Lecture 20 — Mon May 1

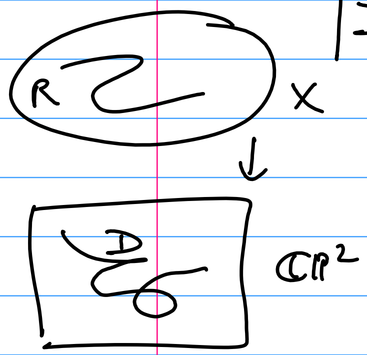
$X \subset \mathbb{CP}^N$ complex proj. surface, $p : \mathbb{CP}^N - \mathbb{CP}^{N-3} \to \mathbb{CP}^2$ linear proj$^{\text{n}}$.
Can assume $\mathbb{CP}^{N-3} \cap X = \emptyset$, so $p_{|X} = f : X \to \mathbb{CP}^2$ well-def$^{\text{d}}$ map, $\deg f = \deg X$.

- We'll assume X is in generic position wrt p. In fact this can be ensured by choosing p well.

<u>Prop:</u> (classical) For a generic choice of the linear proj. p, $f : X \to \mathbb{CP}^2$ is a branched covering whose branch curve has only nodes & ordinary cusp singularities.

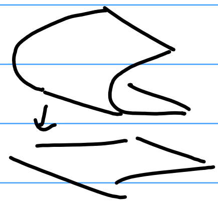
In fact; (1) $R \subset X$ ramification curve $= \{ p \in X \,/\, df_p \text{ not an iso} \}$ is a <u>smooth</u> alg. curve $\subset X$

$D = f(R) \subset \mathbb{CP}^2$ discriminant curve $= \{ z \in \mathbb{CP}^2 \,/\, \# f^{-1}(z) < \deg X \}$ <u>plane alg curve</u> w/ <u>cusps & nodes</u>.

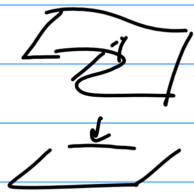


(2) <u>Local models</u>: $\forall p \in X$, $\exists$ local holom. coords on $U(p) \subset X$, $U(f(p)) \subset \mathbb{CP}^2$ in which f is

- $(x,y) \mapsto (x,y)$ if $p \notin R$ (local diffeo.!)



- $(x,y) \mapsto (x^2, y)$ at generic pts of R "<u>simple branching</u>"



- $(x,y) \mapsto (x^3 - xy, y)$ "<u>cusp</u>".

here $R :\ \det df = 3x^2 - y = 0$ smooth

$D = f(R) = \{(-2x^3, 3x^2)\} = \{ 27 z_1^2 = 4 z_2^3 \}$ ordinary cusp

- Where do <u>nodes</u> come from?

They correspond to 2 distinct pts of R where simple branching occurs and which happen to map to the same point in $\mathbb{CP}^2$.

Status of the result is dubious. See Kulikov & Kulikov 2000 for an attempt

<u>Idea pf:</u> transversality theory — in fact the 3 local models are precisely those for generic holomorphic maps $\mathbb{C}^2 \to \mathbb{C}^2$ in sing. theory,

???

so could try to achieve them by perturbing f.
However need to do that using finite dim. space of linear fns. on $\mathbb{CP}^N$!!!)

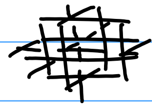
If allow ourselves to re-embed X into a larger proj space (increasing the degree) I'm sure this works, otherwise... here's how the proof should go. – dim. counting.

Get the proj. to $\mathbb{CP}^2$ by successively projecting $\mathbb{CP}^{r+1} - \{q\} \to \mathbb{CP}^r$
($\mathbb{CP}^r \sim$ family of lines through chosen point q). $r = N-1, \dots, 2$
(at each stage, take $q \notin X$ so proj.ⁿ is well def.ᵈ on X)
actually, image of X by previous projections...

- as long as $r \geq 5$, can assume projⁿ is a diffeo on X (gives an embedding $X \hookrightarrow \mathbb{P}^r$)
  Indeed: {lines tangent to X} = 3-dimˡ family of lines (indexed by $\mathbb{P}TX$)
  {lines through 2 points of X} = 4 dimˡ —— (—— $X \times X$)
  The set of all pts on all such lines is of $\dim_{\mathbb{C}} \leq 5 \Rightarrow$
  a generic choice of $q \in \mathbb{CP}^{r+1}$ ensures all lines through q intersect X in at most 1 pt, & transversely.
- for a generic projⁿ $\mathbb{CP}^5 - \{q\} \to \mathbb{CP}^4$, image of X will have double pts at most. (in particular it's immersed) because: can take $q \notin$ any line tangent to X (these form a 3-dimˡ family of lines → can't fill $\mathbb{P}^4$)
  $\dim \leq 5$
  and $\{(x,\ell) / x \in \ell,\ \ell$ passes through 2 pts of $X\} \to \mathbb{CP}^5$ is generically finite-to-one
  → for generic q, ∃ finitely many ℓ through q & 2 pts of X (double pts of proj/X)
  & {pts on lines hitting X 3 times} is actually of $\dim \leq 4$ so generic q gives no triple pts.
- next, $\mathbb{CP}^4 - \{q\} \to \mathbb{CP}^3$: no longer an immersion, but a generic q is a regular value of $pr_x : \{(x,\ell) / x \in \ell,\ \ell$ tangent to $X\} \to \mathbb{CP}^4$ → finitely many lines through q are tgt to X.
  set of ℓ's is parametrized by $\mathbb{P}(TX)$
  Most of the tangents are single tangents ie. contact order w/ X is 2.
  So we can avoid stationary tangents, ie. those with contact order ≥ 3
  Lemma: | the set of all stationary tangents to X is of dim. ≤ 2.

  PF: – at a "generic" pt of X (where 2ⁿᵈ fund. form non degenerate), there are 2 stationary tangents if $X \subset \mathbb{P}^3$; none if $X \subset \mathbb{P}^n, n > 3$
  – set of pts of X with infinitely many stationary tangents (X osculates its tangent plane to order 3) is an alg. subvar $\subset X$, hence of $\dim_{\mathbb{C}} \leq 1$ unless it's all of X – but that happens only if X = linear $\mathbb{CP}^2$ & then the tangents $\subset \mathbb{CP}^2$ ◢

So can take $q \in \mathbb{CP}^4$ s.t. finitely many single tgts pass through q
no stationary tangents
This controls the non-immersed points (also one can control the lines through several pts of X, to control self-intersections.

→ **Prop:** For generic proj^n to $\mathbb{CP}^3$, image of X is $Y \subset \mathbb{CP}^3$ with
- self-intersections along a curve D ("double curve")
  ($Y \sim \{z_1 z_2 = 0\} \subset \mathbb{C}^3$)
- triple points  (these are immersed sings.)
  ($\{z_1 z_2 z_3 = 0\}$)
- "pinches" (or 'Whitney umbrellas') $\{z_1^2 = z_2 z_3^2\}$.
  (locally proj from X is like $(x,y) \mapsto (xy, x^2, y)$)

So need to study how a surface $Y \subset \mathbb{P}^3$ projects to $\mathbb{P}^2$...

• First ignore the sings., i.e. assume Y embedded.

$\{(x,y) \in \mathbb{CP}^3 \times Y \,/\, x \neq y \text{ \& line through } x \text{ and } y \text{ is tangent to } Y \text{ at } y\}$
(dim 4, smooth away from diagonal)

$\downarrow$

$\mathbb{CP}^3$ Take a regular value q of this proj^n, $q \notin Y$

Then its preimage $= \{y \in Y \,/\, \text{line through } q \text{ \& } y \text{ tangent at } Y\}$
$=$ non-immersed pts of proj^n centred at q
is a <u>smooth curve</u> $R \subset Y$.

The proj^n of R to $\mathbb{CP}^2 \equiv$ those lines through q which are tangent to Y.

Claim: a generic choice of q ensures that among the lines through q:
- finitely many stationary tangents, all of contact order $= 3$ (not higher)
- finitely many bitangents, all simple at both points
- no tritangents (lines tangent to Y in 3 points).

Idea: <u>dim counting</u>: above lemma $\Rightarrow \{(x,\ell),\ x \in \ell,\ \ell \text{ stationary tangent}\}$ is of dim. ≤ 3, so proj. to 1st factor is generically finite-to-one
(→ finitely many stationary tangents)

Similarly for bitangents

& <u>can show</u> all the worse cases
(- stationary tgts with order ≥ 4
 - stationary tgts which intersect Y nontransversely somewhere else
 - tritangents)
are at most dim. 2, i.e. don't hit a generic $q \in \mathbb{CP}^3$.

Kulikov-Kulikov claim some of these not so clear-cut...

The stationary tangents of order $= 3$ yield the cusp points

The simple bitangents yield the nodes

- <u>Sing. of Y:</u> — double curve, triple pts not a problem if choose q not on any tangent to the double curve (generically ok) nor in any of the tangent planes at triple pts (______)

  → then $f: X \to \mathbb{CP}^2$ doesn't "see" these immersed sings. of Y (e.g.: f unramified near triple pt)

  — pinches: if $q \notin$ tangent cone to the pinch (generically ok!) then proj. to $\mathbb{CP}^2$ is b.c. of degree 2 and has simple branching ✓

<u>Another viewpt:</u> $R = \{x \in X \,/\, \mathrm{Jac}_f(x) = 0\}$, $\quad df: TX \to f^*T\mathbb{CP}^2$

$$\mathrm{Jac}_f = \Lambda^2 df \in H^0(\Lambda^2 T^*X \otimes \Lambda^2 f^*T\mathbb{CP}^2)$$
$$= K_X \otimes \mathcal{O}(3)_{|X}$$

(think of: $\mathrm{Jac}_f \sim x_0\, \partial x_1 \wedge \partial x_2 + \dots$) $\quad$ "canonical bundle" (2,0)-forms on X

→ 2-form in $\mathcal{O}(3)$ on X

This line bundle is nontrivial & its section Jacf must vanish along a curve.

We can ensure it vanishes transversely along a smooth curve R

Get: $[R] = [K_X] + 3[H]$

↳ hyperplane section class

Also, cusps $\equiv$ points where $f_{|R}$ not an immersion $\equiv$ zeroes of $df_{|TR}$

This lets us compute various things:

- $\deg f = \deg X$
- $\deg D = [R].[H] = [K_X].[H] + 3[H].[H]$
- $2g(D) - 2 = (K_X + [R]).[R]$ (adjunction) $\quad$ ($g(D) = g(R)$)
- $\#$ cusps $= 12[H].[H] + 9[K_X][H] + 2[K_X].[K_X] - e(X)$
- $\#$ nodes given by: $g(D) = \dfrac{(d-1)(d-2)}{2} - \#\text{cusps} - \#\text{nodes}$

- Now, characterize $f: X \to \mathbb{CP}^2$ by ① branch curve $D \subset \mathbb{CP}^2$

  → <u>braid monodromy</u>

  ② $\pi_1(\mathbb{CP}^2 \setminus D) \xrightarrow{\theta} \mathfrak{S}_{n = \deg f}$.

given ①, $\exists$ finitely many choices of ② (π_1 finitely generated, $\mathfrak{S}_n$ finite).