Lecture 11) ev: $\phi \mapsto (\phi(z_i^0))$ defines a loc. triv. fibration $\mathrm{Homeo}_n^+(\Sigma) \to \mathrm{Homeo}^+(\Sigma)$

$$\downarrow$$

$$\widetilde{\mathcal{C}}_n(\Sigma)$$

- This induces a L.E.S.

$$\to \pi_1 \mathrm{Homeo}^+(\Sigma) \xrightarrow[ev_*]{} \pi_1 \widetilde{\mathcal{C}}_n(\Sigma) \xrightarrow[\delta]{} \pi_0 \mathrm{Homeo}_n^+(\Sigma) \xrightarrow{i_*} \pi_0 \mathrm{Homeo}^+(\Sigma) \to \pi_0 \widetilde{\mathcal{C}}_n(\Sigma)$$

$$\pi_1 \widetilde{\mathcal{C}}_n(\Sigma) = P_n(\Sigma) \text{ pure braid grp. of } \Sigma, \quad \pi_0 \mathrm{Homeo}_n^+(\Sigma) = \mathrm{Map}_n(\Sigma), \quad \pi_0 \mathrm{Homeo}^+(\Sigma) = \mathrm{Map}(\Sigma), \quad \pi_0 \widetilde{\mathcal{C}}_n(\Sigma) = 1$$

$\delta : P_n(\Sigma) \to \mathrm{Map}_n(\Sigma)$ (for the disk, this was an isom!)

geometric pure braid β = motion of $\{z_i^0\}$: start at Id, construct an isotopy so that z_i^0 moves along strands of braid β, time 1 map gives $\delta(\beta)$.
(by constr. isotopic to Id, but not among homeos that fix z_i^0!)

Thm: (Birman) $i_* : \mathrm{Map}_n(\Sigma_{g,r}) \to \mathrm{Map}(\Sigma_{g,r})$ is a surjective homomorphism, with $\ker(i_*) = \mathrm{Im}(\delta) = \begin{cases} P_n(\Sigma) & \text{if } g \geq 2 \text{ or } r \geq 1, \text{ any } g. \\ P_n(\Sigma)/\text{center} & \text{if } \Sigma = T^2,\ n \geq 2 \\ & \text{or } \Sigma = S^2,\ n \geq 3 \end{cases}$

The same statement holds for $\dots \to B_n(\Sigma) \xrightarrow[\delta]{} \mathrm{Map}_{\{n\}}(\Sigma) \xrightarrow[i_*]{} \mathrm{Map}(\Sigma) \to 1$
($\ker i_* = B_n(\Sigma)$, or $B_n(\Sigma)/\text{center}$)

Most of the thm follows from the LES induced by ev. fibration; we just need to understand $\mathrm{Im}(\delta) \cong P_n(\Sigma)/\ker \delta$.

Lemma: $\ker \delta \subset \text{center}(P_n(\Sigma))$

Pf: let $\alpha \in \ker \delta = \mathrm{Im}(ev_* : \pi_1 \mathrm{Homeo}^+(\Sigma) \to \pi_1 \widetilde{\mathcal{C}}_n(\Sigma))$, let $H \in \pi_1 \mathrm{Homeo}^+(\Sigma)$ s.t. $ev_*(H) = \alpha$, $H = \{h_t\}_{0 \leq t \leq 1}$, $h_0 = h_1 = Id$.
Consider another pure braid $\beta \in P_n(\Sigma)$, represented by $\beta = (\beta_1(s), \dots, \beta_n(s))$
let $G : [0,1] \times [0,1] \to \widetilde{\mathcal{C}}_n(\Sigma)$

$$(t,s) \longmapsto (h_t(\beta_1(s)), \dots, h_t(\beta_n(s)))$$

$G|_{\partial([0,1]\times[0,1])}$ represents the homotopy class $\alpha\beta\alpha^{-1}\beta^{-1}$, hence $[\alpha,\beta] = 1$
$\Rightarrow \alpha \in \text{center}(P_n \Sigma)$ ◀

Lemma: If $g \geq 2$ or $(g=1, r\geq 1)$ or $(g=0, r\geq 3)$ then $\text{center}(P_n(\Sigma)) = \{1\}$

Pf: induction on n. Recall two things from our study of braid gps:
- $\pi_2(\Sigma - \{q_1 \dots q_n\}) = \pi_2(\text{—}) = 0 \ \forall n \geq 0 \Rightarrow \pi_2 \widetilde{\mathcal{C}}_n(\Sigma) = 0 \ \forall n$

- fibration $\Sigma - \{q_1 \dots q_{n-1}\} \hookrightarrow \widetilde{\mathcal{C}}_n(\Sigma)$ ← (forget last point)
$$\downarrow$$
$$\widetilde{\mathcal{C}}_{n-1}(\Sigma)$$

$\Rightarrow \quad 1 \to \pi_1(\Sigma - \{q_1 \dots q_{n-1}\}) \to \underset{= P_n(\Sigma)}{\pi_1(\widetilde{\mathcal{C}}_n(\Sigma))} \to \underset{= P_{n-1}(\Sigma)}{\pi_1(\widetilde{\mathcal{C}}_{n-1}(\Sigma))} \to 1$

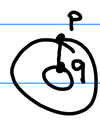
- assumption on $\Sigma \Rightarrow \Sigma$, $\Sigma - \{q_1 \dots q_{n-1}\}$ are h.e. to wedge of ≥ 2 circles, or closed genus ≥ 2 surfaces
So $P_1(\Sigma) = \pi_1 \Sigma$ is centerless
- assume $P_{n-1}(\Sigma)$ centerless $\Rightarrow$ then $\mathrm{center}(P_n(\Sigma)) \subset \pi_1(\Sigma - \{q_1 \dots q_{n-1}\})$
but that subgp is centerless $\Rightarrow$ center $= \{1\}$. ▲

This implies the thm for those cases.
Remaining cases: $\underbrace{S^2, T^2}, \underbrace{D^2, \text{annulus}}$

show $\ker \delta = \mathrm{center}(P_n(\Sigma))$
$(n \geq 2, n \geq 3)$

show $\ker \delta = \{1\}$
- for D^2, we've seen $\pi_1 \mathrm{Homeo}^+(D^2) = \{1\}$
(retract to $\{Id\}$ by radial rescaling)
so $\ker \delta = \mathrm{Im}\, ev_* = \{1\}$.

- for Σ = annulus, use loc. trivial fibration induced by restr. to an arc



$\mathrm{Homeo}^+(D^2) \hookrightarrow \mathrm{Homeo}^+(\Sigma)$
$$\downarrow$$
$\mathcal{P}(p,q) = \{\text{embedded arcs } p \to q\}$

$\Rightarrow \pi_1 \mathrm{Homeo}^+(\Sigma) \simeq \pi_1 \mathcal{P}(p,q) = 1.$
so $\ker \delta = \mathrm{Im}\, ev_* = 1.$

- for S^2: we've seen
  - $B_n(S^2) = B_n / \langle \sigma_1 \cdots \sigma_{n-1} \sigma_{n-1} \cdots \sigma_1 \rangle$

  - $\mathrm{center}(B_n) = \mathrm{center}(P_n) = \langle \Delta^2 \rangle$
(I maybe not stated, but the argument we gave also proves this since we used property of commut. w/ generators of P_n ...)

some work ⇝ (see eg. Birman's book)

lemma: for $n \geq 3$, center $B_n(S^2)$ = center $P_n(S^2)$ = cyclic of order 2 gen'd by $\bar{\Delta}^2 = (\bar{\sigma}_1 \cdots \bar{\sigma}_{n-1})^n$

Idea
cent ⊂ pure, but RS method ⇒ {braids in S^2 which preserve q_n} $\cong B_{n-1} / (\sigma_1 \cdots \sigma_{n-2})^{2(n-1)}$ (placing q_n at ∞)
so center $= (\bar{\sigma}_1 \cdots \bar{\sigma}_{n-2})^{n-1} = (\bar{\sigma}_1 \cdots \bar{\sigma}_{n-2})^{n-1} \bar{\sigma}_{n-1} \cdots \bar{\sigma}_1 \, \bar{\sigma}_1 \cdots \bar{\sigma}_{n-1}$
$= (\bar{\sigma}_1 \cdots \bar{\sigma}_{n-1})^{n-1} (\bar{\sigma}_1 \cdots \bar{\sigma}_{n-1}) = \bar{\Delta}^2$

(note: $(\bar{\sigma}_1 \cdots \bar{\sigma}_{n-1})^{2n} = (\bar{\sigma}_1 \cdots \bar{\sigma}_{n-1})^n (\bar{\sigma}_{n-1} \cdots \bar{\sigma}_1)^n = 1$ in $B_n(S^2)$).
since Δ^2 central, eq. invt under conj. by Δ, $\sigma_i \mapsto \sigma_{n-i}$

and $\bar{\Delta}^2$ is exactly ev_* of loop generating $\pi_1 SO(3)$ (rotation $0 \to 2\pi$)
in $\mathrm{Homeo}^+(S^2)$

so $$\ker \delta \subset \mathrm{Center}\, P_n(S^2) = \langle \bar{\Delta}^2 \rangle \subset \mathrm{Im}\, ev_*$$
equal by LES $\Rightarrow$ all equal

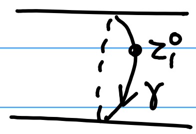
- for T^2, $n \geq 2$: can show
$$\ker \delta = \mathrm{Im}\, ev_* = \mathrm{Center}\, P_n(T^2) \simeq \mathbb{Z}^2 \text{ gen}^d \text{ by } ev_* \text{ of translation loops}$$
in $\mathrm{Homeo}^+(T^2)$ (translations by $0 \rightsquigarrow$ an elt of $\mathbb{Z}^2$)

The pf for $\mathrm{Map}_{\{n\}}(\Sigma) \to \mathrm{Map}(\Sigma)$ and $B_n(\Sigma)$ $\equiv$ same idea.

Rmks: • what does $\ker(i_*: \mathrm{Map}_1(\Sigma) \to \mathrm{Map}(\Sigma))$ ($\simeq P_1(\Sigma) = \pi_1(\Sigma)$ when $g \geq 2$) look like ??

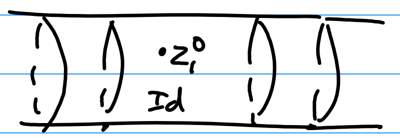
Recall it's $\mathrm{Im}(\delta)$ $\rightsquigarrow$ understand $\delta(\gamma)$, $\gamma \in$ generators of $\pi_1(\Sigma)$:

can assume γ = embedded loop at base pt z_1^0, so

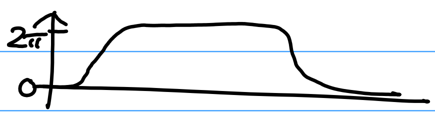


$\delta(\gamma)$ = result of isotopy gen^d by this motion of z_1^0

= "spin map"



rotate each circle by



- similarly $\ker(i_*: \mathrm{Map}_n(\Sigma) \to \mathrm{Map}(\Sigma))$ is gen^d by spin maps along embedded loops through the various z_i^0.

for $\mathrm{Map}_{\{n\}}$, add homeos that permute the z_i^0.

- The structure of $\mathrm{Map}_n(S^2)$, $\mathrm{Map}_{\{n\}}(S^2)$ [recall $\mathrm{Map}_n(D^2) = P_n$, $\mathrm{Map}_{\{n\}}(D^2) = B_n$]

- first, recall $\mathrm{Map}(D^2) \overset{i_*}{\twoheadrightarrow} \mathrm{Map}_1(S^2) \overset{i_*}{\twoheadrightarrow} \mathrm{Map}(S^2)$
∂ component $\to$ marked pt $\to$ forget pt

but we've seen $\mathrm{Map}(D^2) = 1$ ($\equiv$ Braids on 0 strings... or, quotient of $\mathrm{Map}_1(D^2) = B_1 = \{1\}$)

So $\|$ $\mathrm{Map}(S^2) = \mathrm{Map}_1(S^2) = 1$.

Thm: For $n \geq 2$, $\mathrm{Map}_{\{n\}}(S^2)$ has presentation with generators $\tau_1, \dots, \tau_{n-1}$ & relations
$$\tau_i \tau_j = \tau_j \tau_i \quad |i-j| \geq 2$$
$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$$
$$\tau_1 \cdots \tau_{n-1} \tau_{n-1} \cdots \tau_1 = 1$$
$$(\tau_1 \cdots \tau_{n-1})^n = 1$$

ie, for $n \geq 3$, $\mathrm{Map}_{\{n\}}(S^2) = B_n(S^2)/\text{center} = B_n / \langle \tau_1 \cdots \tau_{n-1} \tau_{n-1} \cdots \tau_1, \ (\tau_1 \cdots \tau_{n-1})^n \rangle$

$\hookrightarrow$ recall this $\cong \mathbb{Z}_2$ gen^d by $\bar{\Delta}^2 = (\tau_1 \cdots \tau_{n-1})^n$

Pf: • for $n=2$: check by hand that $\mathrm{Map}_{\{2\}}(S^2) = \mathbb{Z}/2$:

it's at most $\mathbb{Z}/2$ since $\cdots \to B_2(S^2) \to \mathrm{Map}_{\{2\}}(S^2) \to \mathrm{Map}(S^2) \to 1$, with $B_2(S^2) = \mathbb{Z}/2$ (by known presentation) and $\mathrm{Map}(S^2) = 1$

but $\mathrm{Map}_{\{n\}} \twoheadrightarrow \mathfrak{S}_n$ (induced permutation of $\{z_i^0\}$) so at least $\mathbb{Z}/2$

• for $n \geq 3$, use $1 \to B_n(S^2)/\text{center} \to \mathrm{Map}_{\{n\}}(S^2) \xrightarrow{i_*} \mathrm{Map}(S^2) \to 1$, with $\mathrm{Map}(S^2) = 1$

$\Rightarrow \mathrm{Map}_{\{n\}}(S^2) \cong B_n(S^2)/\text{center}$ but we calculated center $= \langle \bar{\Delta}^2 \rangle$ before. ■

Another pf: use inclusion homomorphisms $\mathrm{Map}_{\{n\}}(D^2) \twoheadrightarrow \mathrm{Map}_{\{n\}+1}(S^2) \twoheadrightarrow \mathrm{Map}_{\{n\}}(S^2)$, with $\mathrm{Map}_{\{n\}}(D^2) = B_n$

$\mathrm{Map}_{\{n\}+1}(S^2)$: homeos fixing $\{z_1^0 \dots z_n^0, \infty\}$ and $\infty \mapsto \infty$.

– the first one has kernel gen^d by boundary twist map

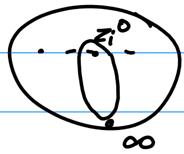
which is exactly the homeo induced by Δ^2

$\Rightarrow \mathrm{Map}_{\{n\}+1}(S^2) \simeq B_n / \langle \Delta_n^2 \rangle$ (and incidentally $\mathrm{Map}_{n+1}(S^2) \simeq P_n / \langle \Delta_n^2 \rangle$)

– the second one has kernel $\mathrm{Im}(\delta : \pi_1 C_1(S^2 - \{z_1^0 \dots z_n^0\}) \to \mathrm{Map}_{\{n\}}(S^2))$

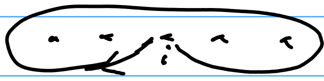
(recall fibration $\mathrm{Homeo}^+_{\{n\}+1}(S^2) \hookrightarrow \mathrm{Homeo}^+_{\{n\}}(S^2) \xrightarrow{\mathrm{ev}_\infty} S^2 - \{z_1^0 \dots z_n^0\}$)

ie gen^d by spin maps along



(move ∞ around z_i^0, take corresponding isotopy)

$\Leftrightarrow$ corresponds to moving z_i^0 around ∞, ie. it's the homeo corresponding to the braid



$= \sigma_{i-1} \sigma_{i-2} \cdots \sigma_1 \sigma_1 \cdots \sigma_{n-1} \sigma_{n-1} \cdots \sigma_i$

$=$ conj. of $\sigma_1 \cdots \sigma_{n-1} \sigma_{n-1} \cdots \sigma_1$

$\Rightarrow \mathrm{Map}_{\{n\}}(S^2) = B_n / \langle \Delta_n^2, \sigma_1 \cdots \sigma_{n-1}^2 \cdots \sigma_1 \rangle$. ■