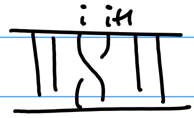
18.937 lecture 3 — Wed Feb 15

Study braid group of the plane: let $\widetilde{\mathcal{C}}_n = \widetilde{\mathcal{C}}_n(\mathbb{R}^2)$ ordered configs, $\mathcal{C}_n = \mathcal{C}_n(\mathbb{R}^2)$ unordered

**Thm** (Artin): $\pi_1 \mathcal{C}_n \cong \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i \quad \forall\, |i-j| \geq 2 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{array} \right\rangle$

**Pf:** for now, let
- $B_n$ = abstract group def$^{d}$ by this presentation
- $\tilde{\sigma}_i$ = element of $\pi_1 \mathcal{C}_n$ represented by 
- $\iota: B_n \to \pi_1 \mathcal{C}_n$, $\sigma_i \mapsto \tilde{\sigma}_i$ (well def$^{d}$ since we know $\tilde{\sigma}_i \tilde{\sigma}_j = \tilde{\sigma}_j \tilde{\sigma}_i$ and $\tilde{\sigma}_i \tilde{\sigma}_{i+1} \tilde{\sigma}_i = \tilde{\sigma}_{i+1} \tilde{\sigma}_i \tilde{\sigma}_{i+1}$)

Goal: show $\iota$ is an iso.

Reduce to pure braids by considering:
- $\nu: B_n \to \mathfrak{S}_n$, $\sigma_i \mapsto (i, i+1)$ (well def$^{d}$ since $(i,i+1)$ satisfy the relations)
- $\tilde{\nu}: \pi_1 \mathcal{C}_n \to \mathfrak{S}_n$ monodromy of the $\mathfrak{S}_n$-covering $\widetilde{\mathcal{C}}_n \to \mathcal{C}_n$, i.e.

given a geom braid $\alpha$ (repr. $b \in \pi_1 \mathcal{C}_n$) with base pt $\{(1,0), \dots, (n,0)\}$, lift to an arc in $\widetilde{\mathcal{C}}_n$ starting at $((1,0), \dots, (n,0))$ then the end pt is $((\tau(1),0), \dots (\tau(n),0))$ for some $\tau \in \mathfrak{S}_n$, $\tau =: \tilde{\nu}(b)$.

- know $\ker \tilde{\nu} = \pi_1 \widetilde{\mathcal{C}}_n$; define $P_n := \ker \nu$.

**Lemma:** $\iota: B_n \to \pi_1 \mathcal{C}_n$ is an isom. iff $\iota|_{P_n}: P_n \to \pi_1 \widetilde{\mathcal{C}}_n$ is an isom.

**Pf:** clearly $\nu$ is surjective, so $1 \to P_n \to B_n \xrightarrow{\nu} \mathfrak{S}_n \to 1$

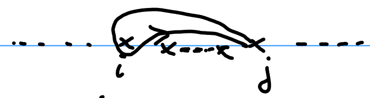
also know (last time) $1 \to \pi_1 \widetilde{\mathcal{C}}_n \to \pi_1 \mathcal{C}_n \xrightarrow{\tilde{\nu}} \mathfrak{S}_n \to 1$

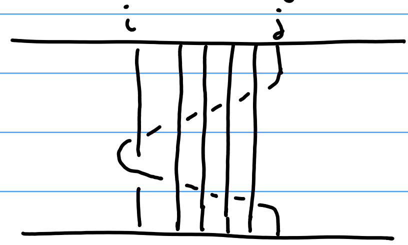
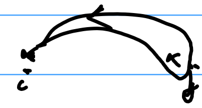
Moreover $\nu(\sigma_i) = \tilde{\nu}(\tilde{\sigma}_i)$ so get a commutative diagram
$$\begin{array}{ccc} B_n & \xrightarrow{\nu} & \mathfrak{S}_n \\ \downarrow \iota & & \| \\ \pi_1 \mathcal{C}_n & \xrightarrow{\tilde{\nu}} & \mathfrak{S}_n \end{array}$$
In particular $\iota(\ker \nu) \subset \ker \tilde{\nu}$, so get
$$\begin{array}{ccccccccc} 1 & \to & P_n & \to & B_n & \xrightarrow{\nu} & \mathfrak{S}_n & \to & 1 \\ & & \downarrow \iota|_{P_n} & & \downarrow \iota & & \| & & \\ 1 & \to & \pi_1 \widetilde{\mathcal{C}}_n & \to & \pi_1 \mathcal{C}_n & \to & \mathfrak{S}_n & \to & 1 \end{array}$$
Five lemma $\Rightarrow$ $\iota$ isom. iff $\iota|_{P_n}$ isom. (or: $\iota$ onto $\Leftrightarrow$ $\iota|_{P_n}$ onto ✓ & $\ker \iota = \ker \iota|_{P_n}$ ✓) $\blacktriangle$

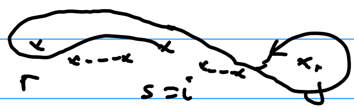
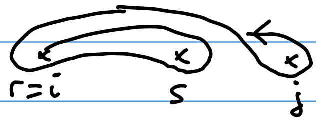
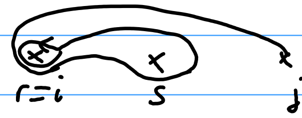
**Lemma:** $P_n$ admits a presentation w/ generators

$$A_{ij} = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-1}^{-1} \qquad 1\le i<j\le n$$

& relations
$$\begin{aligned} A_{rs}^{-1}A_{ij}A_{rs} &= A_{ij} \quad \text{if } r<s<i<j \text{ or } i<r<s<j \\ &= A_{rj}A_{ij}A_{rj}^{-1} \quad \text{if } r<i=s<j \\ &= A_{ij}A_{sj}A_{ij}A_{sj}^{-1}A_{ij}^{-1} \quad \text{if } r=i<s<j \\ &= A_{rj}A_{sj}A_{rj}^{-1}A_{sj}^{-1}A_{ij}A_{sj}A_{rj}A_{sj}^{-1}A_{rj}^{-1} \\ &\qquad \text{if } r<i<s<j \end{aligned}$$

Rmk: Geometrically, $z(A_{ij}) =$ 

$=$  $(=$ 

- case 1 clear (motions $\subset$ disjoint subsets commute)
- case 2 LHS $=$  , RHS $=$ 
- case 3 LHS $=$  RHS $=$ 
- case 4: exercise

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Pf. uses Reidemeister-Schreier method to get presentation of the subgroup $P_n \subset B_n$.

Let $\underset{\text{subgp}}{H} \subset G = \langle a_1,\dots, a_n \mid R_\mu \dots \rangle$
$\quad$ ↖ relations = words in $a_i^{\pm1}$.

**Def:** $\mathcal{R}$ = a set of words in $a_1 \dots a_n$ is a Schreier system if
(i) every right coset of $H$ in $G$ contains exactly one word in $\mathcal{R}$
(ie. $\forall g\in G$, $\exists!\, h\in H$, $\exists!\, w\in\mathcal{R}$ s.t. $g=hw$)
(ii) $\forall w\in\mathcal{R}$, any initial segment of $w$ (ie. subword $w_1$ s.t. $w=w_1w_2$) is also an element of $\mathcal{R}$.

Given $W$ word in $a_1\dots a_n$, let $\overline{W}$ = unique elt. of $\mathcal{R}$ in $H.W$
(ie. decomposition of $W$ is $W = \underset{\in H}{h}.\underset{\in\mathcal{R}}{\overline{W}}$)

Given $a_i$ $(1\le i\le n)$ & $K\in\mathcal{R}$, let $S_{K,i} = Ka_i.\overline{Ka_i}^{-1} \in H$ ("H-part" of $Ka_i$)

**Thm** (Reidemeister-Schreier)

$H$ has a presentation with

- generators: $(s_{K,i})$, $K \in \mathcal{R}$, $i \in \{1..n\}$, s.t. $s_{K,i} \not\equiv 1$. (simplifies as a word in $a_i$)

  (ie. s.t. $K a_i \notin \mathcal{R}$).

- relations: $\tau(K R_\mu K^{-1})$, $K \in \mathcal{R}$, $R_\mu$ relator in pres$^n$ of $G$,

  $\tau$ = "Reidemeister rewriting fn"

  $$\tau(a_{i_1}^{\varepsilon_1} \cdots a_{i_p}^{\varepsilon_p}) = s_{K_1, i_1}^{\varepsilon_1} \cdots s_{K_p, i_p}^{\varepsilon_p}$$

  where $K_j = \overline{a_{i_1}^{\varepsilon_1} \cdots a_{i_{j-1}}^{\varepsilon_{j-1}}} \in \mathcal{R}$ if $\varepsilon_j = 1$

  $K_j = \overline{a_{i_1}^{\varepsilon_1} \cdots a_{i_j}^{\varepsilon_j}}$ if $\varepsilon_j = -1$.

  (Point: $U \in H$ given as a word in $a_j \rightsquigarrow \tau(U)$ = same elt rewritten in terms of the $s_{K,i}$)

In the case of $P_n = \mathrm{Ker}(\nu : B_n \to \mathfrak{S}_n)$; coset representatives = any set of $n!$ words s.t. images by $\nu$ = all of $\mathfrak{S}_n$.

can take Schreier system $\mathcal{R} = \left\{ \prod_{j=2}^{n} M_{j,k_j} \,/\, 1 \le k_j \le j \right\}$, $M_{j,i} = \sigma_{j-1} \cdots \sigma_i$ $(i<j)$ or $1$ $(i=j)$

$\equiv$ bubble sort algorithm:

given any $\tau \in \mathfrak{S}_n$,
- use $M_{2,k_2}$ to send $\tau(2)$ into the correct pos$^n$ compared to $\tau(1)$ (ie. $1$ if $\tau(1) < \tau(2)$, $\sigma_1$ otherwise)
- use $M_{3,k_3}$ to insert $\tau(3)$ into right place wrt $\tau(1), \tau(2)$.

etc...

Clearly, prefix is again $\in \mathcal{R}$ (stop somewhere in coll$^n$ of $M_{j,i}$)

**Hard exercise:** apply Reidemeister-Schreier to get a pres$^n$ of $P_n$
(Birman gets it wrong, I can't do it, ...)
(Rmk in book gives hint on doing it by induction on $n$ ...)

**Look at case $n=3$:** generators of $B_3$ = $\sigma_1, \sigma_2$   Relator $\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}$

$$\mathcal{R} = \{1, \sigma_1, \sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_1\}$$

$s_{1,1}$, $s_{1,2}$, $s_{\sigma_1,2}$, $s_{\sigma_2,1}$, $s_{\sigma_1\sigma_2,1}$ trivial ($K\sigma_i \in \mathcal{R}$)

$s_{\sigma_1,1} = \sigma_1^2 = A_{12}$ , $s_{\sigma_2,2} = \sigma_2^2 = A_{23}$

$s_{\sigma_2\sigma_1,1} = \sigma_2\sigma_1^2\sigma_2^{-1} = A_{13}$ , $s_{\sigma_2\sigma_1,2} = \sigma_2\sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}$

$s_{\sigma_1\sigma_2,2} = \sigma_1\sigma_2^2\sigma_1^{-1}$

$s_{\sigma_1\sigma_2\sigma_1,1} = \sigma_1\sigma_2\sigma_1^2\sigma_2^{-1}\sigma_1^{-1}$, $s_{\sigma_1\sigma_2\sigma_1,2} = \sigma_1\sigma_2\sigma_1\sigma_2\,\sigma_1^{-1}\sigma_2^{-1}$

7 generators

Relations: for each of the six $K \in R$, get $\tau(K\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}K^{-1})$

e.g. for $K=1$, $\tau(\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}) = s_{\overline{1},1}\, s_{\overline{\sigma_1},2}\, s_{\overline{\sigma_1\sigma_2},1}\, s^{-1}_{\overline{\sigma_1\sigma_2\sigma_1\sigma_2^{-1}},2}\, s^{-1}_{\overline{\sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}},1}\, s^{-1}_{\overline{\cdots},2}$

$= s_{1,1}\, s_{\sigma_1,2}\, s_{\sigma_1\sigma_2,1}\, s^{-1}_{\sigma_2\sigma_1,2}\, s^{-1}_{\sigma_2,1}\, s^{-1}_{1,2}$

$\Rightarrow$ $\underline{s_{\sigma_2\sigma_1,2} = 1}$ (kill 1 gen.)

for $K=\sigma_1$ , $\tau(\sigma_1^2\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}) = s_{1,1}\, s_{\sigma_1,1}\, s_{1,2}\, s_{\sigma_2,1}\, s^{-1}_{\sigma_1\sigma_2\sigma_1,2}\, s^{-1}_{\sigma_1\sigma_2,1}\, s^{-1}_{\sigma_1,2}\, s^{-1}_{1,1}$

$= A_{12} \cdot s^{-1}_{\sigma_1\sigma_2\sigma_1,2}$

$\rightarrow$ $\underline{s_{\sigma_1\sigma_2\sigma_1,2} = A_{12}}$

etc. (The 6 relations kill 4 of the 7 generators & give 2 relations among the $A_{ij}$.

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Pf. Thm: show $\iota_{1P_n}: P_n \to \pi_1 \tilde{\mathcal{C}}_n$ isom. Use induction on $n$:

$\boxed{\text{Group theoretic side}}$

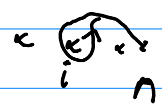
- $P_{n-1} \cong$ subgroup gen$^d$ by $(A_{ij},\ 1\le i<j\le n-1)$ inside $P_n$ (same presentation!)
- $\eta: P_n \twoheadrightarrow P_{n-1}$
  $A_{ij} \mapsto A_{ij}$ if $j \neq n$
  $A_{in} \mapsto 1$
- Let $U_n$ = subgp gen$^d$ by $(A_{in},\ 1\le i\le n-1)$
  presentation of $P_n$ (case $j=n$) $\Rightarrow$ $U_n \triangleleft P_n$ $\Rightarrow$ $U_n = \operatorname{Ker}\eta$.

$\boxed{\text{Geometric side}}$

Recall $1 \to \underbrace{\pi_1(\mathbb{R}^2 - Q_{n-1})}_{n^{th}\text{ strand only}} \to \pi_1 \tilde{\mathcal{C}}_n \xrightarrow{\pi_*} \underbrace{\pi_1 \tilde{\mathcal{C}}_{n-1}}_{\text{forget } n^{th} \text{ strand}} \to 1$ (fibration LES)

$$\begin{tikzcd}
1 \arrow[r] & U_n \arrow[r] \arrow[d, "\iota_n|U_n"] & P_n \arrow[r, "\eta"] \arrow[d, "\iota_n"] & P_{n-1} \arrow[r] \arrow[d, "\iota_{n-1}"] & 1 \\
1 \arrow[r] & \pi_1(\mathbb{R}^2 - Q_{n-1}) \arrow[r] & \pi_1 \tilde{\mathcal{C}}_n \arrow[r, "\pi_*"] & \pi_1 \tilde{\mathcal{C}}_{n-1} \arrow[r] & 1
\end{tikzcd}$$

Commutative w/ exact rows

( $\iota_{|U_n}$ maps to $\underbrace{\pi_1(\mathbb{R}^2 - Q_{n-1})}_{= \ker \pi_*}$ because each $A_{in}$ maps to braid 

ie. $\exists$ geometric representative where first $n-1$ pts don't move

$\Rightarrow \{\iota(A_{in})\}_{1 \le i \le n-1}$ = basis for free group on $n-1$ generators $\pi_1(\mathbb{R}^2 - Q_{n-1})$

Hence $U_n$ is also free (if had relations, would get $\pi_1(\mathbb{R}^2 - Q_{n-1})$ has relations too)

- $P_1 = 1$, and $\pi_1 \tilde{\mathcal{C}}_1 = \pi_1 \mathbb{R}^2 = 1 \Rightarrow \iota_1$ isom.

$\to$ by induction, $\iota_{n-1}$ isom, $\iota_{n|U_n}$ isom $\Rightarrow \iota_n$ isom. ∎

$\Rightarrow$ from now on, identify $\sigma_i$ w/ $\tilde{\sigma}_i$, $B_n$ w/ $\pi_1 \mathcal{C}_n$, $P_n$ w/ $\pi_1 \tilde{\mathcal{C}}_n$ ✓.

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$\boxed{\text{Artin's Solution to word problem}}$

- The pf also shows: Corollary: $P_n = P_{n-1} \ltimes \underset{(\text{free})}{U_n}$

Def: permutation braid = one of the $n!$ coset representatives above

Corollary: (normal form): $\forall \beta \in B_n$ can be written uniquely as
$\beta = \beta_2 \beta_3 \cdots \beta_n \pi_\beta$, $\pi_\beta$ permutation braid
$\beta_i \in U_i$.
& this normal form is algorithmically computable. ("Combing" $\beta$).

$\boxed{\text{Word pb: compare normal forms}}$

Pf: • permutation braids = coset representatives $\Rightarrow \exists!\ \beta = \bar{\beta}_n \pi_\beta$, $\bar{\beta}_n \in P_n$
(Computable from word: induced perm$^{\text{n}}$, bubble sort) $\pi_\beta$ permut braid

• $\bar{\beta}_n = \bar{\beta}_{n-1} \beta_n$ for some $\bar{\beta}_{n-1} \in P_{n-1}$ ($\bar{\beta}_{n-1} = \eta(\bar{\beta}_n)$, forget $n^{th}$ strand)
$\beta_n \in U_n$ free group on $A_{in}$ ($\Rightarrow$ normal form ✓)
& induction on $n$. ∎