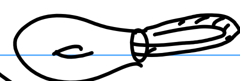
Lecture 16 - Wed Apr 12 - see Etnyre's lecture notes math SG/0409402

Recall:

- open book decomp. of M^3 is $\pi : M - B \to S^1$ fibration w/ fiber Σ (page), $\partial\Sigma = B$, and monodromy $\phi \in Map(\Sigma)$.
  (B: oriented link (binding))
- contact str. on M^3 (oriented) is ξ oriented plane field, $\xi = \ker\alpha$ contact form, $\alpha \wedge d\alpha > 0$

Thm (Giroux, 2000):

$$\{\text{contact structs on } M^3\}/\text{isotopy} \xleftrightarrow{1-1} \{\text{open book decomps. of } M^3\}/\text{positive stabilization}$$



$\Sigma' = \Sigma \cup$ 1-handle

$\phi' = \phi \cdot \tau_\gamma$

twist / s.c.c. passing once through the handle.

Def: A contact str. ξ on M is supported by open book (B,π) if (up to isotoping ξ), $\xi = \ker\alpha$ contact 1-form s.t.

(1) $d\alpha$ is a positive area form on each page Σ_θ

(2) $\alpha > 0$ on B

(recall: ξ, B, Σ_θ all oriented...)

Lemma: equivalent characterizations:

(A) ξ is supported by (B,π)

(B) ξ can be isotoped to ξ' s.t.
- ξ' positively transverse to B
- ξ' arbitrarily close to $T\Sigma_\theta$ outside a neighborhood of B

(C) $\exists$ contact 1-form α s.t. Reeb v.f. X_α defined by $\begin{cases}\alpha(X) = 1\\ d\alpha(X,\cdot) = 0\end{cases}$ is positively tangent to B and positively transverse to pages in $M - B$.

NB: α contact form $\to$ $d\alpha$ 2-form has max. rank (i.e. rank $2n$ - here 2)
- nondegenerate on contact planes ξ by definition.

So $\ker(d\alpha)$ is a line field $\pitchfork \xi$ (depends on choice of α!)
and α is nonzero on that line $\to$ the Reeb v.f. is the unique elt of that line s.t. $\alpha(X) = 1$. E.g. if $M \subset \mathbb{C}^n$ convex hypersurface, with normal vector $\vec{n}$, and $\xi = \max\ \mathbb{C}$ tangencies, can take $\alpha = \iota_{\vec{n}}\omega_0$, $R = J\vec{n}$

Dynamics of Reeb v.f. is important in contact geometry ($\to$ invts of contact mflds)
Here, (C) says: Reeb v.f. $\pitchfork$ pages, so its dynamics are encoded by return map $\Sigma \to \Sigma$ (which is isotopic to monodromy of open book except it can be wild near $\partial\Sigma$, not Id ...)

Pf. (A) $\Leftrightarrow$ (C):

· (C)$\Rightarrow$(A): if α contact form w/ X tgt to B and $X \pitchfork$ pages:
- X tangent to $B \Rightarrow \alpha > 0$ on B
- $X \pitchfork_+$ pages $\Rightarrow$ given v_1, v_2 pos. basis of $T_p\Sigma$,
$0 < (\alpha\wedge d\alpha)(X, v_1, v_2) = \underbrace{\alpha(X)}_{1} \cdot d\alpha(v_1,v_2) \quad \Rightarrow d\alpha_{|T_p\Sigma} > 0$

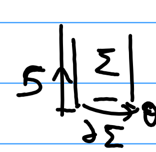
· (A)$\Rightarrow$(C): if α contact form s.t. $\alpha > 0$ on B, $d\alpha > 0$ on pages: let X = Reeb v.f.
- $d\alpha > 0$ on pages $\Rightarrow$ given v_1, v_2 pos. basis of $T_p\Sigma$,
$(\alpha\wedge d\alpha)(X, v_1, v_2) = \underbrace{\alpha(X)}_{1}\, \underbrace{d\alpha(v_1,v_2)}_{>0} > 0 \Rightarrow X \pitchfork T_p\Sigma$ positively
- along B: take coords. (t, r, θ) & use (x,y) instead of (r,θ)
t: along B; r: $\perp B$ in page; θ: angle of open book

write $X = f\frac{\partial}{\partial t} + g\frac{\partial}{\partial x} + h\frac{\partial}{\partial y}$; show $g=0$, $h=0$ for $(x,y)=(0,0)$?
Assume e.g. $g>0$ at some $(t_0,0,0)$: then $g>0$ at $(t_0,0,\varepsilon)$ for ε small, and this violates positive transversality to page $\theta = \pi/2$.
Similarly for $g<0$, and for h; so X is tangent to B.
$\alpha > 0$ on B, $\alpha(X) = 1 \Rightarrow X \in$ positive direction ✓.

Thm (Thurston-Winkelnkemper 1975):
|| Every open book supports a contact structure.

Pf: Consider the open book given by (Σ, ϕ), $Y = \Sigma\times[0,1]/\sim$ mapping torus,
$M = Y \cup_\partial (\partial\Sigma \times D^2)$

① build a contact str. on Y:
let $S = \{ \lambda$ 1-form on $\Sigma \mid$ (1) Near $\partial\Sigma$, $\lambda = (1+s)\,d\theta$; (2) $d\lambda$ is an area form on $\Sigma \}$



- This is nonempty: take λ_1 = any 1-form which equals $(1+r)d\theta$ near $\partial\Sigma$
then $\int_\Sigma d\lambda_1 = \int_{\partial\Sigma}\lambda_1 = 2\pi\cdot(\#\partial\Sigma)$.

Let ω = some area form on Σ with $\left[\begin{array}{l}\int_\Sigma \omega = \int_\Sigma d\lambda_1 \\ \omega = d\lambda_1 \text{ near } \partial\Sigma\end{array}\right.$

$\rightarrow [\omega - d\lambda_1] = 0$ in $H^2(\Sigma, \partial\Sigma)$ $\Rightarrow \exists\, \beta \in \Omega^1(\Sigma, \partial\Sigma)$ s.t. $\omega - d\lambda_1 = d\beta$

Then $\lambda = \lambda_1 + \beta$ has the right model near $\partial\Sigma$ and $d\lambda = \omega$ ✓

- This is convex $\left(\underset{t\in[0,1]}{\lambda, \lambda' \in S} \Rightarrow t\lambda + (1-t)\lambda' \in S\right)$

Given $\lambda \in S$, $\phi^*\lambda \in S$ (since $\phi = Id$ near $\partial\Sigma$), and we can define

$\tilde{\lambda}_{(x,t)} = t\lambda_{(x)} + (1-t)(\phi^*\lambda)_{(x)}$ on $\Sigma \times [0,1]$; by construction

this defines a 1-form $\tilde{\lambda}$ on Y, with $d\tilde{\lambda}_{|\Sigma \times \{t\}} > 0$ and $\tilde{\lambda} = (1+s)\,d\theta$ near ∂Y.

Then $\alpha = \tilde{\lambda} + K\,dt$ is a contact form $\forall K > 0$ suff. large

since $\alpha \wedge d\alpha = \underbrace{\tilde{\lambda} \wedge d\tilde{\lambda}}_{\substack{\text{bounded below} \\ \text{by compactness} \\ \geq -C_1}} + K \underbrace{dt \wedge d\tilde{\lambda}}_{\substack{\geq C_2 > 0 \\ \text{since } d\tilde{\lambda}_{|\text{fiber}} \text{ area form} \\ \text{\& bounded below near boundary}}}$ $\Rightarrow > 0$ if K large.

moreover, $d\alpha_{|\text{page}} > 0$ and $\alpha = (1+s)\,d\theta + K\,dt$ near ∂Y.

(2) Extend it over $\underset{\theta,\ r, \varphi}{S^1 \times D^2}$, in compatible manner near boundary, (gluing: polar coord of $D^2 \Leftrightarrow$ variable t from mapping torus?)

mapping torus $\left[\begin{array}{l}\theta \longleftrightarrow -\theta \\ s \longleftrightarrow r = 1+s \\ t \longleftrightarrow \varphi = t\end{array}\right.$ ⚠ this is so $(s, \theta, t) \mapsto (-\theta, 1+s, t)$ preserves orientation.

- ie. near $r=1$, must have $\alpha = -r\,d\theta + K\,d\varphi$
- near $r=0$, want $\alpha = d\theta + r^2\,d\varphi$ (so $d\alpha = r\,dr \wedge d\varphi$ at $r=0$ $\rightarrow X = \frac{\partial}{\partial\theta}$ ✓).

Take $\alpha = f(r)\,d\theta + g(r)\,d\varphi$

$\alpha \wedge d\alpha = (f\,d\theta + g\,d\varphi) \wedge (f'\,dr \wedge d\theta + g'\,dr \wedge d\varphi) = (fg' - f'g)\,d\theta \wedge dr \wedge d\varphi$

It's a +ve contact form iff $fg' - f'g > 0$ ie. $\left(\frac{f}{g}\right)$ is a strictly $\downarrow$ fn of r.

$\rightarrow$ prescribe $\frac{f}{g} \downarrow$ s.t. $\frac{f}{g} = \frac{1}{r^2}$ near 0 and $-\frac{r}{K}$ near 1. ✓.

- also want $d\alpha > 0$ on pages φ = constant, ie. $f' < 0$

$\rightarrow$ near $r=0$, actually take eg. $\alpha = (1-r^4)\,d\theta + r^2\,d\varphi$, doesn't change Reeb vf @ $r=0$.

Clear: Can choose f & g with prescribed behavior near $r=0$ & 1, s.t. $f \downarrow$ and $(f/g) \downarrow$.

Prop. (Giroux)

Two contact structures supported by the same open book are isotopic.

Pf: start with α_0, α_1 contact forms for ξ_0, ξ_1 supported by open book $\pi: M - B \to S^1$
$\alpha_0, \alpha_1 > 0$ on B and $d\alpha_0, d\alpha_1 > 0$ on pages.

* consider $\alpha_{0,K} = \alpha_0 + K \underbrace{f(r)}\, d\varphi \leftarrow$ angular variable of open book

$f(r) = 1$ outside ε-nbd of binding

$f(r) \sim r^2$ near binding

- $\forall K \geq 0$ this is still a contact form

$$\alpha_{0,K} \wedge d\alpha_{0,K} = \underbrace{\alpha_0 \wedge d\alpha_0}_{>0} + K\Big[f(r) \underbrace{d\varphi \wedge d\alpha_0}_{\substack{>0 \text{ since} \\ d\alpha_0 > 0 \text{ on pages} \\ (=0 \text{ on binding})}} + \underbrace{\alpha_0 \wedge (f'(r)\, dr) \wedge d\varphi}_{\substack{\geq 0, \text{ and } >0 \text{ near binding} \\ \text{since } \alpha_0\left(\frac{\partial}{\partial\theta}\right) > 0 \text{ for } r=0 \\ \text{and hence for } r < \varepsilon.}}\Big]$$

- it also still satisfies $d\alpha_{0,K}|_{\text{page}} = d\alpha_0|_{\text{page}} > 0$
  $\alpha_{0,K}|_{\text{binding}} = \alpha_0|_{\text{binding}} > 0.$

* Now let $\alpha_{s,K} = s\,\alpha_{1,K} + (1-s)\,\alpha_{0,K}$ (linear interpolation).
  - if K large enough this is still a contact form $\forall s \in [0,1]$
    (because $\alpha_{s,K} \wedge d\alpha_{s,K}$ involves terms $\alpha_0 \wedge d\alpha_1$, $\alpha_1 \wedge d\alpha_0$
    which may be negative but are $\geq -C$
    & other terms involving the perturbation $K f(r)\, d\varphi$ are $\geq K C' > 0$)
  - $d\alpha_{s,K}|_{\text{page}} > 0$, $\alpha_{s,K}|_{\text{binding}} > 0$ by convexity.

$\blacktriangle$

Prop (Giroux)

(Σ_+, ϕ_+) positive stabilization of (Σ, ϕ) $\Rightarrow$ the corresponding contact structures ξ, ξ' on M are isotopic.

(Idea: think of stab^n as a contact connected sum w/ $(S^3, \xi_\pm)$; in the case of positive stab^n this operation is trivial in contact category. However neg. stab^n modifies the homotopy class of the 2-plane ξ !!)

So... we have a map $\{\text{open books}\}/\text{+ stab}^{n} \longrightarrow \{\text{contact structs}\}/\text{isotopy}$.
want to show it's a 1-1 correspondence.

- First show surjectivity:

Thm (Giroux): Every contact structure admits a supporting open book.

There are 2 proofs, both due to Giroux; one purely 3-dim^l, the other valid in any dimension!

We sketch the basic idea of the topological (3-dim^l) proof

⚠ There are some approximations, to make things easier to follow...

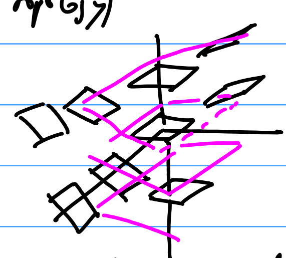
- Step 0: start w/ a triangulation of M^3 (suff. fine: simplices $\subset$ Darboux charts)
- Step 1: ensure the 1-skeleton is Legendrian ie. tangent to ξ everywhere

(NB: $L^n \subset M^{2n+1}$ Legendrian if $TL \subset \xi$

This is the max. possible dimension:
indeed, $\alpha_{|L}=0$, so $d\alpha_{|L}=0$, so $TL \subset \xi$ is isotropic for the sympl. form $d\alpha$ on ξ...)

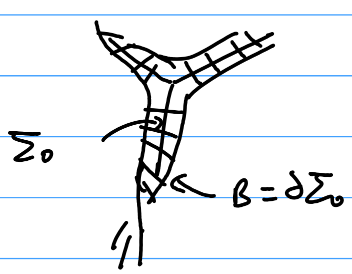
Idea: non-integrability of $\xi \Rightarrow$ can always perturb a given arc to make it Legendrian. (perturb^n small in C^0 topology)

(zigzag around the given arc, following ξ)



(works on interior of edges; near vertices, just move along ξ in any reasonable way).

- Step 2: construct a page by thickening the 1-skeleton in dir^n of ξ.
to get a "ribbon surface"



(NB: the ribbon surface tends to "twist" around the 1-skeleton because of non-integrability of ξ.)

Note: • $T\Sigma_0$ is close to ξ, so in particular $d\alpha > 0$ on it ✓
(recall $d\alpha_{|\xi} > 0$)

- $B = \partial\Sigma_0$ has $\alpha > 0$ (if chose orientations properly) because the core (1-skeleton) is Legendrian so has $\alpha = 0$, and by Stokes thm, using $d\alpha|_{\Sigma_0} > 0$, this implies $\alpha > 0$ on $\partial\Sigma_0$. ✓

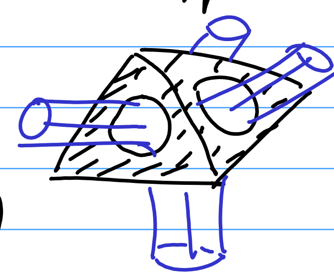
In fact, get half of the open book in this way, filling a tubular nbd of the 1-skeleton: in normal slice to an edge it's 

- Step 3: for each 2-face Δ, look at $(B \cap \Delta)$
  - can assume rotation of B around $\partial\Delta$ is more or less always in the same dir^n

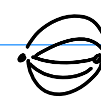
$\#(B\cap\Delta) \longleftrightarrow$ (twice) linking number of $\partial\Delta$ with its pushoff $/\!/ \xi$.
= "Thurston Bennequin invt" of the Legendrian (un)knot $\partial\Delta$.

Bennequin inequality $\rightarrow$ this is always <0, and in fact taking a sufficiently fine triangulation we can assume $tb(\partial\Delta) = -1$ and $|B\cap\Delta| = 2$ for all faces.

- Step 4: $M -$ (tubular nbd of 1-skeleton) retracts onto a 1-complex dual to the chosen triangulation!



& by above assumption, the boundary picture in a normal slice to edge ($\leftrightarrow$ interior of 2-face Δ) is

Then fill by  to complete the open book. ∎

- Finally, injectivity:

Thm (Giroux 2002): Two open books supporting the same contact structure ξ are related by positive stabilizations

Idea: • Lemma: Given an open book supporting ξ, up to positive stab^{n}s we can assume it comes from a cell decomposition as above

(idea: the core of the page Σ (a graph) admits a Legendrian representative.
So can assume Σ = ribbon on a Legendrian graph.
May need to add extra edges to get 1-skeleton of an adapted

cell decomposition – can add them by doing positive stab$^{\text{ns}}$.)

- So, can assume our 2 open books come from leg. cell decompositions
  Take common refinement of these 2 decompositions
  $\rightsquigarrow$ subdivision corresponds to + stabilization, so get a common stabilization. ∎

<u>Conclusion:</u> $\left\| \{\text{open books}\} / \text{positive stab}^{\text{n}} \xleftrightarrow{1-1} \{\text{contact structures}\} / \text{isotopy.} \right.$