MIRROR SYMMETRY: LECTURE 24

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0.1. **General Approach to Special Lagrangian Fibrations.** The idea is to degenerate X to a union of toric varieties, build a degenerate fibration there, and try to smooth it: the approach is due to Haase-Zharkov, WD Ruan, Gross-Siebert, etc. This is a special type of LCSL. We first sketch this in the K3 case: as last time,

$$X_\lambda = \{P_\lambda = x_0x_1x_2x_3 + \lambda P_4(x_0 : \cdots : x_3) = 0\} \subset \mathbb{CP}^3 \tag{1}$$

with $\omega_\lambda = \omega_{\mathbb{CP}^3}|_{X_\lambda}, \Omega_\lambda = \mathrm{res}_{X_\lambda}(\frac{dx_1 dx_2 dx_3}{P_\lambda})$. As $\lambda \to 0$, this degenerates to X_0, a union of 4 $\mathbb{CP}^2$s, with ω_0 the standard form on each component and $\Omega_0 = \prod \frac{dx_i}{x_i}$. Now, we find that $\{|x_i| = \text{constants}\}$ are special Lagrangian (looking at $T^2 \subset \mathbb{CP}^2$), but they degenerate to S^1 at the edges and points at the vertices.

We would like to smooth this for $\lambda \neq 0$ small. The model in dimension 1 is as follows: we smooth $\{xy = 0\} \subset \mathbb{C}^2$ to $\{xy = \lambda\}$, and $\Omega = \frac{dx}{x} = -\frac{dy}{y}$ gives that $|x| = \text{const}, |y| = \text{const}$ are special Lagrangian tori. In dimension one higher, we model along the edge ($|z| = \text{const}$ gives S^1_z times this model) except that we perturb $xy = 0$ to $xy + \lambda P_4(z) = 0$. The four roots of P_4 give 4 singularities of the T^2 fibration on each edge of the torus, giving S^2 an affine structure on $S^2 \setminus \{24 \text{ points}\}$. This same procedure holds in greater generality, and gives affine structures and a way of building a candidate mirror (Gross-Siebert). However, it is not clear if the affine manifold built this way is the base of a special Lagrangian fibration (probably not, according to [Joyce]).

0.2. **Landau-Ginzburg models and non-Calabi Yau manifolds.** Our motivating example is the mirror symmetry between $\mathbb{CP}^1$ and $(\mathbb{C}^*, W = z + \frac{1}{z})$. A *Landau-Ginzburg* model is a noncompact Kähler manifold and a holomorphic function W (the "superpotential"), which measures the obstruction to being Calabi-Yau and affects the geometric interpretation of mirror symmetry. The general idea is that the geometry of X corresponds to the geometry of the critical points of W in $X^\vee$.

Returning to our example, we start with $\mathbb{C}^*$ with any ω, $\Omega = \frac{dz}{z}$ (an open Calabi Yau): we have a special Lagrangian fibration by circles $S^1(r) = \{|z| = r\}$ with base $\mathbb{R}$. Dualizing gives back $\mathbb{C}^*$, and mirror symmetry works well as in SYZ (e.g. $HF(L_p, L_p) \cong H^*(S^1, \mathbb{C}) \cong \mathrm{Ext}^*(\mathcal{O}_p, \mathcal{O}_p)$). However, we need to incorporate the noncompact Lagrangians [Seidel's "wrapped Fukaya category": we perturb by a rotation at ∞, obtaining $HW^*(L_0, L_0) \cong \mathbb{C}[t^{\pm 1}] \cong \mathrm{Hom}(\mathcal{O}, \mathcal{O})$ (holomorphic functions over $\mathbb{C}^*$)].

Now we look at $\mathbb{CP}^1 = \mathbb{C}^* \cup \{0, \infty\}$, with standard ω, $\Omega = \frac{dz}{z}$ (with poles at 0 and ∞). We can still consider a family of special Lagrangian circles, but typically $HF^*(L, L) = 0$ gives the zero object in the bounded derived Fukaya category. Furthermore, the Floer homology is obstructed, as the circles bound disks: recall that, when L, L' bound disks, ∂ on $CF(L, L')$ squares to $\partial^2(a) = m_0' \cdot a - a \cdot m_0$,

$$m_0 = \sum_{\beta \in \pi_2(X,L)} \mathrm{ev}_*[\overline{\mathcal{M}}_1(X, L; J, \beta)] T^{\omega(\beta)} \mathrm{hol}_\nabla(\partial \beta) \in CF(L, L) \tag{2}$$

These features of Floer homology are encoded in the superpotential, namely if $X = \mathbb{CP}^1$ is a Kähler manifold, $D = \{0, \infty\}$ the anticanonical divisor (so $s_D \in H^0(K_X^{-1})$), $\Omega = s_D^{-1} \in H^0(X \setminus D, K_X)$ where $\Omega = \frac{dz}{z}$ on $\mathbb{C}^*$, then

$$M = \{(L, \nabla) | L \text{ special Lagr. torus in} X \setminus D, \nabla \text{ flat } U(1) - \text{connection}\} \tag{3}$$

is the SYZ mirror to the almost-Calabi-Yau manifold $X \setminus D$. For $L \subset X \setminus D$ special Lagrangian, $\beta \in \pi_2(X, L)$ has Maslov index $\mu(\beta) = 2(\beta \cdot D)$. Note that s_D gives a trivialization of det (TM) away from D. Now, the expected dimension of $\overline{\mathcal{M}}(x, L, J, \beta) = n - 3 + \mu(\beta)$: in our case, the positivity of the intersection implies that $\mu(\beta) \geq 0$ for holomorphic disks.

Assume that there do not exist nonconstant $\mu = 0$ holomorphic disks in (X, L), i.e all disks hit D. This is ok for $\mathbb{CP}^1$, as the maximum principle implies that there are no disks in $(\mathbb{C}^*, S^1(r))$. Assume further that $\mu = 2$ disks (which hit D once) are regular, which is also ok for $\mathbb{CP}^1$. These two assumptions are also ok for toric Fano manifolds, e.g. products of $\mathbb{CP}^n$s. Then $\mu = 2$ moduli spaces are compact (there is no bubbling of disks) of dimension $n - 1$. We can define $n_\beta = \deg\ (\mathrm{ev}_{0*}[\overline{\mathcal{M}}_1(\beta)])$ to be the number of holomorphic disks in the class β where the boundary contains a generic point in L.

Definition 1.

$$\omega(L, \nabla) = \sum_{\substack{\beta \in \pi_2(X, L) \\ \mu(\beta) = 2}} n_\beta z_\beta(L, \nabla) \tag{4}$$

where $z_\beta = e^{-2\pi \int_\beta \omega} \mathrm{hol}_{\partial\beta}(\nabla)$.

In our example, the Lagrangian L bounds two $\mu = 2$ disks D and D' centered at $0, \infty$ respectively: D contributes z while D' contributes z', and the two are related by

$$[D] + [D'] = [\mathbb{CP}^1] \implies zz' = e^{-2\pi \int_{\mathbb{CP}^1} \omega} = e^{-\Lambda} \tag{5}$$

Hence $W = z + z' = z + \frac{e^{-\Lambda}}{z}$.

Homological mirror symmetry provides two isomorphisms

$$\begin{aligned} D^\pi \mathrm{Fuk}(\mathbb{CP}^1) &\cong H^0 MF(W) \\ D^b \mathrm{Coh}(\mathbb{CP}^1) &\cong D^b \mathrm{Fuk}(\mathbb{C}^*, W) \end{aligned} \tag{6}$$

with matrix factorizations and "Fukaya-Seidel" category respectively. The first one explains our construction of the mirror. The Fukaya category is actually a collection indexed by "charge" $\lambda \in \mathbb{C}$, and $\mathrm{Fuk}(\mathbb{CP}^1, \lambda)$ is the set of weakly unobstructed Lagrangians with $m_0 = \lambda \cdot [L]$. This is an honest A_∞-category, as the m_0's cancel and the Floer differential squares to zero, whereas from λ to λ' we'd have $\partial^2 = \lambda' - \lambda$. For instance, for $L \cong S^1$, (L, ∇) is weakly unobstructed, with $m_0 = W(L, \nabla) \cdot [L]$. However, $HF(L, L) = 0$ unless L is the equator and $\mathrm{hol}(\nabla) = \pm \mathrm{id}$. For $p \in L$,

(7)

$$\partial([p]) = z \cdot \mathrm{ev}_{0*}([\mathcal{M}_2(L, [D])] \cap \mathrm{ev}_1^{-1}(p)) + z' \cdot \mathrm{ev}_{0*}([\mathcal{M}_2(L, [D'])] \cap \mathrm{ev}_1^{-1}(p)) = z \cdot [L] - z' \cdot [L]$$

Hence the unit $[L]$ is in the image of ∂ unless $z = \frac{e^{-\Lambda}}{z}$, i.e. $z = \pm e^{-\Lambda/2}$, i.e. L is the equator. In this case, the contributions of pairs of symmetric disks cancel exactly, and $HF^*(L, L) \cong H^*(S^1; \mathbb{C})$ as a $\mathbb{Z}/2\mathbb{Z}$-graded vector space. However, the product structure is deformed, as $m_2([p], [p]) = \pm e^{\Lambda/2}[1]$, i.e. multiplicatively $HF^*(L, L) \cong \mathbb{C}[t]/t^2 = \pm e^{-\Lambda/2}$.