Lecture 17 – Wed April 19

→ Leftovers from lec. 16: Giroux's construction of open books

Braid monodromy of complex plane curves (Zariski, Moishezon, ...)

Setup:

$C \subset \mathbb{C}^2$ complex algebraic plane curve (possibly singular!)
$P(x,y) = 0$.

· Assume: $\forall x \in \mathbb{C}$, $P(x,\cdot) =: P_x \in \mathbb{C}[y]$ is a nonzero polynomial of degree d (indep^t of x).

This means $P(x,y) = y^d + Q_{d-1}(x)\, y^{d-1} + \dots + Q_0(x)$
for some $Q_0, \dots, Q_{d-1} \in \mathbb{C}[x]$.

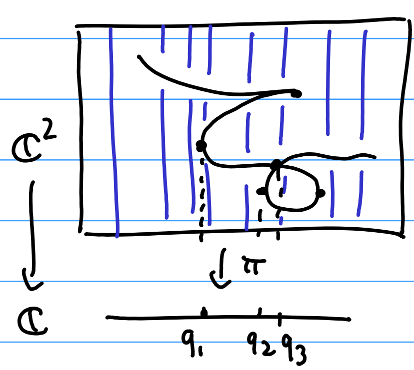
Geometrically: C doesn't have any vertical asymptotic branches
(the projective compactification $\overline{C} \subset \mathbb{P}^1 \times \mathbb{P}^1$ does not pass through $(x,\infty)\ \forall x \in \mathbb{C}$)

· Discriminant: $\Delta(x) =$ discriminant of the degree d polynomial $P_x \in \mathbb{C}[y]$
(whose coefficients depend polynomially on x!)

→ $\Delta \in \mathbb{C}[x]$, its roots $\equiv$ those x s.t. P_x has multiple roots in y
$\equiv$ values of x s.t. $C \cap (\{x\} \times \mathbb{C})$ consists of fewer than d distinct points

Assume: C does not contain any multiple components, i.e. $\Delta \not\equiv 0$.
(avoid: $y^3 = 0$ triple line ═══)

Let $\{q_1, \dots, q_r\} \subset \mathbb{C} :=$ the distinct roots of Δ
(NB: often $r < \deg \Delta$).



The projection $\pi : \mathbb{C}^2 \longrightarrow \mathbb{C}$
$(x,y) \longmapsto x$
restricts to C as a (singular, ramified) d-fold covering, unramified over $\mathbb{C} - \{q_1 \dots q_r\}$

($\pi_{|C}$ is of degree d, and by def^n, $\{q_1 \dots q_r\}$ = pts with fewer than d preimages

NB: The only way $C \cap (\{x\}\times\mathbb{C})$ can have fewer than d points is if the alg. intersection multiplicity at one of these pts ($\Leftrightarrow$ top. intersection number) is >1, which occurs iff C is either singular, or tangent to vert. line.
(mult. root of $P_x \Leftrightarrow P=0,\ \frac{\partial P}{\partial y}=0$; if $\frac{\partial P}{\partial x}=0$, sing. pt, else tangency)

So: $\{q_i\} \equiv$ projections of "special pts" of C = $\begin{cases} \bullet \text{ points where } C \text{ is not smooth} \\ \bullet \text{ points where } C \text{ is tangent to the fiber of } \pi. \end{cases}$

• We have a natural map $\sigma : \mathbb{C}\setminus\{q_1..q_r\} \longrightarrow \mathcal{C}_d$ unordered config. space.
$\forall\, x \in \mathbb{C}\setminus\{q_1..q_r\}$, $\sigma(x) := \pi_{|C}^{-1}(x) \in \mathcal{C}_d(\mathbb{R}^2)$ is an unordered config of d pts in the plane.

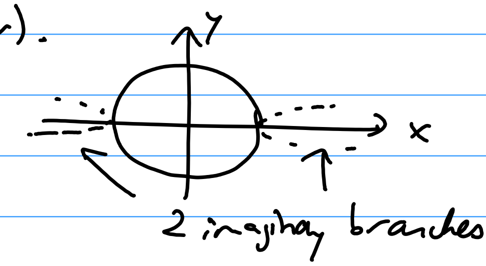
Fix a base point $x_* \in \mathbb{C}\setminus\{q_1..q_r\}$, and consider a loop

$$\gamma \in \pi_1(\mathbb{C}\setminus\{q_1..q_r\}, x_*) \longmapsto \rho(\gamma) := [\sigma_*(\gamma)] \in \pi_1(\mathcal{C}_d, \sigma(x_*)) \cong B_d.$$

Def^n: $\rho : \pi_1(\mathbb{C}\setminus\{q_1..q_r\}) \longrightarrow B_d$ is the braid monodromy of C
(map on fundamental groups induced by σ)

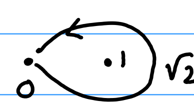
⚠ This depends on choice of an isom. $\pi_1(\mathcal{C}_d, \sigma(x_*)) \xrightarrow{\sim} B_d$, induced by choice of homeo $(\mathbb{C}, \pi_{|C}^{-1}(x_*)) \simeq (\mathbb{R}^2, \{1..d\})$.
Different choices $\leftrightarrow$ replace ρ by its composition w/ an inner aut. of B_d
(conjugation by some braid = "change of triv^n" of the fiber).

Ex. 1: • conic $x^2+y^2=1$

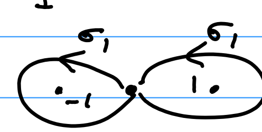


- $y^2 = 1-x^2$ has a double root iff $1-x^2=0 \Leftrightarrow x=\pm 1$
- at $x_*=0$, $\sigma(0)=\{\pm 1\}$
- consider the loop $x(\theta) = (1-e^{i\theta})^{1/2}$ (the square root with $\mathrm{Re}\, x \geq 0$).
$0 \leq \theta \leq 2\pi$

$\rightarrow$ above $x(\theta)$ we have $y^2 = 1 - x(\theta)^2 = e^{i\theta}$
i.e. $\sigma(x(\theta)) = \{\pm e^{i\theta/2}\}$



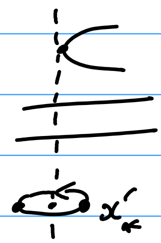
$\Rightarrow$ The braid monodromy along this loop is σ_1
- similarly (by symmetry) around -1. $\Rightarrow$ get:



More generally: if at some point the curve C is smoothly tangent to the fiber of π in a nondegenerate manner

$\rightarrow$ monodromy around this q_i is a <u>half-twist</u> $\in B_d$

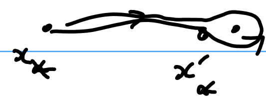
Indeed, in nearby fibers of π:



$\}$ $y^2 = x$, i.e. $y = \pm e^{i\theta/2}$

$\}$ nothing happens

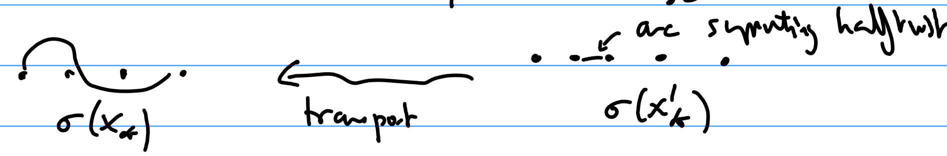
"local monodromy" is a half-twist.

Monodromy along 

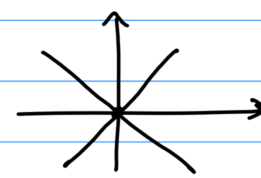
is the same braid up to isomorphism $\pi_1(\mathcal{C}_d, \sigma(x_*)) \xrightarrow{\sim} \pi_1(\mathcal{C}_d, \sigma(x'_*))$.

induced by moving base pt along arc $\sigma(x_*) \rightsquigarrow \sigma(x'_*)$.

(so it's still a halftwist – it still exchanges 2 strands CCW – but it may look more complicated than in the local picture because



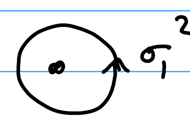
Ex. 2: two lines $y^2 = x^2$



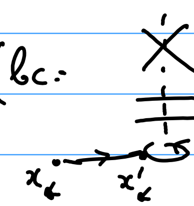
monodromy around 0:

$\sigma(x) = \{\pm x\}$

$x = e^{i\theta} \rightsquigarrow \sigma(x) = \{\pm e^{i\theta}\}$ monodromy is σ_1^2



More generally, if at some point C has a <u>node</u> (transverse double pt) & both branches are transverse to the proj^n π,

local monodromy = square of a halftwist (loc.: 

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Similarly, common types of singularities are recognizable from their braid monodromies! – braid monodromy is the natural way of describing the sing. of a plane curve & how they fit together.

<u>Next:</u> – setup for projective curves

– Zariski–Van Kampen