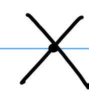
Lecture 23 – May 10

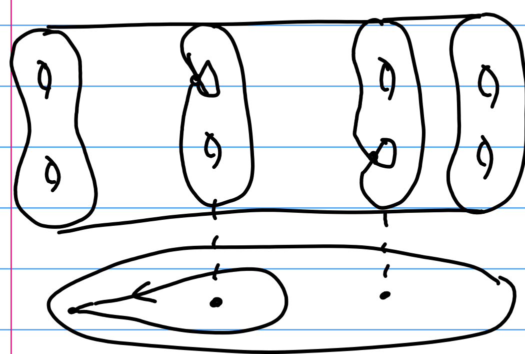
Lefschetz fibrations:

Def. M^4 oriented, $f: M \to S^2$ is a <u>Lefschetz fibration</u> if critical pts of f are isolated & near each of them $\exists$ local orientation-preserving coordinates in which $f(z_1, z_2) = z_1^2 + z_2^2$.

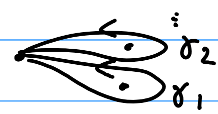
- We'll also assume the critical values are distinct.
- The singular fibers (crit. levels) of f have sing. $\sim \{z_1^2 + z_2^2 = 0\}$

$$z_1^2+z_2^2 = (z_1+iz_2)(z_1-iz_2)$$

$\Rightarrow$  ODP, or <u>"node"</u>



- <u>Monodromy</u> of f: $\pi_1(S^2 - \text{crit} f) \longrightarrow \text{Map}_g$ mapping class group of regular fiber.
  - defined up to simult. conjugation by an element of Map_g
  - as before, can choose a basis of $\pi_1(S^2 - \text{crit} f) = \langle \gamma_1 \dots \gamma_r \mid \prod \gamma_i = 1 \rangle$

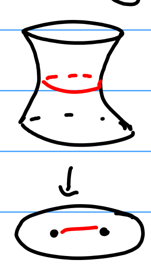


$\Rightarrow$ get a <u>factorization of Id</u> as a product of the monodromies around the individual crit. values.

Choice of basis $\Rightarrow$ this is up to <u>Hurwitz moves</u>. (B_r-action)

$$\langle \gamma_1, \dots, \gamma_r \rangle \longleftrightarrow \langle \gamma_1, \dots, \gamma_i \gamma_{i+1} \gamma_i^{-1}, \gamma_i, \dots, \gamma_r \rangle$$

- near a sing. fiber: regular levels $\{z_1^2 + z_2^2 = t\}$



$$\{z_1^2+z_2^2=t\} \xrightarrow[2:1]{pr_{z_1}} \mathbb{C} \quad \text{branched at } z_1 = \pm t^{1/2}$$

"Neck": assuming $t \in \mathbb{R}_+$: $pr_{z_1}^{-1}([-\sqrt{t}, \sqrt{t}]) = \{(z_1, z_2) \in \underline{\underline{\mathbb{R}^2}}, z_1^2 + z_2^2 = t\}$

This loop shrinks $\to$ origin as $t \to 0$: it's called the <u>vanishing cycle</u> at the critical value.

More generally, given an arc  q_* —— q_{crit} associate a vanishing cycle = s.c.c. in $f^{-1}(q_*)$ = collapses as $q \to q_{crit}$.

Monodromy around sing. fiber: for $t = \varepsilon e^{i\theta}$,

$\{z_1^2 + z_2^2 = t\}$
$\downarrow pr_{z_1}$
$\mathbb{C}$

the branch points $\pm \varepsilon^{1/2} e^{i\theta/2}$ rotate by a <u>half-twist</u>.

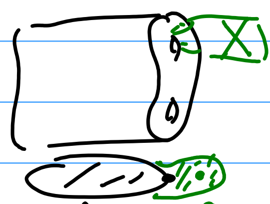
Recall lifting homomorphism: half-twist $\mapsto$ Dehn twist.

Hence: ‖ the monodromy of f around a crit val is the Dehn twist along the vanishing cycle.

- Given the monodromy over a large disc, $\pi_1(D^2 - \text{crit } f) \to \mathrm{Map}_g$
  generators $\mapsto$ Dehn twists ---

<u>Prop</u> (Kas): ‖ the monodromy determines the LF f over disc D^2 up to isomorphism

(<u>Idea</u>: • start with small disc $\Delta_0 \not\ni$ any crit. val., $f^{-1}(\Delta_0) \simeq \Delta_0 \times \Sigma_g$
then successively larger discs; when add a crit. fiber, must glue



Δ_i q_{i+1}

$\leftarrow$ local model (ball $\subset \mathbb{C}^2$) near vanishing cycle, trivial (Σ - annulus) $\times$ disc on rest of fiber).

- However, to recover f over S^2 (assuming boundary monodromy trivial):
  boundary of $f_{|D^2}$ is $\simeq \Sigma_g \times S^1$. Need to glue $\Sigma_g \times D^2_\infty$ in a fibration-preserving manner

$\}D^2$ containing crit f
$\}D^2_\infty$

Requires choice: an element of $\pi_1 \mathrm{Diff}(\Sigma_g)$.
Homeo

We've seen a while ago: if $g \geq 2$ then this π_1 is trivial.

Hence: <u>Prop</u>: ‖ for $g \geq 2$, $\{\text{genus } g \text{ L. fibrations}\}/\text{isom}$
$\Updownarrow$ 1-1
$\{\text{fact}^{\text{ns}} \text{ of Id as } \prod(\text{Dehn twists})\}/\text{conj. / Hurwitz}$

If $g = 0$ or 1, also need to specify an elt of $\pi_1 \mathrm{Diff}(\Sigma_g)$
(although, if monodromy of f is nontrivial, diff^t choices may become equivalent)

Symplectic point of view:

Thm (Thurston, 70s). $\Sigma \to M^4 \xrightarrow{f} \Sigma'$ loc. triv. bundle, [fiber] not torsion in $H_2(M)$
$\Rightarrow \exists$ sympl. structure on M s.t. $\omega_{|\text{fibers}} > 0$

generalizes to

Thm (Gompf) 1998: $f: M \to S^2$ L.F.; assume [fiber] is not a torsion element in $H_2(M,\mathbb{Z})$. Then $\exists$ sympl. form ω on M s.t. the fibers of f are sympl. submflds; such ω is unique up to deformation (ie. $(\omega_t)_{t\in[0,1]}$, $[\omega] \neq$ cst)

Note: the only case where the assumption on fiber class can fail is if f is actually a torus bundle $/S^2$ (trivial monodromy).

Ex: Hopf fibration $S^1\times S^3 \to S^2$ fiber $S^1\times S^1$
$H_2(S^1\times S^3) = 0$ ... so can't be symplectic ! ($[\omega]=0$ ...)

Pf: • [fiber] $\neq$ torsion $\Rightarrow \exists\, c \in H^2(M,\mathbb{R})$, $c\cdot[\text{fiber}] > 0$.

Goal: build $\alpha \in \Omega^2(M)$, closed, $[\alpha] = c$, $\alpha_{|\text{fiber}} > 0$ on all fibers

Indeed: Given such α, let $\omega = \alpha + K f^*\omega_{S^2}$, $K \gg 0$

Then ω is closed, $\omega_{|\text{fiber}} = \alpha_{|\text{fiber}} > 0$, and

$$\omega\wedge\omega = \underbrace{\alpha\wedge\alpha}_{\text{bounded}} + 2K\,\underbrace{\alpha\wedge f^*\omega_{S^2}}_{\substack{>0 \text{ since } \alpha_{|\text{fiber}}>0;\text{ so}\\ \geq c>0 \text{ by compactness}}}$$

$\Rightarrow \omega$ symplectic for K large enough.

Moreover, space of $\{\alpha \text{ closed} \mid \alpha_{|\text{fibers}} > 0\}$ is convex
$\Rightarrow$ given α_0 & α_1, consider $\alpha_t = t\alpha_1 + (1-t)\alpha_0$ and
$\omega_t = \alpha_t + K_t f^*\omega_{S^2}$, K_t suff$^{\text{ly}}$ large.
This gives uniqueness up to deformation.

• Building α: start w/ any representative η_0, $[\eta_0] = c$.

near a smooth fiber $\Sigma = f^{-1}(p)$, trivialize a nbd $f^{-1}(U_p) \simeq \Sigma\times U_p$ (disc at p), and consider an area form σ on Σ s.t. $[\sigma] = i^*c$, $i: \Sigma \hookrightarrow M$
(ie. $\int_\Sigma \sigma = c\cdot[\text{fiber}] > 0$)

Then $\alpha_p = pr_1^*\sigma$ 2-form on $f^{-1}(U_p)$, $\begin{cases}\alpha_p \text{ positive on fibers,}\\ [\alpha_p] = c_{|f^{-1}(U_p)} \quad (\text{since } pr_1 \simeq \text{ on } H^2)\end{cases}$

Near a sing. fiber, local model $\rightsquigarrow$ can also build a closed 2-form α_p on $f^{-1}(U_p)$ s.t. $\begin{cases} \alpha_{p|\text{fibers}} > 0 \\ [\alpha_p] = c_{|f^{-1}(U_p)}. \end{cases}$

(start w/ $d(\text{cut-off}\cdot(x_1\,dy_1 + x_2\,dy_2))$ on nbd of crit. pt. in coords. st. f standard and extend to rest of fiber where things are loc. trivial)

Remaining task: patch these α_p's together (for a covering $\bigcup_{p_i} U_{p_i} = S^2$)
$\to$ need to use partition of unity smartly (so linear combination remains closed)

Recall we chose $[\alpha_p] = c_{|f^{-1}(U_p)}$

$\to \exists\, \beta_p$ 1-form on $f^{-1}(U_p)$ s.t. $\alpha_p = \eta_0 + d\beta_p$.

Let ρ_p = partition of unity on S^2 subordinate to open cover U_p.

$\Sigma \rho_p = 1$ $\rightsquigarrow$ let $\alpha = \eta_0 + d\left[\Sigma\,(\rho_p \circ f)\,\beta_p\right]$.

Clearly, α is closed, $[\alpha] = c$.

$\alpha_{|\text{fiber}} = \eta_{0|\text{fiber}} + d\left(\Sigma\,\underbrace{(\rho_p\circ f)_{|\text{fiber}}}_{\text{constant}}\cdot \beta_{p|\text{fiber}}\right) = \Sigma\,(\rho_p\circ f)\,\underbrace{(\eta_0 + d\beta_p)_{|\text{fiber}}}_{\alpha_p}$

> 0. ✓

QED

Q: How many sympl. 4-mflds admit Lefschetz fibrations?

A. not so many, because need a slight generalization: Lefschetz pencils.

E.g.: $X \subset \mathbb{CP}^N$ proj. surface, take generic linear proj. $\mathbb{CP}^N - \mathbb{CP}^{N-2} \xrightarrow{\pi} \mathbb{CP}^1$

e.g. $(x_0 : \dots : x_N) \mapsto (x_0 : x_1)$

"fibers" = intersections of X w/ pencil of hyperplanes $\{x_0 = \alpha x_1\}_{\alpha \in \mathbb{C}\cup\infty = \mathbb{CP}^1}$

$\rightsquigarrow$ generic fiber is a smooth proj. curve $\subset X$
some isolated fibers may be singular (can show: at most nodes, in generic situation).

This looks like the previous setup, except... $f = \pi_{|X}$ not defined at $X \cap \mathbb{CP}^{N-2} =: B$ "base points" (finite set).

This is because all hyperplanes $\{x_0 = \alpha x_1\}$ contain $\{x_0 = x_1 = 0\}$, so all "fibers" of f contain B ...

Def: $X \supset B = \{b_1 \dots b_n\}$ finite set, $f : X - B \to \mathbb{CP}^1$ is a Lefschetz pencil if
* near $b_i \in B$, $\exists$ loc. coords where $f(z_1, z_2) = (z_1 : z_2)$
* outside B, f is a L. fibration, ie. isolated crit pts where $\sim z_1^2 + z_2^2$.