MIRROR SYMMETRY: LECTURE 25

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Last time, we were considering $\mathbb{CP}^1$ mirror to $\mathbb{C}^*, W = z + \frac{e^{-\Lambda}}{z}$ for $\Lambda = 2\pi \int_{\mathbb{CP}^1} \omega$: the latter object is a Landau-Ginzburg model, i.e. a Kähler manifold with a holomorphic function called the "superpotential". Homological mirror symmetry gave

$$\begin{aligned} D^\pi \mathrm{Fuk}(\mathbb{CP}^1) &\cong H^0 MF(W) \\ D^b \mathrm{Coh}(\mathbb{CP}^1) &\cong D^b \mathrm{Fuk}(\mathbb{C}^*, W) \end{aligned} \tag{1}$$

We stated that the Fukaya category of $\mathbb{CP}^1$ was a collection indexed by "charge" $\lambda \in \mathbb{C}$, and defined $\mathrm{Fuk}(\mathbb{CP}^1, \lambda)$ to be the set of weakly unobstructed Lagrangians with $m_0 = \lambda \cdot [L]$. This is an honest A_∞-category, as the m_0's cancel and the Floer differential squares to zero, whereas from λ to λ' we'd have $\partial^2 = \lambda' - \lambda$. For instance, for $L \cong S^1$, (L, ∇) is weakly unobstructed, with $m_0 = W(L, \nabla) \cdot [L]$. However, $HF(L, L) = 0$ unless L is the equator and $\mathrm{hol}(\nabla) = \pm \mathrm{id}$. Then $L_\pm$ has $HF \cong H^*(S^1, \mathbb{C})$ with deformed multiplicative structure, $HF^*(L, L) \cong \mathbb{C}[t]/t^2 = \pm e^{-\Lambda/2}$.

We now look at the matrix factorizations of $W - \lambda, \lambda \in \mathbb{C}$. These are $\mathbb{Z}/2\mathbb{Z}$-graded projective modules Q over the ring of Laurent polynomials $R = \mathbb{C}[\mathbb{C}^*] \cong \mathbb{C}[z^{\pm 1}]$ equipped with $\delta \in \mathrm{End}^1(Q)$ s.t. $\delta^2 = (W - \lambda) \cdot \mathrm{id}_Q$. That is, we have maps $\delta_0 : Q_0 \to Q_1, \delta_1 : Q_1 \to Q_0$ given by matrices with entries in the space of Laurent polynomials s.t. $\delta_0 \circ \delta_1 = (W - \lambda) \cdot \mathrm{id}_{Q_1}, \delta_1 \circ \delta_0 = (W - \lambda) \cdot \mathrm{id}_{Q_0}$. Now $\mathrm{Hom}(Q, Q')$ is $\mathbb{Z}/2\mathbb{Z}$ graded, with

$$\mathrm{Hom}^0 = \left\{ \begin{tikzcd} Q_0 \arrow[r, shift left, "\delta_0"] \arrow[d, "f_0"'] & Q_1 \arrow[l, shift left, "\delta_1"] \arrow[d, "f_1"] \\ Q_0' \arrow[r, shift left, "\delta_0"] & Q_1' \arrow[l, shift left, "\delta_1"] \end{tikzcd} \right\} \tag{2}$$

This has a differential ∂ s.t. $\partial(f) = \delta' \cdot f \pm f \cdot \delta$ and $\partial^2 = 0$. We obtain a homology category $H^0 MF(W - \lambda)$: hom $= H^0(\mathrm{Hom}, \partial)$, i.e. "chain maps" up to "homotopy".

Theorem 1. $H^0(MF(W - \lambda)) = 0$, *i.e. all matrix factorizations are nullhomotopic, unless λ is a critical value of W.*

Warning: again, looking at homomorphisms from $MF(W-\lambda)$ to $MF(W-\lambda')$, then $\partial^2 \neq 0$, $\partial^2(f) = \partial'^2 \cdot f - f \cdot \partial^2 = (\lambda - \lambda')f$.

Example. $W = z + \frac{e^{-\lambda}}{z}$ has critical points $\pm e^{-\Lambda/2}$ with critical values $\pm 2e^{-\Lambda/2}$. Then

$$\begin{gathered} W \pm 2e^{-\Lambda/2} = z \pm 2e^{-\Lambda/2} + \frac{e^{-\lambda}}{z} = (z \pm e^{-\Lambda/2})(1 \pm \frac{e^{-\Lambda/2}}{z}) \\ Q_\pm = \{\ \mathbb{C}[z^{\pm 1}] \underset{1 \pm e^{-\Lambda/2} z^{-1}}{\overset{z \pm e^{-\Lambda/2}}{\rightleftarrows}} \mathbb{C}[z^{\pm 1}]\ \} \end{gathered} \tag{3}$$

Then

$$\mathrm{End}_{H^0 MF}(Q_\pm) = \left\{ \begin{tikzcd} R \arrow[r, shift left] \arrow[d, "f"] & R \arrow[l, shift left] \arrow[d, "f"] \\ R \arrow[r, shift left] & R \arrow[l, shift left] \end{tikzcd} \right\} / \text{homotopy} \tag{4}$$

is multiplication by $f \in \mathbb{C}[z^{\pm 1}]$. The maps ∂ sends

$$\begin{tikzcd} R \arrow[r, shift left] & R \arrow[l, shift left] \arrow[dl, "h"] \\ R \arrow[r, shift left] & R \arrow[l, shift left] \end{tikzcd} \mapsto \begin{tikzcd} R \arrow[r, shift left] \arrow[d, "(x \pm e^{-\Lambda/2})h"'] & R \arrow[l, shift left] \arrow[d] \\ R \arrow[r, shift left] & R \arrow[l, shift left] \end{tikzcd} \tag{5}$$

and similarly on the other side, so

$$\mathrm{End}(Q_\pm) = \mathbb{C}[z^{\pm 1}]/(z \pm e^{-\Lambda/2}, 1 + \pm e^{-\Lambda/2} z^{-1}) \cong (\mathbb{C}[z^{\pm 1}]/z \pm e^{-\Lambda/2}) \cong \mathbb{C} \tag{6}$$

Similarly $\mathrm{Hom}_{H^0 MF}(Q_\pm, Q_\pm[1]) \cong \mathbb{C}$.

Indeed, in the case of the two maps $z - c, 1 - cz^{-1}$, we take vertical maps $z, 1$, so

$$\begin{tikzcd} R \arrow[r, shift left, "z-c"] \arrow[d, "z"'] & R \arrow[l, shift left, "1-cz^{-1}"] \arrow[d, "1"] \\ R \arrow[r, shift left, "1-cz^{-1}"] & R \arrow[l, shift left, "z-c"] \end{tikzcd} \tag{7}$$

giving us $\mathbb{C}[z^{\pm 1}]/\langle z - c \rangle$.

Next, $D^b\mathrm{Coh}(\mathbb{CP}^1)$ is generated by $\mathcal{O}(-1)$ and $\mathcal{O}$, i.e. the smallest full subcategory containing $\mathcal{O}, \mathcal{O}(-1)$ and closed under shifts and cones contains all of D^b. More generally, via Beilinson we have that

$$D^b\mathrm{Coh}(\mathbb{CP}^n) = \langle \mathcal{O}(-n), \ldots, \mathcal{O}(-1), \mathcal{O} \rangle \tag{8}$$

The idea is the the diagonal $\Delta \subset \mathbb{CP}^n \times \mathbb{CP}^n$ is the (transverse) zero set of $s = \sum_{i=0}^n \frac{\partial}{\partial x_i} \otimes y_i$, which is a section of $E = T(-1) \boxtimes \mathcal{O}(1) = \pi_1^*(T\mathbb{CP}^n \otimes$

$\mathcal{O}(-1)) \otimes \pi_2^*\mathcal{O}(1)$. Recall that $T\mathbb{CP}^n$ is spanned by the vector fields $x_i\frac{\partial}{\partial x_i}$ on $\mathbb{C}^{n+1}$ under the relation $\sum_{i=0}^n x_i\frac{\partial}{\partial x_i} = 0$. Taking the Koszul resolution

$$0 \to E^* = \Omega^1(1) \boxtimes \mathcal{O}(-1) \to \mathcal{O} \boxtimes \mathcal{O} \to \mathcal{O}_\Delta \to 0 \tag{9}$$

in $D^b\mathrm{Coh}(\mathbb{P}^1 \times \mathbb{P}^1)$. On the other hand, $\mathcal{E} \in D^b\mathrm{Coh}(X \times Y)$ gives $\phi^{\mathcal{E}} : D^b(\mathrm{Coh}(X) \to D^b\mathrm{Coh}(Y), \mathcal{F} \mapsto R\pi_{2*}(L\pi_1^*\mathcal{F} \overset{L}{\otimes} \mathcal{E})$. Exactness implies that $\phi^{\mathcal{O}_\Delta}(\mathcal{F}) \cong \mathcal{F}$ sits in an exact triangle with

$$\begin{aligned} \phi^{\Omega^1 \boxtimes \mathcal{O}(-1)}(\mathcal{F}) &\cong R\Gamma(\mathcal{F} \otimes \Omega^1(1)) \otimes_{\mathbb{C}} \mathcal{O}(-1) \\ \phi^{\mathcal{O} \boxtimes \mathcal{O}}(\mathcal{F}) &\cong R\Gamma(\mathcal{F}) \otimes_{\mathbb{C}} \mathcal{O} \end{aligned} \tag{10}$$

which completes the proof.

The algebra of the exceptional collection $\langle \mathcal{O}(-1), \mathcal{O} \rangle$ is given by

$$\mathcal{A} = \mathrm{End}^*(\mathcal{O}(-1) \oplus \mathcal{O}) \tag{11}$$

and $D^B\mathrm{Coh}(\mathbb{CP}^1)$ is isomorphic to the derived category of finitely-generated $\mathcal{A}$-modules.

Finally, the Fukaya category of $(\mathbb{C}^*, W = z + \frac{e^{-\Lambda}}{2})$ is the category whose objects are admissible Lagrangians with flat connections, i.e. L is a (possibly noncompact) Lagrangian submanifold with $W|_L$ proper, $W|_L \in \mathbb{R}_+$ outside a compact subset. We can perturb such L: for $a \in \mathbb{R}$, let $L^{(a)}$ be Hamiltonian isotopic to L, $W(L^{(a)}) \in \mathbb{R}_+ + ia$ near ∞. In good cases, it will be the Hamiltonian flow of $X_{\mathrm{Re}\ (W)} = \nabla \mathrm{Im}\ W$. Then $\mathrm{Hom}(L, L') = CF^*(L^{(a)}, L'^{(a')})$ for $a > a'$ (the Floer differential is well-defined), and we obtain $m_k, k \geq 2$ similarly, perturbing the Lagrangians so they are in decreasing order of $\mathrm{Im}\ (W)$.

Example. Consider $L_0 = \mathbb{R}_+$, L_{-1} = an arc joining 0 to $+\infty$ and rotating once clockwise around the origin. Then $e^{-\Lambda/2} \in L_0, -e^{-\Lambda/2} \in L_{-1}$, so under $W = z + \frac{e^{-\Lambda}}{z}$, we have $W(L_0)$ being the interval $[2e^{-\Lambda/2}, +\infty)$ on the positive real axis, while $W(L_{-1})$ is an arc that joins $-2e^{-\Lambda/2}$ to $+\infty$ in the lower half plane. Furthermore, $\mathrm{hom}(L_0, L_0) \cong \mathbb{C} \cdot e, e = \mathrm{id}_{L_0}$, and same for L_{-1}, while $\mathrm{hom}(L_0, L_{-1}) = 0$ and $\mathrm{hom}(L_{-1}, L_0) = V$ has dimension 2. Then $\mathrm{Fuk}(\mathbb{C}^*, W)$ is generated by L_{-1}, L_0 (Seidel)

Similarly, one can obtain homological mirror symmetry for toric Fano manifolds: see M. Abouzaid.