MIRROR SYMMETRY: LECTURE 15

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1. Lagrangian Floer Homology (contd)

Recall first our approaches to $CF^*(L,L)$ with the A_∞ algebraic structure:

(1) Hamiltonian perturbations $CF^*(L,L) = \Lambda^{|L\cap\phi_H(L)|}$
(2) FOOO: $CF^*(L,L) = C_*(L,\Lambda)$ the space of "chains" on L. We have evaluation maps $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,k+1}(M,L;J,\beta) \to L$, giving multiplication maps

$$m_k(C_k,\dots,C_1) = \sum_{\beta\in\pi_2(X,L)} T^{\omega(\beta)}(\mathrm{ev}_0)_*([\overline{\mathcal{M}}_{0,k+1}(M,L;J,\beta)]\cap ev_1^*C_1\cap\cdots\cap ev_k^*C_k)$$

(3) Cornea-Lalonde approach: "clusters". Pick a Morse function $f : L \to \mathbb{R}$, and set $CF^*(L,L) = \Lambda^{\mathrm{crit}(f)}$. m_k counts "clusters" of J-holomorphic disks and gradient flowlines.

1.1. Disks and Obstruction.

We've seen that, if L_0 or L_1 bound holomorphic disks, then $\partial^2 \neq 0$ (the moduli space of index 2 strips has disk bubbling on the boundaries in addition to strips). Counting the contribution of disk bubbles gives $m_0 \in CF^*(L,L)$. In FOOO theory, $m_0 = \sum_{\beta\neq 0}\mathrm{ev}_*[\overline{\mathcal{M}}_{0,1}(M,L;J,\beta)]\cdot T^{\omega(\beta)}$. A bubble on the boundary of the disk on L_1 is $m_2(m_0,p)$, for $p \in CF^*(L_0,L_1)$, $m_0 \in CF^*(L_1,L_1)$. Hence m_0 is the obstruction to $\partial^2 = 0$. More generally, A_∞-equations still hold if we include the terms $m_k(\cdots,m_0,\cdots)$, which we can interpret as disks with $k+1$ marked points developing disk bubbles on the boundary. This is called a "curved A_∞-category". We say that L is unobstructed if $m_0 = 0$, and weakly unobstructed if $m_0 \in \Lambda.1_L$, where 1_L is the fundamental chain $[L]$. This implies centrality, and $m_1^2 = 0$ on $CF(L,L)$. Weakly unobstructed L's with a given "charge" form an honest A_∞-category.

In FOOO, one tries to cancel the obstruction by a formal deformation $b \in CF^1(L,L)$. For $\nabla = d + b$ on $CF^*(L,L)$, write

$$m_k^b(C_k,\dots,C_1) = \sum m_{k+\ell}(b\dots b, c_k, b\dots b,\dots,b\dots b,c_1,b\cdots b) \tag{1}$$

This is still a curved A_∞-algebra, and we look for b, s.t. $m_0^b = m_0 + m_1(b) + m_2(b,b) + \cdots = 0$. Such a b is called a "bounding cochain". One can similarly define weakly bounding cochains, and define our obmjects to be equivalence classes of pairs (L,b) for b a weakly bounding cochain.

1.2. **Coherent Sheaves on a Complex Manifold.** Let X be a complex manifold, $\mathcal{O}_X$ the sheaf of holomorphic functions on X. Recall that a coherent sheaf $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules s.t.

- $\mathcal{F}$ is of finite type, i.e. there is an open cover by affines U_i s.t. $\mathcal{F}_{U_i}$ is generated by a finite number of sections, i.e. $\exists$ surjective maps $\mathcal{O}_X|_{U_i}^{\oplus n} \to \mathcal{F}|_{U_i}$.
- For all $U \subset X$ open, $\phi : \mathcal{O}_X|_U^{\oplus n} \to \mathcal{F}|_U$ a homomorphism of $\mathcal{O}_X$-module, Ker ϕ is of finite type.

If X is nice enough, $\mathcal{F}$ has *finite presentation*, i.e. $\exists$ an open cover s.t. there is an exact sequence

$$\mathcal{O}_X^{\oplus r}|_U \to \mathcal{O}_X^{\oplus n}|_U \to \mathcal{F}|_U \to 0 \tag{2}$$

i.e. a coherent sheaf is the cokernel of a morphism of vector bundles. Coherent sheaves form an abelian category, i.e. they contain kernels and cokernels.

Example. Any vector bundle E can be thought of as a locally free sheaf of holomorphic sections. For D a hypersurface defined by $s = 0$ for s a section of some line bundle $\mathcal{L}$, we have a short exact sequence

$$0 \to \mathcal{L}^{-1} \xrightarrow{s} \mathcal{O}_X \to \mathcal{O}_D \to 0 \tag{3}$$

For $Z \subset X$ a codimension r subvariety defined transversely as $s^{-1}(0)$, for s a section of a rank r vector bundle $\mathcal{E}$, we have a Koszul resolution

$$0 \to \bigwedge^r \mathcal{E}^* \xrightarrow{s} \bigwedge^{r-1} \mathcal{E}^* \xrightarrow{s} \cdots \xrightarrow{s} \mathcal{E}^* \xrightarrow{s} \mathcal{O}_X \to \mathcal{O}_Z \to 0 \tag{4}$$

For X smooth (proper?), coherent sheaves always have a finite resolution by vector bundles.

The category of sheaves has both an internal $\mathscr{H}$ (which is a sheaf) and an external Hom (just a group, and in fact the global sections for the former). A functor $F : \mathcal{C} \to \mathcal{C}'$ is left exact if $0 \to A \to B \to C \to 0 \implies 0 \to F(A) \to F(B) \to F(C)$. If the category $\mathcal{C}$ has enough injectives (objects such that $\mathrm{Hom}_C(-, I)$ is exact), there are right-derived functors R^iF s.t.

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to \cdots \tag{5}$$

To compute $R^iF(A)$, resolve A by injective objects as $0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$, we get a complex $0 \to F(I^0) \to F(I^1) \to F(I^2) \to \cdots$. Taking cohomology gives $R^iF(A) = \mathrm{Ker}\,(F(I^i) \to F(I^{i+1}))/\mathrm{im}\,(F(I^{i-1}) \to F(I^i))$. Note that $R^0F(A) = F(A)$.

Example. Sheaf cohomology arises as the right derived functor of the global section functor, and can be computed by acyclic sheaves (e.g. flasque sheaves).