MIRROR SYMMETRY: LECTURE 12

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0.1. **Lagrangian Floer Homology (contd).** Let (M, ω) be a symplectic manifold, L_0, L_1 compact Lagrangian submanifolds intersecting transversely. We defined $CF(L_0, L_1) = \Lambda^{|L_0 \cap L_1|}$ and the differential

$$\partial(p) = \sum_{\substack{q \in L_0 \cap L_1 \\ \phi \in \pi_2(M, L_0 \cup L_1) \\ \mathrm{ind}(\phi) = 1}} \#(\mathcal{M}(p, q, \phi, J)/\mathbb{R}) T^{\omega(\phi)} \cdot q \tag{1}$$

where $\mathcal{M}$ is the set of finite energy J-holomorphic maps $u : \mathbb{R} \times [0, 1] \to M, u(s, 0) \in L_0, u(s, 1) \in L_1, \lim_{s \to +\infty} u = p, \lim_{s \to -\infty} u = q$. The limits of sequences in $\mathcal{M}$ exhibit sphere bubbling, disk bubbling, and broken strips. If there was no bubbling (e.g. $\omega \cdot \pi_2(M, L_i) = 0$, we stated that $\partial^2 = 0$.

Example. Consider $T^*\mathbb{R} \cong \mathbb{R}^2$ again, with L_0 the zero section and L_1 the unit circle intersecting L_0 at points $p = (1, 0), q = (-1, 0)$. Then $CF(L_0, L_1) = \Lambda p \oplus \Lambda q$, $\partial(p) = \pm T^{\omega(u)} q, \partial(q) = \pm T^{\omega(v)} p$, and $\partial^2 \neq 0$. We have an index 2 $\mathcal{M}(p, p)$ isomorphic to the interval, consisting of holomorphic maps whose image is the unit disk with a slit entering at the point p and ending at a position $\alpha \in (-1, 1)$. More precisely, using the upper half of the unit disk as our domain, we can write $u_\alpha(z) = \frac{z^2 + \alpha}{1 + \alpha z^2}$. There are two endpoints: $\alpha \to -1$, in which we obtain a broken trajectory $p \to q \to p$, and $\alpha \to 1$, where we obtain the constant strip at p and a disk bubble.

0.2. **More about grading.** Question: can we define deg (p), deg (q) s.t. deg $(q)-$ deg $(p) = \mathrm{ind}([u])$? Maslov index comes from $\pi_1(\Lambda\mathrm{Gr}) \cong \mathbb{Z}$: if $2c_1(M) = 0$, then the $\Lambda\mathrm{Gr}$-bundle of Lagrangian planes in TM has a fiberwise universal cover, the $\Lambda\widetilde{\mathrm{Gr}}$-bundle of "graded Lagrangian planes". Then, if at $p \in L_0 \cap L_1$, we fix graded lifts of T_pL_0, T_pL_1, then deg (p) is the Maslov index at p. Locally, in the basic configuration $L_0 = \mathbb{R}^n \subset \mathbb{C}^n, L_1 = (e^{-i\theta}\mathbb{R})^n \subset \mathbb{C}^n$ for small θ, deg $(p) = 0$: in general, deg (p) will be the Maslov index of the path from this reference configuration.

The obstructions to the existence of a global graded lift of L are $2c_1(TM)$ and, if it vanishes, the Maslov class $\mu_L \in H^1(L, \mathbb{Z})$. If the latter vanishes as well, then the index $\mathrm{ind}(u) = $ deg $(q) -$ deg (p) independently of $[u]$ and HF is $\mathbb{Z}$-graded. If we have nonzero Maslov class, we can do modifications at the boundary of u away from p, q to change the index. In such a case, we find that HF is $\mathbb{Z}/N\mathbb{Z}$ graded, where N is the minimal Maslov number (if L_i are oriented, N is always even, so can reduce to a $\mathbb{Z}/2\mathbb{Z}$ grading which coincides with the signs of intersections $L_1 \cdot L_0$). We can also work over a larger graded ring, with a new parameter that keeps track of how the index of the strip differs from the difference of degrees. In the monotone case, i.e. when the area and the Maslov index are proportional to each other, we just need to assign our parameter T some nonzero degree.

0.3. **Hamiltonian isotopy invariance.** Say $H : [0,1] \times M \to \mathbb{R}$ generates $\phi_H^t =$ the flow of X_H ($\iota_{X_H}\omega = dH$).

Proposition 1. *If there is no bubbling, then* $HF^*(\phi_H^1(L_0), L_1) \cong HF^*(L_0, L_1)$.

We want to count finite energy ($E(u) = \iint \left|\frac{\partial u}{\partial s}\right|^2$) solutions of:

$$\begin{gathered} u : \mathbb{R} \times [0,1] \to M \\ \frac{\partial u}{\partial s} + J(\frac{\partial u}{\partial t} - \beta(s) X_H(t,u)) = 0 \\ u(s,0) \in L_0, u(s,1) \in L_1 \end{gathered} \tag{2}$$

where β is a cutoff function that goes to 0 for $s \gg 0$, 1 for $s \ll 0$. For $s \to +\infty$, u converges to a point in $L_0 \cap L_1$, while for $s \to -\infty$, it converges to a trajectory $\dot\gamma(t) = X_H(t,\gamma), \gamma(0) \in L_0, \gamma(1) \in L_1 \Leftrightarrow \gamma(1) \in \phi_H^1(L_0) \cap L_1$. If $\tilde u(s,t) = \phi_H^{(t,1)}(u(s,t))$, then we can modify $J \mapsto \tilde J$ to obtain $\frac{\partial \tilde u}{\partial s} + \tilde J \frac{\partial \tilde u}{\partial t} = 0$ for $s \ll 0$. Counting isolated index 0 solutions gives $\Psi_H : CF(L_0, L_1) \to CF(\phi_H(L_0), L_1)$. In the absence of bubbling, we can show that Ψ_H is a chain map, i.e. $\Psi_H \circ \partial = \partial' \circ \Psi_H$ (look at the index 1 moduli space, with the ends given by broken trajectories). The breaking of strips can occur at $s \to -\infty$, where we obtain a $\tilde J$-holomorphic strip contributing to ∂' between $\phi_H(L_0)$ and L_1, or at $s \to +\infty$, where we obtain a J-holomorphic strip contributing to ∂ between L_0 and L_1. The signed number of ends of a 1-manifold is 0, so the contributions of $\Psi_H \circ \partial, \partial' \circ \Psi_H$ cancel out. Then Ψ_H induces a map on HF. To see that it is an isomorphism, we build a homotopy Θ between $\Psi_{-H} \circ \Psi_H$ and id, i.e. $\Psi_{-H} \circ \Psi_H - \mathrm{id} = \partial \circ \Theta + \Theta \circ \partial$.

Example. Let $M = T^*N, \omega = \sum dp_i \wedge dq_i$. Equip N with a Riemannian metric g which induces a metric and almost-complex structure on T^*N: along the zero section, $TM = TN \oplus T^*N$ (the two components isomorphic via g), and $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. Let L_0 be the zero section, L_1 the graph of ϵdf for $\epsilon > 0$ small, f a Morse function on N, Morse-Smale for g. Then $L_0 \cap L_1$ is the set of critical points of f, and the Maslov index is $n-$ the Morse index.

Theorem 1 (Fukaya-Oh et al)**.** *For $\epsilon \to 0$, holomorphic strips between L_0 and L_1 are in one-to-one correspondence with the gradient trajectories of f, and $HF^*(L_0, L_1) \cong HM_{n-*}(f) \cong H^*(N)$.*

(L_0, L_1 are exact Lagrangian, hence $\omega \cdot \pi_2 = 0$, and all strips $p \to q$ have $\int u^*\omega = \epsilon(f(p) - f(q))$, so we can forget about T up to $\tilde{p} = T^{\epsilon f(p)}p$.) By the Weinstein Lagrangian neighborhood theorem, this is a universal local calculation. If L doesn't bound any holomorphic disks in M, $HF(L, L) = HF(L, \psi(L)) = H^*(L, \Lambda)$. Otherwise, we can try to filter CF, ∂ by the symplectic area of disks, e.g. if L is monotone (ω, c_1 positively proportional on $\pi_2(M, L)$). Then we get the *Oh spectral sequence* which starts at $H^*(L, \Lambda)$ and converges to $HF^*(L, L)$.