Lecture 4 - Wed Feb 22

<u>Recall:</u> we proved Artin's presentation of B_n (& got a presentation of P_n along the way). This also gave us Artin's sol^{n} to the word problem.

<u>Another way to think about braids:</u>

$$G = \mathrm{Homeo}_c^+(\mathbb{R}^2, Q_n) = \{ \varphi \in \mathrm{Homeo}(\mathbb{R}^2),\ \varphi(Q_n) = Q_n,$$
$$\varphi \text{ compactly supported (\& hence orient preser)} \}$$

<u>Thm:</u> $B_n \simeq \pi_0(G)$

(I.e. B_n is also the <u>mapping class group</u> in genus 0 w/ 1 ∂ component and n marked pts, anticipating on future terminology).

<u>Pf:</u>
$$\mathrm{Homeo}_c^+(\mathbb{R}^2, Q_n) \hookrightarrow \mathrm{Homeo}_c^+(\mathbb{R}^2) \qquad \varphi$$
$$\downarrow ev \qquad\qquad \downarrow ev$$
$$\mathcal{C}_n \qquad\qquad \varphi(Q_n)$$

Equip Homeo_c^+ with compact-open topology

(i.e. nbd of $\{\varphi_0\}$ = prescribe: $\forall K$ compact subset, want to map it to a given nbd of $\varphi_0(K)$.)

$\Rightarrow$ evaluation map is continuous

<u>Lemma:</u> ev is a loc. trivial fibration

<u>Pf:</u> essentially the same as when we compared P_n w/ P_{n-1} ...

fix $\{z_1^0, \dots, z_n^0\} \in \mathcal{C}_n$, let U_i = nbds of z_i^0 (mutually disjoint),

$U = (U_1 \times \dots \times U_n)$ (or rather its image under $\widetilde{\mathcal{C}}_n \to \mathcal{C}_n$)

show $ev^{-1}(U) \underset{\text{homeo}}{\simeq} U \times \mathrm{Homeo}_c^+(\mathbb{R}^2, \{z_1^0 \dots z_n^0\})$

$$\varphi \longmapsto \left(ev(\varphi),\ H_{ev(\varphi)}^{-1} \circ \varphi \circ H_{ev(\varphi)}\right)$$

$H_{ev(\varphi)}$ = homeo of $\mathbb{R}^2$ supported in $\coprod U_i$, mapping each z_i^0 to $z_i :=$ the pt of $ev(\varphi)$ which lies in U_i.

depending continuously on $ev(\varphi) \in U$ (usual construction)

(composition is C^0 in compact-open topology ✓) ◾

So: $\cdots \to \pi_1 \mathrm{Homeo}_c^+(\mathbb{R}^2) \to \underset{\overset{\|}{B_n}}{\pi_1 \mathcal{C}_n} \to \pi_0 \mathrm{Homeo}_c^+(\mathbb{R}^2, Q_n) \to \pi_0 \mathrm{Homeo}_c^+(\mathbb{R}^2) \to \cdots$

conclude using

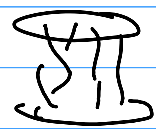
<u>Lemma</u>: $\mathrm{Homeo}_c^+(\mathbb{R}^2)$ is contractible

<u>Pf</u>: retract to $\{Id\}$ using $\rho_t(\varphi) = z \longmapsto t\,\varphi(z/t)$

(as $t \to 0$, $\rho_t(\varphi)$ tends continuously to Id wrt compact open topology
- clearly depends C^0 on t as long as $t > 0$
- for t small, $= Id$ outside of a smaller & smaller disc centered at 0) ▲

How to think about it:

- given a geometric braid, get a homeo $\varphi \in \mathrm{Homeo}_c^+(\mathbb{R}^2, Q_n)$ (up to isotopy) by flowing



id

start from id & deform to move the n given pts as prescribed.

[if braid is smooth, flow a v.f. s.t. $X_t(z_i(t)) = \dfrac{dz_i}{dt}$]

- conversely, given $\varphi \in \mathrm{Homeo}_c^+(\mathbb{R}^2, Q_n)$, consider isotopy $Id \xrightarrow[\varphi_t]{} \varphi$ (<u>not</u> fixing Q_n of course), and get a geometric braid by $\{\varphi_t(Q_n)\}_{0 \le t \le 1}$

<u>Thm</u>: $\exists$ natural right action of B_n on the free group $F_n = \langle x_1, \dots x_n \rangle = \pi_1(\mathbb{R}^2 - Q_n)$

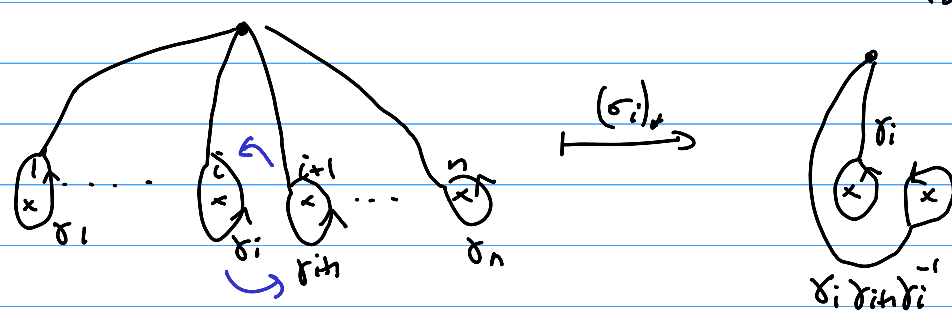
given by $(\sigma_i)_*$: $x_i \mapsto x_i x_{i+1} x_i^{-1}$
$x_{i+1} \mapsto x_i$
$x_j \mapsto x_j \quad j \ne i, i+1$

This induces a <u>faithful</u> representation $B_n \hookrightarrow \mathrm{Aut}(F_n)$.

This right action $=$ given $b \in B_n$, associate a homeomorphism $\varphi_b \in \mathrm{Homeo}_c^+(\mathbb{R}^2, Q_n)$
induces by restriction a homeo of $\mathbb{R}^2 - Q_n$ $\Rightarrow b_* = \varphi_{b*} : \pi_1(\mathbb{R}^2 - Q_n) \circlearrowleft$
(clearly indep. of choice of φ_b in its isotopy class; <u>right</u> action: because $\varphi_{bb'} = \varphi_{b'} \circ \varphi_b$)

Action of σ_i:

$x_i = [\gamma_i]$,



on the pure braid group, $(A_{rs})_*: x_i \mapsto x_i \quad (i<r \text{ or } i>s)$

$$x_s \mapsto x_r x_s x_r^{-1}$$
$$x_r \mapsto x_r x_s x_r x_s^{-1} x_r^{-1}$$
$$x_i \mapsto x_r x_s x_r^{-1} x_s^{-1} x_i x_s x_r x_s^{-1} x_r^{-1} \quad (r<i<s)$$

(exercise: check - on picture
- from def of A_{rs} in terms of σ_i's).

<u>Pf. faithfulness:</u> assume $b \in B_n$ s.t. $b_* = Id \to$ show $b=1$.

- first look at action on conjugacy classes in F_n:

$\sigma_i: [x_i] \mapsto [x_{i+1}]$ so b_* <u>permutes</u> the n conj. classes $[x_1], \dots, [x_n]$
$[x_{i+1}] \mapsto [x_i]$
$[x_j] \mapsto [x_j]$ by the permutation $\pi(b)$, $\pi: B_n \to \mathfrak{S}_n = B_n/P_n$
$\sigma_i \mapsto (i, i+1)$.

In particular if $b_* = Id$ then $\pi(b) = Id \Rightarrow b \in P_n$.

- $b \in P_n \Rightarrow$ recall we have seen $P_n = P_{n-1} \ltimes U_n$ free gp gen^d by $A_{i,n}$ $i=1,\dots,n-1$

and so $b = \beta_1 \cdots \beta_n$, $\beta_i \in U_i$.

Assume $\beta_i \neq 1$, $\beta_{i+1} = \cdots = \beta_n = 1$.

Observe: $\beta_1, \dots, \beta_{i-1}$ gen^d by A_{rs} with $s<i$, so act trivially on x_i.

So $b_*(x_i) = \beta_{i*}(x_i) = x_i$. Claim: this implies $\beta_i = 1$ (contradiction).
$(\to b=1)$

- To see this, think of the action in a slightly $\neq$ way:

$F_n \simeq U_{n+1}$ (free subgp of P_{n+1})

$x_i \longmapsto A_{i,n+1}$

loop in $\mathbb{R}^2 - Q_n \longleftrightarrow$ motion of an $(n+1)^{st}$ pt leaving the first n untouched.

<u>Lemma:</u> Under this isom.
the action of P_n on $F_n \longleftrightarrow$ the action of P_n on U_{n+1} by conjugation
(formulas above) (presentation of P_{n+1} seen last time)

$(A_{rs})_* x_i$ as a word in x_j's $\longleftrightarrow A_{rs}^{-1} A_{i,n+1} A_{rs}$ as word in $A_{j,n+1}$'s

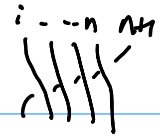
(ok from formulas, also ok geometrically)

So: what we get is: $\beta_i^{-1} A_{i,n+1} \beta_i = A_{i,n+1}$

But now $\underbrace{A_{1i}, \dots, A_{i-1\,i}}_{U_i}, A_{i,i+1} \cdots\cdots A_{i\,n+1}$ generate a free group!!

($\leftrightarrow$ motion of the i^{th} pt around the others)

(Pf: conjugate by $\pi = \sigma_n \sigma_{n-1} \cdots \sigma_i$



$$\pi A_{i,j+1} \pi^{-1} = A_{j,n+1} \qquad j = i, \dots, n$$

$$\pi A_{k,i} \pi^{-1} = A_{k,n+1} \qquad k = 1, \dots, i-1.$$

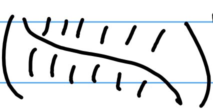
so conj by π maps this subgroup to U_{n+1}, hence free too).

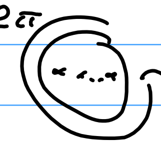
Hence: if β_i, $A_{i,n+1}$ elts of a free group commute, then β_i is a power of $A_{i,n+1}$.
($\uparrow$ one of the generators)

But actually $\beta_i \in U_i = \langle A_{1i} \dots A_{i-1,i} \rangle$. So $\beta_i = 1$ ◭

Corollary: Another sol^{n} to the word problem: given a braid $b \in B_n$ as a word in $\sigma_1 \dots \sigma_{n-1}$, compute $b_*(x_i)$ $1 \le i \le n$ (action of b on free group) Then $b = 1$ iff $b_*(x_i) = x_i \ \forall i$.
– fast from formula!

Corollary (Chow 1948): for $n \geq 3$ the center of B_n is the ∞ cyclic group gen^d by $\Delta_n^2 = (\sigma_1 \dots \sigma_{n-1})^n = (A_{12})(A_{13}A_{23}) \cdots (A_{1n}A_{2n} \cdots A_{n-1n})$

Pf: • $\Delta_n^2 = (\sigma_1 \dots \sigma_{n-1})^n = \Big($  $\Big)^n$ = rotation of whole disc by 2π

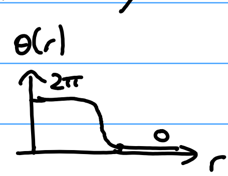


also check for other eq^n in A_i by induction on n:

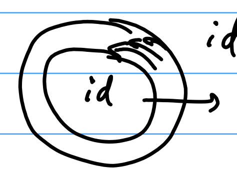
(rotate whole disc by first rotating inside w/ $n-1$ pts then annulus of n^{th})

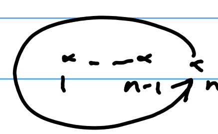
• so: clearly Δ_n^2 is central, as any homeo$_c(\mathbb{R}^2)$ commutes w/

rotate by $\theta(r)$



$\Delta_n^2 \equiv$



• $A_{1n} A_{2n} \cdots A_{n-1n}$ =  belongs to the centralizer of P_{n-1} in P_n (braids on first $n-1$ pts)
(ie. it commutes w/ every elt of P_{n-1}).

(can also check these claims using presentation of P_n ...).

- Let $\beta \in Z(B_n)$ central: for $n\geq 3$, $Z(\mathfrak{S}_n)=\{1\}$, so necess. $\beta\in P_n$ (else fails to commute w/ braids whose $\mathfrak{S}_n$ reduction doesn't commute).

- now: $\beta = \overline{\beta}_{n-1}\beta_n$, $\overline{\beta}_{n-1}\in P_{n-1}$, $\beta_n\in U_n$

β central $\Rightarrow$ $\overline{\beta}_{n-1}\beta_n A_{in}\beta_n^{-1}\overline{\beta}_{n-1}^{-1} = A_{in}$ $\forall i=1\dots n-1$

so $\beta_n A_{in}\beta_n^{-1} = \overline{\beta}_{n-1}^{-1} A_{in}\overline{\beta}_{n-1}$ $(*)$

$(*)$ $\Rightarrow$ $\beta_n(A_{1n}\cdots A_{n-1\,n})\beta_n^{-1} = \overline{\beta}_{n-1}^{-1}A_{1n}\cdots A_{n-1\,n}\overline{\beta}_{n-1}$

$= A_{1n}\cdots A_{n-1\,n}$

by above observ.ⁿ ($A_{1n}\cdots A_{n-1\,n}$ centralizes P_{n-1})

But β_n and $A_{1n}\cdots A_{n-1\,n}$ $\in$ free group U_n

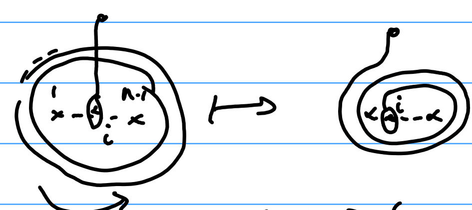
$\Rightarrow$ only way they can commute: $\beta_n = (A_{1n}\cdots A_{n-1\,n})^m$ for some $m\in\mathbb{Z}$.

- Plugging this into $(*)$, get formulas for the action of $\overline{\beta}_{n-1}$ on U_n by conjugation:

$$\overline{\beta}_{n-1}^{-1}A_{in}\overline{\beta}_{n-1} = (A_{1n}\cdots A_{n-1\,n})^m A_{in}(A_{1n}\cdots A_{n-1\,n})^{-m}$$

- But: action of P_{n-1} on U_n by conjugation $\longleftrightarrow$ action of P_{n-1} on F_{n-1} discussed above, in particular it's faithful! $\Rightarrow$ these formulas characterize $\overline{\beta}_{n-1}$ uniquely!

However...



$(\Delta^2_{n-1})^m$ $\qquad x_i \longmapsto (x_1\cdots x_{n-1})^m x_i (x_1\cdots x_{n-1})^{-m}$

as above

$\Rightarrow$ $\overline{\beta}_{n-1} = (\Delta^2_{n-1})^m$, and $\beta = \overline{\beta}_{n-1}\beta_n$

$= (\Delta^2_{n-1})^m (A_{1n}\cdots A_{n-1\,n})^m$

$= (\Delta^2_n)^m$ [using commutation b/w Δ^2_{n-1} & $A_{1n}\cdots A_{n-1\,n}$]

- $B_n \hookrightarrow \mathrm{Aut}(F_n)$: next we'll see a Thm of Artin which characterizes the image.