{M}_{0,3} = \{\text{pt}\}$).

Definition 1. *$u : \Sigma \to X$ is a J-holomorphic map if $J \circ du = du \circ J$, i.e. $\overline{\partial}_J u = \frac{1}{2}(du + Jduj) = 0$. For $\beta \in H_2(X, \mathbb{Z})$, we obtain an associated moduli space*

$$M_{g,k}(X, J, \beta) = \{(\Sigma, j, z_1, \dots, z_k), u : \Sigma \to X | u_*[\Sigma] = \beta, \overline{\partial}_J u = 0\} / \sim \tag{15}$$

where $\sim$ is the equivalence given by ϕ below.

$$\begin{tikzcd} \Sigma, z_1, \dots, z_k \arrow[r, "u"] \arrow[d, "\phi"', "\cong"] & X \\ \Sigma', z_1', \dots, z_k' \arrow[ur, "u'"'] & \end{tikzcd} \tag{16}$$

This space is the zero set of the section $\overline{\partial}_J$ of $\mathcal{E} \to \mathrm{Map}(\Sigma, X)_\beta \times \mathcal{M}_{g,k}$, where $\mathcal{E}$ is the (Banach) bundle defined by $\mathcal{E}_u = W^{r,p}(\Sigma, \Omega_\Sigma^{0,1} \otimes u^* TX)$.

We can define a linearized operator

$$D_{\overline{\partial}} : W^{r+1,p}(\Sigma, u^*TX) \times T\mathcal{M}_{g,k} \to W^{r,p}(\Sigma, \Omega^{0,1}_\Sigma \otimes U^*TX)$$

$$\begin{aligned} D_{\overline{\partial}}(v, j') &= \frac{1}{2}(\nabla v + J\nabla v j + (\nabla_v J) \cdot du \cdot j + J \cdot du \cdot j') \\ &= \overline{\partial} v + \frac{1}{2}(\nabla_v J) du \cdot j + \frac{1}{2} J \cdot du \cdot j' \end{aligned} \tag{17}$$

This operator is Fredholm, with real index

$$\mathrm{index}_{\mathbb{R}} D_{\overline{\partial}} := 2d = 2\langle c_1(TX), \beta\rangle + n(2-2g) + (6g - 6 + 2k) \tag{18}$$

One can ask about transversality, i.e. whether we can ensure that $D_{\overline{\partial}}$ is onto at every solution. We say that u is *regular* if this is true at u: if so, $\mathcal{M}_{g,k}(X, J, \beta)$ is smooth of dimension $2d$.

Definition 2. *We say that a map $\Sigma \to X$ is* simple *(or "somewhere injective") if $\exists z \in \Sigma$ s.t. $du(z) \neq 0$ and $u^{-1}(u(z)) = \{z\}$.*

Note that otherwise u will factor through a covering $\Sigma \to \Sigma'$. We set $\mathcal{M}^*_{g,k}(X, J, \beta)$ to be the moduli space of such simple curves.

Theorem 2. *Let $\mathcal{J}(X, \omega)$ be the set of compatible almost-complex structures on X: then*

$$\mathcal{J}^{reg}(X, \beta) = \{J \in \mathcal{J}(X, \omega) |\ \textit{every simple } J\textit{-holomorphic curve in class } \beta \textit{ is regular}\} \tag{19}$$

is a Baire subset in $\mathcal{J}(X, \omega)$, and for $J \in \mathcal{J}^{reg}(X, \beta)$, $\mathcal{M}^*_{g,k}(X, J, \beta)$ is smooth (as an orbifold, if $\mathcal{M}_{g,k}$ is an orbifold) of real dimension $2d$ and carries a natural orientation.

The main idea here is to view $\overline{\partial}_J u = 0$ as an equation on $\mathrm{Map}(\Sigma, X) \times \mathcal{M}_{g,k} \times \mathcal{J}(X, \omega) \ni (u, j, J)$. Then $D_{\overline{\partial}}$ is easily seen to be surjective for simple maps. We have a "universal moduli space" $\tilde{M M}^* \stackrel{\pi_J}{\to} \mathcal{J}(X, \omega)$ given by a Fredholm map, and by Sard-Smale, a generic J is a regular value of π_J. This universal moduli space is $\mathcal{M}^* = \bigsqcup_{J \in \mathcal{J}(X,\omega)} \mathcal{M}^*_{g,k}(X, J, \beta)$. For such J, $\mathcal{M}^*_{g,k}(X, J, \beta)$ is smooth of dimension $2d$, and the tangent space is Ker $(D_{\overline{\partial}})$. For the orientability, we need an orientation on Ker $(D_{\overline{\partial}})$. If J is integrable, the $D_{\overline{\partial}}$ is $\mathbb{C}$-linear ($D_{\overline{\partial}} = \overline{\partial}$), so Ker is a $\mathbb{C}$-vector space. Moreover, $\forall J_0, J_1 \in \mathcal{J}^{reg}(X, \beta)$, $\exists$ a (dense set of choices of) path $\{J_t\}_{t \in [0,1]}$ s.t. $\bigsqcup_{t \in [0,1]} \mathcal{M}^*_{g,k}(X, J_t, \beta)$ is a smooth oriented cobordism. We still need compactness.