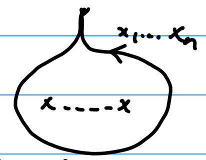
Lecture 5 – Mon Feb 27

Recall last time:
- $B_n = \pi_0 \, \mathrm{Homeo}_c^+(\mathbb{R}^2, Q_n)$
- this induces a right action of B_n on free group F_n, faithful: $B_n \hookrightarrow \mathrm{Aut}(F_n)$.

Now characterize the image of this map:

Thm (Artin 1925): $\beta \in \mathrm{Aut}(F_n)$ is induced by an elt of B_n iff it satisfies:
(1) $\exists$ permutation $\tau \in \mathfrak{S}_n$ & $A_i \in F_n$ st. $(x_i)\beta = A_i x_{\tau(i)} A_i^{-1}$ $\forall i = 1 \dots n$.
(2) $(x_1 \cdots x_n)\beta = x_1 \cdots x_n$.

Pf: – these 2 conditions are clearly necessary (



clearly preserved by braids)

& x_i's mapped to conjugates of each other

– conversely, we'll show that any autom. of F_n st. (1) + (2) can be expressed as product of $(\sigma_i)_*$.

– assume each word $A_i x_{\tau(i)} A_i^{-1}$ is freely reduced, & consider the identity
$$A_1 x_{\tau(1)} A_1^{-1} A_2 x_{\tau(2)} A_2^{-1} \cdots A_n x_{\tau(n)} A_n^{-1} = x_1 \cdots x_n \quad (*)$$

Lemma 1: for $(*)$ to hold, either $\beta = \mathrm{Id}$, or $\exists \nu \in \{1, \dots, n-1\}$ st. either
(a) $x_{\tau(\nu)} A_\nu^{-1}$ is absorbed by $A_{\nu+1}$ (ie. $A_\nu x_{\tau(\nu)}^{-1}$ is a prefix in $A_{\nu+1}$)
or (b) A_ν^{-1} absorbs $A_{\nu+1} x_{\tau(\nu+1)}$ (ie. $A_{\nu+1} x_{\tau(\nu+1)}$ is a prefix in A_ν).

Assume lemma 1 holds: then reduce β to Id by induction on its "length"

Def. length of β = sum of word lengths of $(x_i)\beta = A_i x_{\tau(i)} A_i^{-1}$, $i = 1 \dots n$.

Lemma 2: if (a) holds, $\sigma_\nu \beta$ has length shorter than β
if (b) holds, $\sigma_\nu^{-1} \beta$ has length shorter than β

[right action, so this means: first act by $(\sigma_\nu^{\pm 1})_*$, then by β]

Then, lemma 1 + lemma 2 $\Rightarrow$ can reduce β to Id by composing repeatedly with action of σ_ν or σ_ν^{-1}
$\Rightarrow$ β is indeed in the image of $B_n \hookrightarrow \mathrm{Aut}(F_n)$. $\blacktriangle$

Pf Lemma 2: assume (a) holds, then $(x_\nu)\beta = A_\nu x_{\tau(\nu)} A_\nu^{-1}$

$(x_{\nu+1})\beta = A_\nu x_{\tau(\nu)}^{-1} \tilde{A}_{\nu+1} x_{\tau(\nu+1)} \tilde{A}_{\nu+1}^{-1} x_{\tau(\nu)} A_\nu^{-1}$

↳ [this is a reduced word!]

so recalling that $\sigma_\nu : x_\nu \mapsto x_\nu x_{\nu+1} x_\nu^{-1}$, $x_{\nu+1} \mapsto x_\nu$, we have

$(x_\nu)\sigma_\nu\beta = A_\nu \tilde{A}_{\nu+1} x_{\tau(\nu+1)} \tilde{A}_{\nu+1}^{-1} A_\nu^{-1}$ → [this might not be reduced, but is shorter than above anyway].

$(x_{\nu+1})\sigma_\nu\beta = A_\nu x_{\tau(\nu)} A_\nu^{-1}$

→ total length decreased (since σ_ν doesn't affect the other x_i, $i \notin \{\nu, \nu+1\}$)

Similarly if (b) holds, $(x_\nu)\beta = A_{\nu+1} x_{\tau(\nu+1)} \tilde{A}_\nu x_{\tau(\nu)} \tilde{A}_\nu^{-1} x_{\tau(\nu+1)}^{-1} A_{\nu+1}^{-1}$

$(x_{\nu+1})\beta = A_{\nu+1} x_{\tau(\nu+1)} A_{\nu+1}^{-1}$ ← [reduced]

→ $(x_\nu)\sigma_\nu^{-1}\beta = (x_{\nu+1})\beta = A_{\nu+1} x_{\tau(\nu+1)} A_{\nu+1}^{-1}$

$(x_{\nu+1})\sigma_\nu^{-1}\beta = (x_{\nu+1}^{-1} x_\nu x_{\nu+1})\beta = A_{\nu+1} \tilde{A}_\nu x_{\tau(\nu)} \tilde{A}_\nu^{-1} A_{\nu+1}^{-1}$ ← [shorter]

■

Pf Lemma 1: examine how LHS of (*) reduces to RHS:

$$A_1 x_{\tau(1)} A_1^{-1} A_2 x_{\tau(2)} A_2^{-1} \cdots A_n x_{\tau(n)} A_n^{-1} = x_1 \cdots x_n \quad (*)$$

first assume $\exists \nu$ s.t. letter $x_{\tau(\nu)}$ cancels in the reduction of LHS to RHS & look at how it gets absorbed:

[NB: sequence of cancellations LHS → RHS need not be unique, choose one & stick to it]

- can't be absorbed by a letter in A_ν or A_ν^{-1} since by assumption $A_\nu x_{\tau(\nu)} A_\nu^{-1}$ is reduced.
- if absorbed by a letter in $A_{\nu+1}$, then (a) holds ($x_{\tau(\nu)} A_\nu^{-1}$ absorbed by $A_{\nu+1}$)
- ——————— $A_{\nu-1}^{-1}$, then (b) holds ($A_{\nu-1}^{-1}$ absorbs $A_\nu x_{\tau(\nu)}$)
- otherwise, it's absorbed by a letter to the left of $x_{\tau(n-1)}$ or to the right of $x_{\tau(n+1)}$

but then, $x_{\tau(n-1)}$ or $x_{\tau(n+1)}$ is absorbed by a letter that's closer to it: ie. if $x_{\tau(n)}$ cancels w/ a letter from $A_i x_{\tau(i)} A_i^{-1}$ then $x_{\tau(n+1)}$ cancels w/ a letter from $A_j x_{\tau(j)} A_j^{-1}$,

$[x_{\tau(n)}^{-1}]$ $[x_{\tau(n+1)}^{-1}]$ $x_{\tau(n-1)} \cdots x_{\tau(n)}$

cancels here

$A_i^{\pm 1}$ $A_j^{\pm 1}$ cancels or here (in A_n).

& either $j = n$ ($\Rightarrow$ done by above), or $|j - (n \pm 1)| < |i - n|$

so by induction on $|i-n|$ reduce to previous cases & (a) or (b) holds!

Otherwise: $\forall \nu$, $x_{\tau(\nu)}$ survives the cancellations

→ then $\tau(\nu) = \nu$ $\forall \nu$, and the other letters all cancel!

however this implies $A_{\nu-1}^{-1} A_\nu \mapsto 1$ $\forall \nu$ ie $A_{\nu-1} = A_\nu$

and $A_1 \mapsto 1$, $A_n^{-1} \mapsto 1$ ie A_1, A_n trivial $\Rightarrow$ Id ■

· **Braid group of the sphere..**

Thm (Fadell - Van Buskirk 1962)

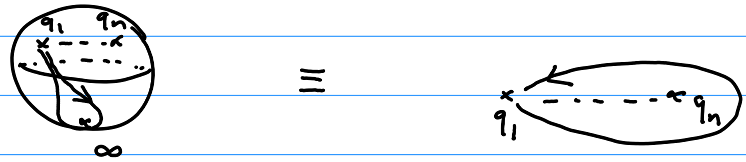
$B_n(S^2) = \pi_1 \mathcal{C}_n(S^2)$ admits a presentation w/ generators $\sigma_1, \dots, \sigma_{n-1}$, & relations
$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad |i-j|\geq 2$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$
$$\sigma_1\sigma_2\cdots\sigma_{n-1}\sigma_{n-1}\cdots\sigma_2\sigma_1 = 1$$

Sketch Pf: $S^2 = \mathbb{R}^2 \cup \{\infty\}$, so letting $D_\infty = \{\{q_1..q_n\} \in \mathcal{C}_n(S^2),\ \{q_1 \dots q_n\} \ni \infty\}$
D_∞ is a smooth submfld of $\mathcal{C}_n(S^2)$, connected, of codim. 2.
$(\simeq \{\infty\} \times \mathcal{C}_{n-1}(\mathbb{R}^2))$ and $\mathcal{C}_n(\mathbb{R}^2) = \mathcal{C}_n(S^2) - D_\infty$.

will see more carefully later

Van Kampen's thm $\Rightarrow$ $\pi_1 \mathcal{C}_n(S^2) \simeq \pi_1 \mathcal{C}_n(\mathbb{R}^2) / \langle m \rangle$, normal subgroup gen^d by m = meridian loop
$(\mathcal{C}_n(S^2) = \mathcal{C}_n(\mathbb{R}^2) \cup U(D_\infty)$, intersection = S^1-bundle$/D_\infty)$

Meridian loop = keep $q_2 \cdots q_n$ fixed, move q_1 around ∞

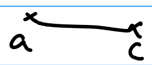
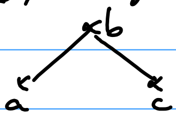


$\equiv \sigma_1 \cdots \sigma_{n-1}\sigma_{n-1} \cdots \sigma_1$.

(can also prove directly using a method similar to what we used for $B_n(\mathbb{R}^2)$:
first understand pure braids $P_n(S^2)$; for that use fibration $\tilde{\mathcal{C}}_n(S^2) \to \tilde{\mathcal{C}}_{n-1}(S^2)$, but beware the l.e.s. reduces to π_1 only when $n\geq 4$. $(\pi_2 \tilde{\mathcal{C}}_{n-1}(S^2) = 0$ only for $n\geq 4)$

Braids and links:

- Work with PL links, ie. union of disjoint polygonal simple closed curves in $\mathbb{R}^3$.
- Combinatorial equivalence of links ($\leftrightarrow$ link isotopy): gen^d by move =

replace an edge  by edges  provided that

- b, a, c not aligned in that order (nor a, c, b)
- the convex hull $\Delta(a,b,c)$ does not intersect the link anywhere outside the segment $[a,c]$.

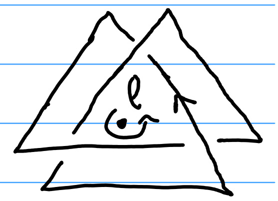
Call this move $\mathcal{E}^b_{a,c}$

(easy: this coincides w/ usual link isotopy)

- fix a line $\ell \subset \mathbb{R}^3$ — call it axis.
Say the link L is in general position if none of its edges is coplanar w/ ℓ.

(can always put a link in general position: if $[ac]$ coplanar w/ ℓ, insert a vertex b very close to $[ac]$ but not in the plane $([ac], \ell)$).

. fix an orientation of L, & assume L is in general position: view L in terms of its projection onto a plane $\mathbb{R}^2 \perp \ell$. fix ↺ positive dirⁿ of rotation around ℓ.



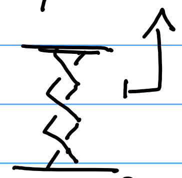
Ex. of closed braid (= trefoil)

<u>Def</u>: an oriented edge ab of L is <u>positive</u> if its proj. to $\mathbb{R}^2$ rotates around origin in $+$ direction, <u>negative</u> if rotates in $-$ direction

L is a <u>closed braid</u> if all its edges are positive

$h(L)$ ("<u>height</u>" of L) $:=$ # neg. edges

. <u>Fact</u>: any geometric braid β can be used to construct a closed braid $\hat{\beta}$ (identify initial & end pts of the strings) [& polygonal approx.]



$\sigma_1^3 \in B_2$

<u>Thm</u> (Alexander 1923):

Every link is isotopic to a closed braid

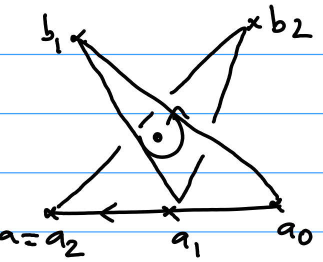
<u>Pf</u>: get rid of negative edges by inserting <u>sawtooths</u>:

given a negative edge $a_0 a$:

1) subdivide it: take $a_0, a_1, \ldots, a_n = a$ aligned in that order
2) replace each negative edge $a_{i-1}a_i$ by 2 <u>positive</u> edges $a_{i-1}b_i$, $b_i a_i$ (s.t. this is an isotopy move $\mathcal{E}_{a_{i-1},a_i}^{b_i}$, ie the triangle $\Delta(a_{i-1},a_i,b_i) \cap \overline{L} = [a_{i-1},a_i]$)

link obtained so far.

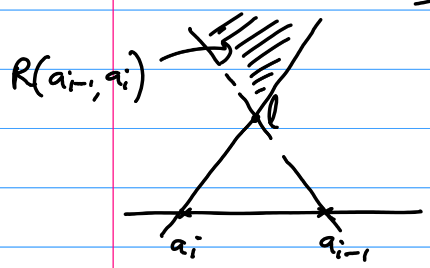
<u>Ex</u>:



<u>Lemma</u>: $[a_0, a]$ negative edge of L = link in general posⁿ.
$\Rightarrow$ can erect a sawtooth on $[a_0, a]$.

($\Rightarrow$ thm, since we can then inductively decrease $h(L)$ by inserting sawtooths on negative edges).

Pf. Lemma: observe: given $[a_{i-1}, a_i]$ negative edge and the line ℓ,
the 2 planes through ℓ & a_{i-1} / ℓ & a_i subdivide $\mathbb{R}^3$ into 4 regions



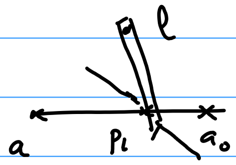
The edges $[a_{i-1}, b_i]$ & $[b_i, a_i]$ will be >0 iff b_i is chosen in the region $R(a_{i-1}, a_i)$

Case 1: if the projection of the edge $[a_0, a]$ has no double pts (crossings).

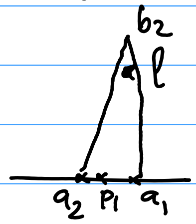
→ choose b in the region $R(a_0, a)$, very far above (or below) the plane of projection so plane (a_0, b, a) is almost $\perp$ projection

⇒ then the triangle $\Delta(a_0, b, a)$ doesn't intersect L
(passes above or below any part of L that passes between $[a_0\, a]$ and ℓ).
⇒ can erect a sawtooth there (w/out subdividing $[a_0, a]$).

Case 2: if $\exists$ double pts: $p_1 \cdots p_r \in [a_0, a]$ corresponding to crossings of the diagram



$P_1 :=$ plane through p_1 and ℓ
Can draw a line segment in P_1 joining p_1 to ℓ & avoiding L otherwise; then make a very thin tooth which lies very close to this line.
(triangle $\Delta(a_1, a_2, b_2)$ stays in nbd of the chosen line segment & hence only intersects L along edge $a_1 - a_2$).



proceed similarly near each p_i, & on remaining segments do as in Case 1. ◁

· Representing links by closed braids is a way to put more algebraic structure into things. As an illustration, can compute π_1 of link complement easily for a closed braid:

Thm (Artin 1925): $\beta \in B_n$, $\hat{\beta} =$ link obtained by closing β
$\Rightarrow \pi_1(S^3 - \hat{\beta}) = \langle y_1, \dots, y_n \mid y_i = (y_i)\beta \ \ \forall i = 1 \dots n \rangle$
where $(y_i)\beta =$ action of B_n on $F_n = \pi_1(\mathbb{R}^2 - Q_n)$

Corollary: A group is a link gp iff it admits a presentation of the form

$$\langle y_1,\dots,y_n \mid y_i = A_i(y_1\dots y_n)\, y_{\tau(i)}\, A_i(y_1\dots y_n)^{-1} \rangle$$

where $\tau \in \mathfrak{S}_n$ permutation

and $A_i \in F_n$ s.t. $\prod_{i=1}^{n} A_i y_{\tau(i)} A_i^{-1} = y_1 \dots y_n$ in F_n.

(using characterization of image of $B_n \hookrightarrow \mathrm{Aut}(F_n)$)

Pf Thm: Let $\varphi \in \mathrm{Homeo}_c^+(\mathbb{R}^2, Q_n)$ representing β (supported in a disc D^2)

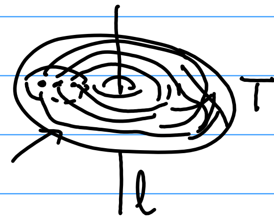
& consider the 3-mfld $Y = (D^2 - Q_n)\times I/\sim \qquad (z,1)\sim(\varphi(z),0)$

— the mapping torus of $\varphi_{|D^2-Q_n}$

$Y \equiv$ the complement of the link $\hat\beta$ in the solid torus $D^2\times I/\sim \; = T$

ie $Y = T - \hat\beta \cong S^3 - (\hat\beta \cup \bar\ell)$

unknot = closure of the axis



Clearly Y fibers above S^1 with fibre $D^2 - Q_n$

$\Rightarrow$ l.e.s. gives $\underset{\pi_2(S^1)}{\overset{\parallel}{1}} \longrightarrow \pi_1(D^2 - Q_n) \xrightarrow{i_*} \pi_1(Y) \xrightarrow{p_*} \pi_1(S^1) \longrightarrow \underset{\pi_0(D^2-Q_n)}{\overset{\parallel}{1}}$

Let $y_i = i_*(x_i) \in \pi_1(Y)$ $(i=1..n)$, and $t \in \pi_1(Y)$ loop going to $\{z\}\times I$, $z \in \partial D^2$

(t = lift of the generator of $\pi_1(S^1)$)

$\to$ then the exact sequence defines a presentation for

$\pi_1(Y) = \mathbb{Z} \ltimes \pi_1(D^2 - Q_n)$ ($\rightsquigarrow$ semidirect product of 2 free groups $\Rightarrow$ just need to specify how to collect t's to the beginning, by describing conjugation action of t on F_n)

generators $= y_1, \dots, y_n, t$

relations: $t y_i t^{-1} = (y_i)\beta$

. Next, $\pi_1(S^3 - \hat\beta) = \pi_1(Y)/\langle t \rangle$, since t = meridian of $\bar\ell$ on $\partial(Y)$.

$(Y = (S^3 - \hat\beta) - \nu(\bar\ell))$

(Van Kampen: $(S^3 - \hat\beta) = Y \cup (S^3 - T)$ intersecting along torus ∂T

so $\pi_1(S^3 - \hat\beta)$ generators $y_1, \dots, y_n, t, \ell, \underline{L, M}$ (M bounds disc $\pitchfork \bar\ell$, $L \sim \bar\ell$)

core of $(S^3 - T)$ (= axis) — longitude & meridian on ∂T generate $\pi_1(\partial T) = \mathbb{Z}^2$

relns:
- $t y_i t^{-1} = (y_i)\beta$ (from Y)
- $L = \ell$, $M = 1$ (from $\partial T \hookrightarrow S^3 - T$)
- $L = y_1 \dots y_n$, $M = t$ (from $\partial T \hookrightarrow Y$) $\Rightarrow \pi_1(S^3 - \hat\beta) = \pi_1(Y)/\langle t \rangle = \langle y_i \mid y_i = (y_i)\beta \rangle$ ✓