Lecture 19 - Wed Apr 26 - Hurwitz curves

Def: $C \subset \mathbb{CP}^2$ closed oriented $\dim_{\mathbb{R}} 2$ submfld w/ isolated singularities is a <u>Hurwitz curve</u> if
- $(0:0:1) \notin C$
- C intersects transversely & positively the fibers of $\pi: (x:y:z) \mapsto (x:y)$ except at finitely many pts $p_1, \dots, p_r \in C$ (singularities & vertical tangencies)

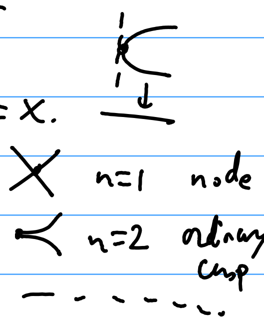
- Given any Hurwitz curve $C \subset \mathbb{CP}^2$, we can still define braid monodromy.
  The "degree" of C is $d = [C]\cdot[\text{line}] > 0$ (intersection number b/w 2 classes in $H_2(\mathbb{CP}^2, \mathbb{Z}) \simeq \mathbb{Z}$)
  (ie. $[C] = d\cdot[\text{line}]$)

$\rightsquigarrow$ factorization in B_d.

- Usually one requires a bit more, by prescribing a class of model behaviors at p_i:
  - near each p_i, $\exists$ nbd U_i, a model curve $\tilde{C}_i \subset \mathbb{C}^2$ (in allowed class of models) and orientation-preserving local diffeos. s.t.

$$\begin{tikzcd} (U_i \cap C) \arrow[r, "\sim"] \arrow[d, phantom, "\subset"] & \tilde{C}_i \arrow[d, phantom, "\subset"] \\ U_i \arrow[r, "\sim"] \arrow[d, "\pi"] & \mathbb{C}^2 \arrow[d, "pr_1"] \\ \pi(U_i) \arrow[r, "\sim"] & \mathbb{C} \end{tikzcd}$$

- <u>Important class of Hurwitz curves:</u> <u>Simple Hurwitz curves</u> := such that
  - the projections of the special pts are distinct
  - all vert. tangencies are nondegenerate: modelled on $y^2 = x$.
  - all singular pts are modelled on A_n-sing: $y^2 = x^{n+1}$ $(n \geq 1)$



<u>Rmk:</u>
- Here models are algebraic, which is the most common setting.
  Later we'll also allow "$\overline{A}_n$ sings": modelled on $y^2 = \bar{x}^{n+1}$, $n \geq 1$
  (non-algebraic: "mirror image").
- a Hurwitz curve can always be perturbed so special pts lie in different fibers of π; and an isotopy of H-curves can be perturbed so this holds at all stages in the isotopy. (so this extra requirement isn't much of an issue).

Prop (---, Kulikov-Kharlamov 2003):

For simple H.curves, the braid monodromy $\rho: \pi_1(\mathbb{C} - \text{pts}) \to B_d$ determines the curve C uniquely up to isotopies of $\mathbb{CP}^2$ preserving the projection π.

Expect this to hold in full generality (all H. curves); e.g. Kulikov-Kharlamov show this remains true if we allow "A_n" sing models, or $y^k = x^\ell$ $\forall k, \ell \geq 1$.

Idea: The braid monodromy describes C above $D^2(R) \setminus \bigsqcup_i D^2(q_i, \varepsilon) \underset{h.e.}{\simeq} \vee S^1$ up to fiber-preserving isotopies (tautologically).

Using constraints about what can happen near q_i → glue in some std local model for nbd of each special pt, to recover all of C.

Corollary:

$$\{\text{simple Hurwitz curves in } \mathbb{CP}^2\} \Big/ \text{(equisingular) isotopy among Hurwitz curves}$$

i.e. preserving types of special pts

$$\Big\updownarrow \text{1-1}$$

$$\{(b_1, \dots, b_r) \in B_d \;/\; \textstyle\prod b_i = \Delta^2, \text{ each } b_i \text{ is conjugate to } \sigma_1^{n+1} \text{ for some } n \geq 0\} \Big/ \text{simult. conj. Hurwitz equiv.}$$

(& similarly if allow $y^k = x^\ell$ or $y^2 = x^{n+1}$ models).

Rmk: • Can extend this discussion to curves in $\mathbb{C}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$, ---- (with suitable modifications so braid monodromy makes sense)

• **Isotopy problem for Hurwitz curves:**

• every alg. curve is Hurwitz (positivity of intersection) (unless it's reduced or passes through pole of projection ...)

but $\exists$ simple Hurwitz curves which are not homotopic to any alg. curve.

$\ast$ in $\mathbb{CP}^2$: various families of examples w/ nodes & cusps (A_1 & A_2)

E.g: curves of degree 18 with 81 cusps (A_2 singularities only)

(Moishezon early 90s. Idea: braiding along an annulus → get curves w/ ∞ly many different $\pi_1(\mathbb{CP}^2 - C_i)$; only finitely many are alg)

Conj: A simple Hurwitz curve in $\mathbb{CP}^2$ which is smooth or nodal (A_1 only) is isotopic through Hurwitz curves to an algebraic curve.

($\leftrightarrow$ "symplectic isotopy problem"; proved by Siebert-Tian 2003 using J-hol. curve methods, for smooth curves of $\deg \leq 17$)

Geom. approach uses J-hol. curve theory, but one could try by gp theory: Conj $\Leftrightarrow$ every fact^n of Δ^2 into halftwists or squares of halftwists is Hurwitz + conj. equivalent to that of an alg. curve. (explicit list of model b.m.f's).

E.g., for smooth curves: $\{$smooth alg. curves$\}$ (in proj. plane of deg d) $= \mathbb{P}(\{$homogeneous deg-d polynomials$\}) \setminus$ divisor (& same for those which are nondegenerately tangent to fibers of π), connected set $\to$ they all have the same b.m.f. up to Hurwitz & conj. Moishezon has shown it is $\Delta^2 = (\sigma_1 \cdot \ldots \cdot \sigma_{d-1})^d$. Isotopy conjecture says $\nexists$ other fact^n into halftwists.

(so far, can't prove conj. by this method except $d=2$, maybe $d=3$).

- This phenomenon is specific to projective curves.

  <u>Thm</u> (Kulikov-Kharlamov 2003):

  Given any $(b_1, \ldots, b_r)$ algebraic braids (ie. monodromies of isolated special points of alg. curves — e.g. positive powers of halftwists), $\exists$ algebraic curve $C \subset \mathbb{C}^2$ s.t. the braid monodromy of $C \cap (D^2(1) \times \mathbb{C})$ can be represented by the factorization $b_1 \ldots b_r$.

<u>Complex projective surfaces</u>:

$X \subset \mathbb{CP}^N$ $\mathbb{C}$ alg. projective surface, smooth

<u>e.g.</u>:
- X defined by alg. equations (e.g. complete intersection)
- X compact complex manifold, L ample line bundle (ie. $c_1(L) = [\omega]$, ω Kähler form: closed 2form / $\omega(\cdot, J\cdot)$ Riem. metric)

  $\to$ for $k \gg 0$, $L^{\otimes k}$ has sufficiently many holomorphic sections so that, choosing a basis $s_0, \ldots, s_N \in H^0(L^{\otimes k})$,

  $$X \longrightarrow \mathbb{CP}^N$$
  $$x \longmapsto (s_0(x) : \ldots : s_N(x))$$

  is an embedding & makes X a proj. surf.

  (<u>Kodaira embedding thm</u>)

- Now, consider a linear projection

  $$P : \mathbb{CP}^N \setminus \mathbb{CP}^{N-3} \longrightarrow \mathbb{CP}^2$$
  $$(x_0 : \ldots : x_N) \longmapsto (x_0 : x_1 : x_2)$$

  (in fact can choose any such proj^n — up to proj. linear transformations)

- Can assume $\mathbb{CP}^{N-3} \cap X = \emptyset$ (for dimensional reasons: $\dim_{\mathbb{C}} X = 2 < \operatorname{codim} \mathbb{CP}^{N-3}$)

  (in fact: given $x \in X$, space of $\mathbb{CP}^{N-3}$'s passing through x has $\mathbb{C}$-codim. 3 in $Gr(N-2, N+1)$ so all $\mathbb{CP}^{N-3}$'s intersecting X form at most codim$_{\mathbb{C}}$ 1 family $\to$ generic $\mathbb{CP}^{N-3}$ avoids X.

Then by restriction, get a well defined map $f = p_{|X} : X \to \mathbb{CP}^2$.

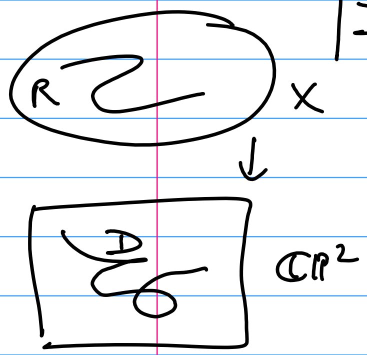
NB: fibers of p = linear $\mathbb{CP}^{N-2}$'s in $\mathbb{CP}^N$ (passing through the given $\mathbb{CP}^{N-3}$)
they intersect X in $[X].[\mathbb{CP}^{N-2}] = \deg(X)$ pts ($\leftrightarrow$ class in $H_4(\mathbb{CP}^N, \mathbb{Z})$) i.e. $\deg(f) = \deg(X)$

- We'll assume X is in generic position wrt p. In fact this can be ensured by choosing p well.

(Zariski 1935 ??) Prop: For a generic choice of the linear proj. p, $f : X \to \mathbb{CP}^2$ is a branched covering whose branch curve has only nodes & ordinary cusp singularities.

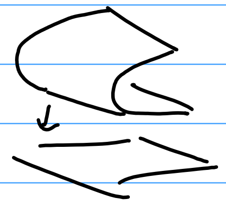
In fact; (1) $R \subset X$ ramification curve $= \{p \in X \,/\, df_p \text{ not an iso}\}$ is a ~~smooth~~ alg. curve $\subset X$

$D = f(R) \subset \mathbb{CP}^2$ discriminant curve $= \{z \in \mathbb{CP}^2 \,/\, \# f^{-1}(z) < \deg X\}$ plane alg. curve w/ cusps & nodes.

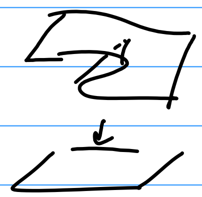


(2) Local models: $\forall p \in X$, $\exists$ local holom. coords on $U(p) \subset X$, $U(f(p)) \subset \mathbb{CP}^2$

in which f is
- $(x,y) \mapsto (x,y)$ if $p \notin R$ (local diffeo. !)



- $(x,y) \mapsto (x^2, y)$ at generic pts of R "simple branching"



- $(x,y) \mapsto (x^3 - xy, y)$ "cusp".

here $R : \det df = 3x^2 - y = 0$ smooth

$D = f(R) = \{(-2x^3, 3x^2)\} = \{27z_1^2 = 4z_2^3\}$ ordinary cusp

- Where do nodes come from?

They correspond to 2 distinct pts of R where simple branching occurs and which happen to map to the same point in $\mathbb{CP}^2$.

Status of the result is dubious. See Kulikov & Kulikov 2000 for an attempt