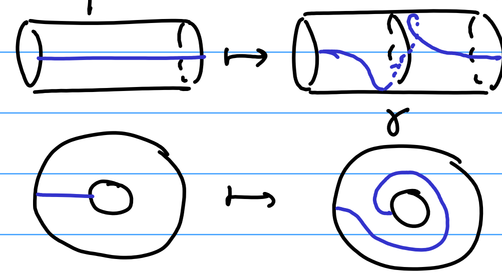
Lecture 12 – Wed Mar 22

Dehn twists & presentation of Map_g:

Def: Σ oriented, $\gamma \subset \Sigma$ simple closed curve

Dehn twist $\tau_\gamma \in Homeo^+(\Sigma) := \begin{cases} \text{Id outside cylinder } U(\gamma) \cong [-\pi,\pi]\times S^1 \\ (x,\theta) \mapsto (x, \theta + (x+\pi)) \text{ in cylinder} \end{cases}$



Notes:
- the isotopy class of τ_γ depends only on the isotopy class of γ
  ($\gamma \sim \gamma' = h_1(\gamma) \Rightarrow h_t \circ \tau_\gamma \circ h_t^{-1} \dots$)
- the def. involves a choice of orientation of Σ — but not of γ
  (given a l.o.p. α, assume $\alpha \pitchfork \gamma$, $\tau_\gamma(\alpha)$ = near each intersection of α w/ γ, cut open α and insert a copy of γ, so that if we approach γ by travelling along α, we turn to our right when we hit γ
  ↳ this notion depends on orientation of Σ).

Thm: (Dehn 1938, Lickorish 1960s, ...)
Every orient-preserving homeo of Σ_g is isotopic to a product of Dehn twists.
$Map(\Sigma_g)$, $Map(\Sigma_{g,r})$ and $Map_n(\dots)$ are all gen^d by Dehn twists.

Strategy pf: * Enough to prove it for $Map(\Sigma_g)$. Indeed,
- recall
  - kernel of $Map(\Sigma_{g,r}) \twoheadrightarrow Map_r(\Sigma_g)$ gen^d by Dehn twists / boundary curves
  - $\ker(Map_n(\Sigma) \twoheadrightarrow Map(\Sigma))$ gen^d by spin maps $= \tau'(\tau'')^{-1}$

  (γ', γ'', z_0)

Hence subgp gen^d by D-twists in $Map_n(\Sigma_{g,r})$ maps onto $Map(\Sigma_g)$ (by thm) & contains kernel of i_* ✓

* given $h \in Homeo^+(\Sigma_g)$, consider disjoint nonseparating curves $\alpha_1, \dots, \alpha_g \subset \Sigma_g$, and $\gamma = h(\alpha_1)$. We'll find a product of D. twists ϕ_1 s.t. $\phi_1(\gamma)$ is isotopic to a curve γ' disjoint from $\alpha_1 \dots \alpha_g$.
Then another product of D. twists ϕ_2 s.t. $\phi_2(\gamma')$ isotopic to α_1.
Then $\phi_2\phi_1 h$ (composed by a suitable isotopy) maps α_1 to itself & we can assume restricts as $Id: \alpha_1 \to \alpha_1$

Then: Cut Σ_g open along α_1 $\to$ get $\Sigma_{g-1,2}$ & a homeo of it.

hence $[\phi_2][\phi_1][h] \in \mathrm{Im}\left(\mathrm{Map}(\Sigma_{g-1,2}) \xrightarrow[\text{ind.}]{} \mathrm{Map}(\Sigma_g)\right)$

and use induction on g to conclude

(by above observation, reduce from $\Sigma_{g-1,2}$ to Σ_{g-1}; and for $\Sigma_0 = S^2$ the statement is trivial).

> In other words, what we'll show is that the subgroup gen^d by Dehn twists acts transitively on set of isotopy classes of nonseparating simple closed curves.

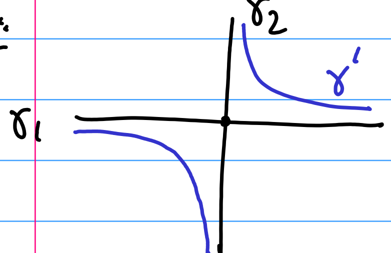
Notation: $\mathcal{T}$ = subgroup of Homeo^+ gen^d by Dehn twists & isotopies

$\gamma \leftrightarrow \gamma'$ if $\exists\, \phi \in \mathrm{Homeo}^+$, $\phi(\gamma) = \gamma'$

$\gamma \underset{\mathcal{T}}{\leftrightarrow} \gamma'$ if $\exists\, \phi \in \mathcal{T}$, $\phi(\gamma) = \gamma'$

Lemma 1: γ_1, γ_2 s.c.c. on Σ_g with $|\gamma_1 \cap \gamma_2| = 1 \Rightarrow \gamma_1 \underset{\mathcal{T}}{\leftrightarrow} \gamma_2$.

Pf:



observe γ' is isotopic to $\tau_{\gamma_2}(\gamma_1)$

but also to $\tau_{\gamma_1}^{-1}(\gamma_2)$

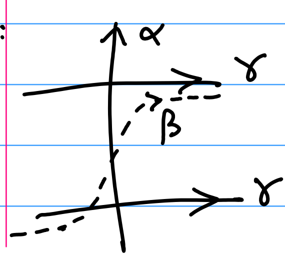
So $\tau_{\gamma_1}\tau_{\gamma_2}(\gamma_1)$ is isotopic to γ_2. ∎

Lemma 2: α, γ simple closed curves on Σ, $N = \mathcal{V}(\alpha)$,

$\Rightarrow \exists\, \gamma'$ s.t.
- $\gamma \underset{\mathcal{T}}{\leftrightarrow} \gamma'$
- $\gamma' \subset \gamma \cup N$
- $|\gamma' \cap \alpha| = 0$, or $|\gamma' \cap \alpha| = 2$ & these 2 intersections have opposite orientations

Pf: induction on $|\gamma \cap \alpha|$.
- If it's 0, or 2 w/ opposite orients, done.
- If it's 1, use Lemma 1, $\gamma \underset{\mathcal{T}}{\leftrightarrow} \alpha$; take a pushoff of α disjoint from α
- Otherwise we have one of 2 configurations; fixing orientations of α, γ

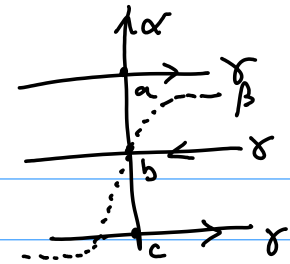
Case 1:



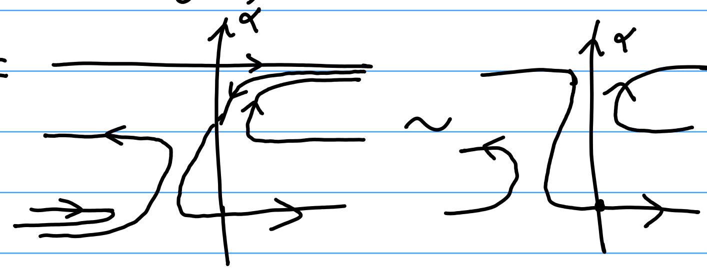
2 consecutive intersections (along α) w/ same orientation:

$|\beta \cap \gamma| = 1 \Rightarrow$ by Lemma 1, $\gamma \underset{\mathcal{T}}{\leftrightarrow} \beta$

$|\beta \cap \alpha| < |\gamma \cap \alpha|$ so apply induction to β

Case 2: 

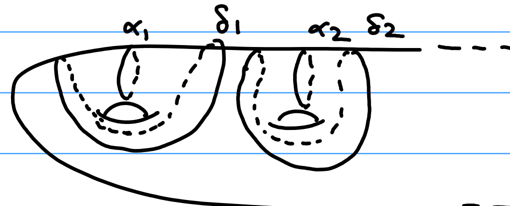
3 consecutive intersections w/ alternating orientations: can go either from a to c or from c to a along γ without hitting b, assume it's $a \to c$.

$T_\beta(\gamma) =$ 

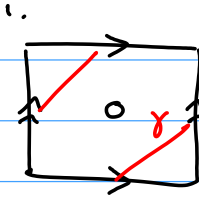
has 2 fewer intersec^{tns} w/ α $\Rightarrow$ apply induction to it. ▲

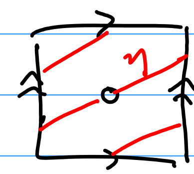
Lemma 3: γ simple closed curve, $\alpha_1 \ldots \alpha_r$ disjoint simple closed curves
$\Rightarrow \exists \gamma'$ s.t. $\gamma \underset{T}{\longleftrightarrow} \gamma'$, and $|\gamma' \cap \alpha_i| = 0$, or 2 w/ opp. orientations $\forall i = 1 \ldots r$.

pf: apply Lemma 2 repeatedly: first reduce intersections w/ α_1 to 0 or 2, then reduce ints w/ α_2 to 0 or 2 — in the process, we either preserve the intersection w/ α_1 (because we modify γ only in a nbd of α_2) or we remove them altogether (because we end up with $|\gamma \cap \alpha_2| = 1$ and replace γ by a parallel copy of α_2). Repeat process until α_r. ▲

We'll apply this lemma to 

& cut things into punctured tori $\Sigma_{1,1}$ along δ_i's.
Need a preliminary lemma about s.c.c's in $\Sigma_{1,1}$.

Lemma 4: (a) γ simple closed curve in $\Sigma_{1,1}$ $\Rightarrow$ γ isotopic to a straight line 

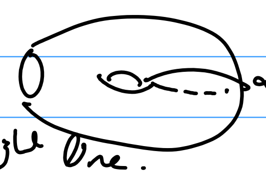
(b) η = simple arc in $\Sigma_{1,1}$ joining ∂ to itself $\Rightarrow$ η isotopic to a straight line through puncture (by isotopies which are not Id on $\partial\Sigma_{1,1}$) 

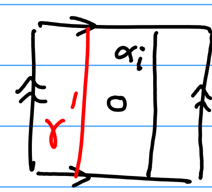
Pf.: (a) – on $\Sigma_1 = T^2$ this is classical
– on $\Sigma_{1,1}$: first deform γ to a straight line on T^2; move puncture in t-dependent way if needed so it's not hit. Then translate all curves in isotopy so the puncture doesn't move ✓

(b) similarly: close η to a s.c.c. on T^2 by adding a disc (whose center becomes a marked pt on η)
deform scc to straight line (moving marked pt if needed).
Undo motion of marked pt by translations; remove small disc there ▲

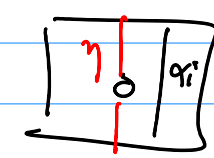
Lemma 5: $\gamma = h(\alpha_1)$, $h \in \mathrm{Homeo}^+(\Sigma_g)$ (α_i, δ_i as in picture above)
$\Rightarrow \exists \gamma'$ s.t. $\gamma \underset{T}{\longleftrightarrow} \gamma'$, $|\gamma' \cap \alpha_i| = 0\ \forall i$,
and $|\gamma' \cap \delta_i| = 0$, or 2 w/ opp. orient, $\forall i$.

Pf: Lemma 3 $\Rightarrow$ get γ' / $|\gamma' \cap \alpha_i|$ and $|\gamma' \cap \delta_i|$ = either 0 or 2 opp. orient. $\forall i$.
If $|\gamma' \cap \alpha_i| = 0\ \forall i$ we're done. Else $\exists i$ / $|\gamma' \cap \alpha_i| = 2$ w/ opp. orientⁿˢ.

(1) Assume $|\gamma' \cap \delta_i| = 0$.
Then $\gamma' \subset \Sigma_{1,1}$ bounded by δ_i 

= 

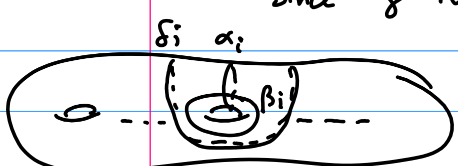
Use lemma 4 $\Rightarrow$ isotope γ' to a straight line.
$|\gamma' \cap \alpha_i| = 2$ w/ opp. orient. $\Rightarrow$ alg. $\cap$ number (isotopy inv't) is zero
$\Rightarrow$ the straight lines are //
can assume $|\gamma' \cap \alpha_i| = 0$ by isotopy.

(2) Assume $|\gamma' \cap \delta_i| = 2$: the 2 pts of $\gamma' \cap \delta_i$ split γ' into 2 arcs, call η the part in $\Sigma_{1,1}$; by Lemma 4, isotope γ' so that η is a straight line. Again, alg. $\cap$ number b/w η and α_i is zero
$\Rightarrow$ lines are //
$\Rightarrow$ can ensure $|\gamma' \cap \alpha_i| = 0$ by isotopy. 

▲

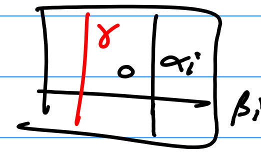
Lemma 6: $\gamma = h(\alpha_1)$ as above $\Rightarrow \gamma \underset{T}{\longleftrightarrow} \alpha_1$

Pf: start with γ' given by Lemma 5: $\gamma \underset{T}{\simeq} \gamma'$, $|\gamma' \cap \alpha_i| = 0$, $|\gamma' \cap \delta_i| = 0$ or 2 opp. orient.
$\rightarrow \gamma' \subset \Sigma_g \setminus (\alpha_1 \cup \dots \cup \alpha_g) \cong \Sigma_{0,2g}$ $\leftarrow$ $2g$ boundaries = 2 copies of each α_i.
every simple closed curve on S^2 is separating
$\exists i$ s.t. the 2 boundaries corresp. to α_i are on $\neq$ sides of γ'
(otherwise glue boundaries together $\rightsquigarrow$ γ' was separating on Σ, contradiction since $\gamma' \sim$ image of γ nonseparating)



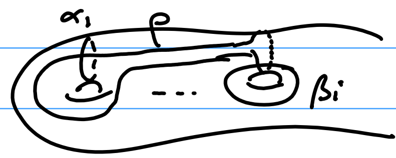
after opening along α_i, β_i joins 2 sides of γ', so $|\gamma' \cap \beta_i|$ is odd, $|\gamma' \cap \alpha_i| = 0$, $|\gamma' \cap \delta_i| = 0$ or 2 opp. orient.

Lemma 4 $\Rightarrow$ can isotop γ' so γ' or its part inside $\Sigma_{1,1}$ is a straight line.
$[\gamma']\cdot[\alpha_i] = 0$ $\Rightarrow$ this line is $\parallel \alpha_i$ and so intersects β_i exactly once.



Hence $\gamma \underset{T}{\Leftrightarrow} \gamma' \underset{T}{\Leftrightarrow} \beta_i$
by Lemma 1.

- if $i=1$: $|\beta_i \cap \alpha_1| = 1$ so $\beta_1 \underset{T}{\leftrightarrow} \alpha_1$
- if $i \geq 2$:



$|\beta_i \cap \rho| = 1$, $|\rho \cap \alpha_1| = 1$

$\beta_i \underset{T}{\leftrightarrow} \rho \underset{T}{\leftrightarrow} \alpha_1$. ∎

Complete pf as explained at beginning: induction on g, $g=0$ trivial. Assume $g-1$ ok. Let $h \in Map(\Sigma_g)$.

- L.6 $\Rightarrow$ $h(\alpha_1) \underset{T}{\leftrightarrow} \alpha_1$, so $\exists$ prod. D. twists ϕ st. $\phi(h(\alpha_1))$ isotopic to α_1.
  Compose with suitable isotopy $\Rightarrow$ assume $\phi \circ h$ is Id on α_1.
- cut open along α_1 $\Rightarrow$ get $\tilde{h} : \Sigma_{g-1,2} \circlearrowleft$, and $i_* : Map(\Sigma_{g-1,2}) \twoheadrightarrow Map(\Sigma_{g-1})$.
  $i_*(\tilde{h})$ = prod. D. twists by induction hypothesis
  $\leadsto$ $\tilde{h}$ = (elt in ker i_*) $\cdot$ (prod D. twists)
  $\hookrightarrow$ = prod. of ∂ twists & spin maps
  ie. prod of D. twists.
  $\Rightarrow$ $\tilde{h}$ is prod. Dehn twists, so $\phi \circ h$ too, so h too ∎