Lecture 24 – May 15

Q: How many sympl. 4-mflds admit Lefschetz fibrations?

A. not so many, because need a slight generalization: Lefschetz pencils.

E.g.: $X \subset \mathbb{CP}^N$ proj. surface, take generic linear proj. $\mathbb{CP}^N - \mathbb{CP}^{N-2} \xrightarrow{\pi} \mathbb{CP}^1$

e.g. $(x_0 : \dots : x_N) \mapsto (x_0 : x_1)$

"fibers" = intersections of X w/ pencil of hyperplanes $\{x_0 = \alpha x_1\}_{\alpha \in \mathbb{C} \cup \infty = \mathbb{CP}^1}$

⇝ generic fiber is a smooth proj. curve $\subset X$

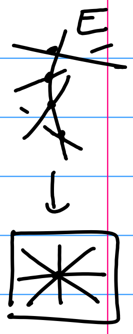
some isolated fibers may be singular (can show: at most nodes, in generic situation).

This looks like the previous setup, except... $f = \pi_{|X}$ not defined at $X \cap \mathbb{CP}^{N-2} =: B$ "base points" (finite set).

This is because all hyperplanes $\{x_0 = \alpha x_1\}$ contain $\{x_0 = x_1 = 0\}$, so all "fibers" of f contain B - - -.

Def: $X \supset B = \{b_1 .. b_n\}$ finite set, $f : X - B \to \mathbb{CP}^1$ is a Lefschetz pencil if
- near $b_i \in B$, $\exists$ loc. coords where $f(z_1, z_2) = (z_1 : z_2)$
- outside B, f is a L-fibration, ie. isolated crit pts where $\sim z_1^2 + z_2^2$.

Blowup construction



$$\begin{tikzcd} \hat{\mathbb{C}}^2 := \{(x,\ell) \in \mathbb{C}^2 \times \mathbb{CP}^1,\ x \in \ell\} \arrow[d, "\pi"'] \\ \mathbb{C}^2 \end{tikzcd}$$

($\equiv$ tautological bundle over $\mathbb{CP}^1$) $\mathcal{O}(-1)$

π is 1-1 except at 0, $\pi^{-1}(0) = \mathbb{CP}^1$

Replace 0 by set of $\mathbb{C}$ lines through it.

Then the pencil of lines through 0 in $\mathbb{C}^2$ lift to a family of disjoint lines (fibers of $\hat{\mathbb{C}}^2 \to \mathbb{CP}^1$). The exceptional curve of the blowup $E = \pi^{-1}(0)$ is the zero section of tautolg. bundle, intersects each line once.

- This is a $\mathbb{C}$ geom. description. Can also "blow up topologically" (given coords. near a point, do this!), or symplectically (beware: the sympl. form on $\hat{\mathbb{C}}^2$ depends on choice of a "size" param. = symplectic area of the exceptional curve.)

- By def. of a Lefschetz pencil, if we blow up X at the base points b_i ($\to$ get $\hat{X} \xrightarrow{\pi} X$ a new 4-mfld): then f extends over all of $\hat{X}$, & gives a Lefschetz fibration $\hat{f}: \hat{X} \to S^2$, with distinguished sections $E_1, \dots, E_n$ (the exc. curves of the blowups).
  (this is exactly the same as above rmk on $\tilde{\mathbb{C}}^2$, near each b_i).
  NB: these exc. sections have normal bundle of deg. -1 :

- Conversely, given a LF with section of square -1, can "blow down" & get a L. pencil.

Adaptation of the results about LFs:

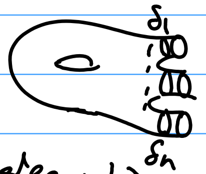
- Monodromy: the distinguished sections $E_1 \dots E_n$ define n marked pts on Σ_g (the fiber of f). So monodromy now takes place in $\mathrm{Map}_n(\Sigma_g)$ (& still consists of Dehn twists / vanishing cycles).

  In fact, can do better: over $\mathbb{C} = \mathbb{CP}^1 - \{\infty\}$
  $\uparrow$ regular value of f,
  trivialize normal bundle to section E_i $\Rightarrow$ remove a small disk around each marked pt, & get monodromy $\pi_1(\mathbb{C} - \mathrm{crit}\, f) \to \mathrm{Map}(\Sigma_{g,n})$.
  $= \mathrm{Map}_{g,n}$.
  (the vanishing cycles now $\subset \Sigma_{g,n}$)

can represent monodromy along loops $\subset$ affine part $\mathbb{C}$ by diffeos that $\equiv \mathrm{Id}$ near base pts.

- Monodromy at ∞ = failure of global triviality = boundary twist
  $\delta \in \ker(\mathrm{Map}\,\Sigma_{g,n} \to \mathrm{Map}_n\,\Sigma_g)$.

  $\delta = \prod_{i=1}^{n} \delta_i$,



  (believable ; we'll see later why).

- So: If genus $g \geq 2$ (or $g=1, n\geq 1$; $g=0, n \geq 3$) then
  $\{$genus g LFs w/ n distinguished -1-sections$\}$ / isom.
  $\xleftrightarrow{1-1}$ $\{$fact$^{\text{ns}}$ of δ as $\prod$(Dehn twists) in $\mathrm{Map}_{g,n}\}$ / conj. Hurwitz.

Ex: ‖ the pencil of conics on $\mathbb{CP}^2$ & the lantern relation.

Fix 4 points in $\mathbb{CP}^2$ (no 3 aligned) & consider family of conics through them.

This ≡ also setup with $\mathbb{CP}^2 \underset{\substack{|O(2)| \\ \text{deg-4 surface}}}{\hookrightarrow} \mathbb{CP}^5$ & family of hyperplanes through a codim 2 space which intersects $i(\mathbb{CP}^2)$ in 4 pts.

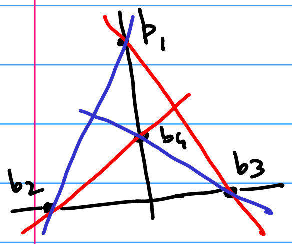
Or also:

Passing through 4 pts = 4 linear cond. on the coeffs. of the conic; get a $\mathbb{CP}^1$-family

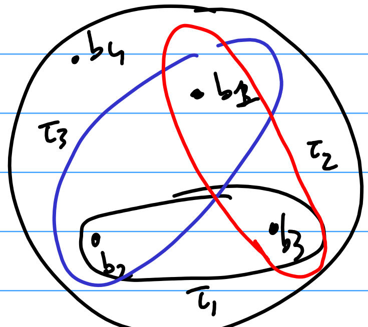
Through any 5 pts ∃! conic – possibly degenerate – so get a map $\mathbb{CP}^2 \setminus \{4\text{ pts}\} \to \mathbb{CP}^1$

The singular fibers of the pencil ≡ degenerate conics through the 4 points $b_1 \dots b_4$

≡ union of 2 lines; there are 3 of them



The 3 vanishing cycles are



Hence: monodromy fact^{n} is: $\tau_1 \tau_2 \tau_3 = \delta$

in $\mathrm{Map}_{0,4}$, where δ = prod. of the 4 boundary twists

≡ LANTERN RELATION

- Thm (Gompf). ‖ $f : M \setminus B \to S^2$ Lefschetz pencil s.t.
  every component of every fiber contains ≥1 base point.
  Then ∃ ω symplectic form on M, $\omega|_{\text{fibers}} > 0$,
  $[\omega]$ = Poincaré dual to fiber class,
  & such ω is unique up to sympl. isotopy.

(Idea: first build $\hat{\omega}$ on $\hat{M}$, choosing $[\alpha] = c =$ P.D. to $\sum [E_i]$
and "blow down" E_i symplectically .... ($c|_{\text{fibers}} > 0$ ✓).)

Existence result: similar to what we've seen about branched covers (but predates it)

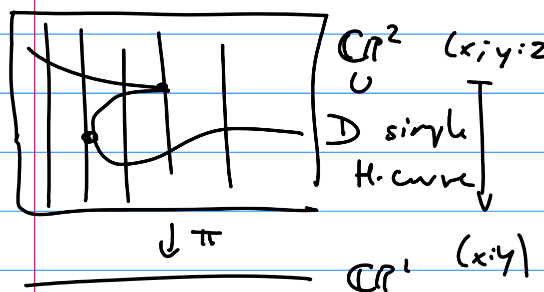
- Thm (Donaldson 1998) ‖ (X^4, ω) sympl. 4-mfld, $[\omega]$ integral class,
  $k \gg 0$ large integer ⇒ ∃ $f_k : X \setminus B_k \to \mathbb{CP}^1$
  Lefschetz pencil w/ symplectic fibers, $[\text{fiber}] = PD(k[\omega])$
  If k large enough, constr. is canonical up to isotopy.

‖ So... again get 2 maps $\{\text{sympl 4-mflds, integral}\}/\sim \;\rightleftarrows\; \{\text{monodromy fact}^{ns}\}/\sim$

Relation w/ branched covers w/ simple Hurwitz branched curves:

X

$\downarrow f$ branched cover



Consider the composition

$$\phi = \pi \circ f : X \setminus \underbrace{f^{-1}(0:0:1)}_{=:B} \longrightarrow \mathbb{CP}^1$$

Fibers of $\phi \equiv$ preimages by f of the fibers of π (vertical lines in $\mathbb{CP}^2$)

If we blow up $\mathbb{CP}^2$ at $(0:0:1)$, and blow up X at $f^{-1}(0:0:1)$, get

$$\hat{X} \xrightarrow{\hat{f}} \underset{(\mathbb{P}^1\text{-bundle}/\mathbb{P}^1)}{\mathbb{F}_1} \xrightarrow{\pi} \mathbb{CP}^1$$

Fact: $\|$ ϕ is a Lefschetz pencil

In fact: singular fibers of $\phi \equiv$ preimages of fibers of π that are tangent to D at a smooth pt of D.

Note: $d\phi = d(\pi \circ f)$ surjective unless df not surj. (i.e. we're along R)

and $\underset{\overset{\shortparallel}{TD}}{\operatorname{Im} df} \cap \ker d\pi \neq \{0\}$ – occurs only where D tangent to fiber of π.

Ex: near a cusp,

$$\begin{tikzcd} \mathbb{C}^2 \supset D = \{y^2 = x^3\} \arrow[d, "{(x,y)}"] \\ \mathbb{C} \quad x \end{tikzcd}$$

$(x,y) \mapsto x$

branch curve of $f : (z_1, z_2) \mapsto \left(z_2, \dfrac{z_1^3 - 3z_1 z_2}{2}\right)$

Composition is $\phi . (z_1, z_2) \mapsto z_2$ ✓.

while, near a tangency,

$$\begin{tikzcd} \mathbb{C}^2 \supset D = \{y^2 = x\} \arrow[d] \\ \mathbb{C} \end{tikzcd}$$

double cover of $\mathbb{C}^2$ branched along D is $\underset{\overset{\Updownarrow}{x = y^2 + z^2}}{\{z^2 = x - y^2\}} \subset \mathbb{C}^3$, $\simeq \mathbb{C}^2$ taking proj. to (y,z) variables.

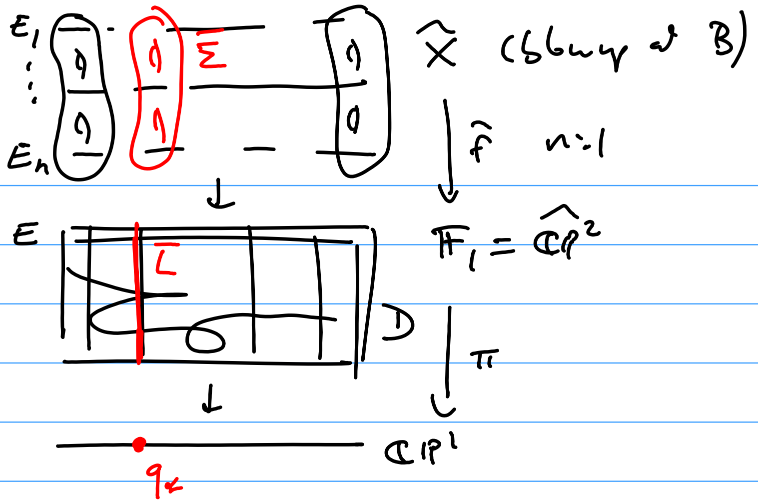
Ie. local model for $\left[\begin{matrix} f \\ \phi \end{matrix}\right]$ is

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "f"] & \mathbb{C}^2 \arrow[r, "\pi"] & \mathbb{C} \\ (y,z) \arrow[r, mapsto] \arrow[rr, bend right, "\phi"'] & (y^2+z^2, y) \arrow[r, mapsto] & y^2+z^2 \end{tikzcd}$$

as expected for a L. pencil.

($\leadsto$ Donaldson's existence result for L. pencils can be deduced from the existence result for branched covers).

Lifting the monodromy:



Fix a base pt $q_* \in \mathbb{CP}^1$, and $\bar L = \pi^{-1}(q_*)$ proj. line
$\bar\Sigma = \hat f^{-1}(\bar L)$ fiber of ϕ

The map $f_{|\bar\Sigma} : \bar\Sigma \to \bar L$ is an n-fold branched covering, with simple branching at the pts of $D \cap \bar L$ & monodromy $\theta_{|\bar L} : \pi_1(\bar L - (\bar L \cap D)) \to \mathfrak{S}_n$.

Let $L = \bar L - \nu(\bar L \cap E)$ = disc containing the pts of $L \cap D$.
$\Sigma = \bar\Sigma - \nu(E_i)$ = complement of nbd of base pts in fiber of ϕ $(\simeq \Sigma_{g,n})$.

If we move to a different fiber of π, the intersection pts in $L \cap D$ move, & this modifies the covering $f_{|\Sigma} : \Sigma \to L$.

In particular, moving along a loop $\gamma \in \pi_1(\mathbb{C} - \mathrm{crit}(\pi_{|D}))$, get the braid monodromy of D along γ $(\in B_d)$, and the monodromy of ϕ along γ $(\in \mathrm{Map}_{g,n})$.

Recall the lifting homomorphism from a subgp $LB_d \subset B_d$ (liftable braids) to $\mathrm{Map}_{g,n}$ (depends on $\theta_{|L}$).

- the braid monodromy of D takes values in the liftable subgroup LB_d.
- the monodromies are related by:

$$\begin{tikzcd}
\pi_1(\mathbb{C} - \mathrm{crit}\,\pi_{|D}) \arrow[r, "\text{braid mon of } D"'] \arrow[rr, bend right, "\text{monodromy of } \phi"'] & LB_d \arrow[r, "\text{lifting hom.}"'] & \mathrm{Map}_{g,n}
\end{tikzcd}$$

- in particular, halftwists (along arcs s.t.  branching in same sheets of covering)
$\mapsto$ Dehn twists (along loop formed by 2 lifts of the arc)
while $(\text{halftwists})^2$, $(\text{halftwists})^3$ (w/ non-matching ends) $\longmapsto 1$.

▶ This is actually often the best way to compute the monodromy of a L-pencil !!