Lecture 10 – March 15 – Mapping class groups

Notation: Σ_g = closed orientable surface of genus g

$\Sigma_{g,r}$ = orientable surface of genus g w/ r boundary components

$z_1^0, \dots, z_n^0$ = n fixed distinct pts on Σ_g or $\Sigma_{g,r}$

$\rightarrow$ $\mathrm{Homeo}^+(\Sigma_g)$ = orient.-preserving homeos of Σ_g (compact-open top.)

$\mathrm{Homeo}^+(\Sigma_{g,r})$ = — ——— $\Sigma_{g,r}$ s.t. $\phi_{|\partial\Sigma_{g,r}} = Id$

$\mathrm{Homeo}^+_n(\Sigma) = \{\phi \in \mathrm{Homeo}^+(\Sigma) \,/\, \phi(z_i^0) = z_i^0 \ \forall i\}$

$\mathrm{Homeo}^+_{\{n\}}(\Sigma) = \{ \text{———} \,/\, \phi(\{z_1^0 \dots z_n^0\}) = \{z_1^0 \dots z_n^0\}\}$

$\mathrm{Map}(\Sigma_g) = \mathrm{Map}(g)$, $\mathrm{Map}(\Sigma_{g,r})$, $\mathrm{Map}_n(\Sigma_{g,r})$, $\mathrm{Map}_{\{n\}}(\Sigma_{g,r})$ $\longleftrightarrow$ DIFFER BY $\mathfrak{S}_n$.

<u>mapping class groups</u> = π_0 Homeo

<u>Goal</u>: understand the structure of these gps & how they relate to each other

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• Relating various flavors of mcg's:

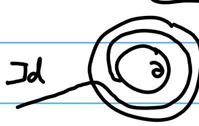
<u>Observ</u>: • $\Sigma \overset{i}{\hookrightarrow} \Sigma'$ $\Rightarrow$ a homeo of Σ ($=Id$ on ∂) naturally extends to Σ' (by Id on $\Sigma' - \Sigma$). This induces nat'l homomorphisms $\mathrm{Map}(\Sigma) \xrightarrow{i_*} \mathrm{Map}(\Sigma')$

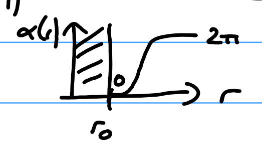
• we'll see: $\mathrm{Map}(\Sigma_{g,r}) \twoheadrightarrow \mathrm{Map}_r(\Sigma_g) \twoheadrightarrow \mathrm{Map}(\Sigma_g)$

① punctures vs. boundary components: $\mathrm{Homeo}^+(\Sigma_{g,r}) \hookrightarrow \mathrm{Homeo}^+_r(\Sigma_g)$ $\left(\Sigma_g = \Sigma_{g,r} \cup \bigcup B_i\right.$, $B_i = D^2(z_i^0)$, & extend by Id on these discs $B_i$$\left.\right)$

This induces $i_*: \mathrm{Map}(\Sigma_{g,r}) \to \mathrm{Map}_r(\Sigma_g)$

<u>Thm</u>: (Wajnryb ??) If $g \geq 1$, or if $g=0$, $r \geq 3$,
$$1 \to \mathbb{Z}^r \to \mathrm{Map}(\Sigma_{g,r}) \xrightarrow{i_*} \mathrm{Map}_r(\Sigma_g) \to 1$$
central extension, kernel gen'd by <u>∂ twist maps</u> (near a ∂, $\underset{=\{r=r_0\}}{}$ $(r,\theta) \mapsto (r, \theta + \alpha(r))$)

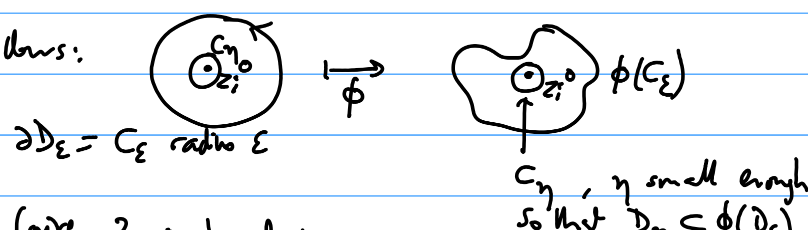




(for $g=0, r=2$, $\mathrm{Map}(\Sigma_{0,2}) = \mathbb{Z}$ (both ∂ twist maps are $\equiv$)
$g=0, r=1$, $\mathrm{Map}(\Sigma_{0,1}) = 1$ (seen: $\mathrm{Map}_{\{n\}}(\Sigma_{0,1}) = B_n$))

<u>Pf (idea)</u>: • given a homeo of Σ_g s.t. $\phi(z_i^0) = z_i^0$, can approximate it by a homeo which is Id on a small nbd $D(\eta)$

<u>Idea</u>: for small $\varepsilon > 0$ build ϕ_ε as follows:



Schoenflies thm (given 2 simple closed curves C, C' in the plane, any homeo $h: C \to C'$ extends to a homeo of the interior region)

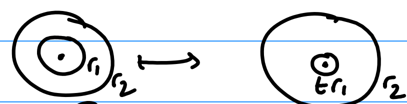
$\Rightarrow$ get a homeo $\overline{D_\varepsilon} \longrightarrow \overline{\phi(D_\varepsilon)}$
which · agrees with $\phi_{|C_\varepsilon}: C_\varepsilon \to \phi(C_\varepsilon)$ on boundary
· is Id over D_η

Then replace ϕ by this homeo inside D_ε
get ϕ_ε, and in compact-open topology $\phi_\varepsilon \xrightarrow[\varepsilon\to 0]{} \phi$

$\Rightarrow$ ϕ, ϕ_ε in same conn. component of $\mathrm{Homeo}^+_r(\Sigma_g)$ if ε small enough!

Moreover, maybe the disc of radius η doesn't contain the disc B_i (= comp. of z_i in $\Sigma_g - \Sigma_{g,r}$)

but: conjugate ϕ_ε with family $(r,\theta) \longmapsto (f_t(r), \theta)$



$D(r_2) \supset D(r_1) \supset B_i$

to deform it continuously so it's Id on a larger and larger disc (go from $t=1$ to t s.t. $tr_1 < \eta$
then conjugating homeo takes B_i into $D(\eta)$ where ϕ_ε is Id).

This implies surjectivity of $i_*: \mathrm{Map}(\Sigma_{g,r}) \to \mathrm{Map}_r(\Sigma_g)$
(we found a homeo fixing B_i pointwise in the same conn component of $\mathrm{Homeo}^+_r(\Sigma_g)$ as ϕ).

- Now understand $\ker(i_*)$: assume $\phi_0, \phi_1 = \mathrm{Id}$ on B_i, in same component of $\mathrm{Map}_r(\Sigma_g)$ ie joined by $(\phi_t)_{t\in[0,1]}$ arc in $\mathrm{Homeo}^+_r(\Sigma_g)$.

LEMMA
Idea: can similarly approximate (ϕ_t) by an arc s.t. $\phi_{t,\varepsilon}$ maps a small disc $D(\eta)$ to itself — but can't ensure $\phi_{t,\varepsilon|D(\eta)} = \mathrm{Id}$ $\forall t$.
However, can assume $\phi_{t,\varepsilon|D(\eta)}$ is a rotation. (and $\phi_{0,\varepsilon} = \phi_0$, $\phi_{1,\varepsilon} = \phi_1$).

Proceed as above, but look at the homeo $\Phi: (t,z) \mapsto (t, \phi_t(z))$ on $[0,1]\times D_\varepsilon$, $D_\varepsilon \subset B_i$ — and use $\phi_0 = \phi_1 = \mathrm{Id}$ on B_i to think of it as a homeo from $S^1 \times D_\varepsilon$ to $\Phi(S^1\times D_\varepsilon)$.

For $\eta \ll \varepsilon$, $S^1\times(D_\varepsilon - D_\eta)$ and $\Phi(S^1\times D_\varepsilon) - S^1\times D_\eta$ are again homeomorphic by a homeo $(t,z) \mapsto (t, \psi_t(z))$

Claim: we can ensure
$$\begin{cases} (1) & \psi_{t=0=1} = \mathrm{Id} \\ (2) & \psi_{t|C_\varepsilon} = \phi_{t|C_\varepsilon} \\ (3) & \psi_{t|C_\eta} \text{ is a rotation} \end{cases}$$

- first compose each ψ_t with $\psi_0^{-1}: D_\varepsilon - D_\eta \to D_\varepsilon - D_\eta$ to ensure (1)

- then ensure (2) by composing ψ_t with a homeo $(t, r, \theta) \mapsto (t, r, f_t(\theta))$
where $f_t : S^1 \to S^1$ is the discrepancy b/w φ_t and ψ_t on C_ε.
($f_0 = Id$).

- then ensure (3) by looking at ψ_t on $S^1 \times C_\eta \to S^1 \times C_\eta$
$(t, \eta, \theta) \mapsto (t, \eta, \theta + \delta(t, \theta))$

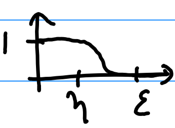
- $\delta : S^1 \times C_\eta \to \mathbb{R}$ continuous;
- $\theta + \delta(t, \theta)$ strictly $\nearrow$ function of θ
- δ lifts to a function on univ cover $\mathbb{R}_t \times \mathbb{R}_\theta$, periodic in θ
however $\delta(t+1, \theta) = \delta(t, \theta) + k$ for some $k \in \mathbb{Z}$!

- WLOG $\delta(t=0, \theta) \equiv 0$, $\delta(t=1, \theta) \equiv k$ since $\psi_{t=0 \text{ or } 1}$ is Id.

Then we can interpolate between $\theta \mapsto \theta + \delta(t, \theta)$ and $\theta \mapsto \theta + kt$ via fns w/ the same property (take: $\lambda \delta(t, \theta) + (1-\lambda) kt$).

Then compose ψ_t with $\left((t, r, \theta) \underset{\gamma_t}{\longmapsto} (t, r, \theta + \lambda(r)(\delta(t, \theta) - kt)) \right)^{-1}$

- for $t = 0$ or 1, $\gamma_t = Id$ ✓
for $r = \varepsilon$, ——— ✓



- for $r = \eta$, $(t, \eta, \theta + \delta(t, \theta) - kt) \underset{\gamma_t^{-1}}{\longmapsto} (t, \eta, \theta) \underset{\psi_t}{\longmapsto} (t, \eta, \theta + \delta(t, \theta))$
rotation by kt ✓

$\to$ replace ψ_t by $\psi_t \circ \gamma_t^{-1}$!

Now, glue together
- ϕ_t outside D_ε
- ψ_t on $D_\varepsilon \setminus D_\eta$
- rotation by kt on D_η.

$\to$ get $\phi_{t,\varepsilon}$ as claimed.

(and $\phi_{t,\varepsilon} \underset{\varepsilon \to 0}{\to} \phi_t$ in compact open topology)

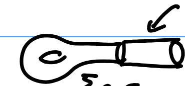
As before, conjugating by radial scaling we can assume $\phi_{t,\varepsilon} = Id$ on a fixed given disc (e.g. B_i) instead of D_η.

- Now assume $\phi \in \ker i_*$ and take an arc in $Homeo^+$ from $\phi_0 = \phi$ to $\phi_1 = Id$
$\to$ use above argument to deform it to an arc $\phi \underset{\phi_t}{\to} Id$ such that ϕ_t is a rotation near z_i° $\forall t$

Replace  R_α by  for all t, near each z_i°

$\Rightarrow$ get an arc $\phi \to \prod \partial$ twist maps inside $Homeo^+(\Sigma_{g,r})$

Hence the ∂ twist maps generate $\ker(i_*)$

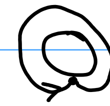
(NB: they clearly commute; in fact they are central in $Map(\Sigma_{g,r})$ — can enlarge  isotope so supp.(twist) $\subset$ this.)

So $\ker(i_*) = \text{Image}\,\big(\underset{\partial\text{ twist maps}}{\mathbb{Z}^r} \longrightarrow \text{Map}(\Sigma_{g,r})\big)$; claim: if $g \geq 1$ or $g=0,\ r\geq 3$ this map is injective.

To see this look at induced action of $\text{Map}(\Sigma_{g,r})$ on $\pi_1(\Sigma_{g,r}, p_i)$

base pt on ith ∂ component

→ this is a free gp on $2g+r-1$ generators, hence nonabelian if $g \geq 1$ or $g=0,\ r\geq 3$
center-free

and twist map at ith boundary acts by conjugation by , hence nontrivially

jth ——, $j \neq i$ acts trivially

→ get $\mathbb{Z}^r \hookrightarrow \text{Map}(\Sigma_{g,r})$. ▲

. More conceptual way to think about it: the above argument $(\phi_t) \rightsquigarrow (\phi_{t,\varepsilon})$ adapted to case where $\phi_0, \phi_1 / V(z_i^0)$ are rotations (not necess. Id) shows:

$$\text{Map}_r(\Sigma_g) \underset{\text{def}}{=} \pi_0\{\underbrace{\phi \in \text{Homeo}^+,\ \phi(z_i^0) = z_i^0}_{\text{Homeo}^+_r}\} \cong \pi_0\{\phi \in \text{Homeo}^+,\ \phi = \text{rotation in a nbd of } z_i^0\}.$$

e.g. B_i

Then $\exists$ loc. trivial fibration $\{\phi \mid \phi = \text{Id on } B_i\} \hookrightarrow \{\phi \mid \phi = \text{rotation on } B_i\}$

$$\{\phi \mid \phi = \text{rotation on } B_i\} \xrightarrow{\text{restr to } B_i} (S^1)^r$$

and l.e.s. $\pi_1\{\phi \mid \phi = \text{rot. on } B_i\} \to \pi_1(S^1)^r \to \pi_0\{\phi \mid \phi = \text{Id on } B_i\} \to \pi_0\{\phi \mid \phi = \text{rot on } B_i\} \to 1$

where $\pi_1(S^1)^r = \mathbb{Z}^r$, $\pi_0\{\phi \mid \phi = \text{Id on } B_i\} = \text{Map}(\Sigma_{g,r})$ by def, $\pi_0\{\phi \mid \phi = \text{rot on } B_i\} = \text{Map}_r(\Sigma_g)$ by above rmk

one can show this map is zero if $g \geq 1$ or $g=0,\ n\geq 3$ by looking at action on homotopy near boundaries

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② Forgetting marked pts. $\text{Map}_n(\Sigma_g) \twoheadrightarrow \text{Map}(\Sigma_g)$ induced by inclusion
$\text{Map}_{\{n\}}(\Sigma_g)$

• The evaluation map ev: $\phi \mapsto (\phi(z_i^0))$ defines loc. trivial fibrations

$$\text{Homeo}^+_n(\Sigma) \to \text{Homeo}^+(\Sigma) \xrightarrow{\text{ev}} \widetilde{\mathcal{C}}_n(\Sigma) \quad \text{(ordered configs. of distinct pts)}$$

$$\text{Homeo}^+_{\{n\}}(\Sigma) \to \text{Homeo}^+(\Sigma) \to \mathcal{C}_n(\Sigma) \quad \text{(unordered configs)}$$

(in case $\Sigma = D^2$, this is exactly what we used to prove $B_n = \text{Map}_{\{n\}}(D^2)$, $P_n = \text{Map}_n(D^2)$).
The argument is the same here!!
(use homeos $p_i \mapsto z_i^0$ for $p_i \in U_i$ disjoint nbds of given $(p_i^0) \in \widetilde{\mathcal{C}}_n(\Sigma)$ to trivialize).

- This induces a L.E.S.

$$\to \pi_1 \mathrm{Homeo}^+(\Sigma) \xrightarrow{ev_*} \pi_1 \widetilde{\mathcal{C}}_n(\Sigma) \xrightarrow{\delta} \pi_0 \mathrm{Homeo}^+_n(\Sigma) \xrightarrow{i_*} \pi_0 \mathrm{Homeo}^+(\Sigma) \to \pi_0 \widetilde{\mathcal{C}}_n(\Sigma)$$

$\pi_1 \widetilde{\mathcal{C}}_n(\Sigma) = P_n(\Sigma)$ (pure braid grp. of Σ), $\pi_0 \mathrm{Homeo}^+_n(\Sigma) = \mathrm{Map}_n(\Sigma)$, $\pi_0 \mathrm{Homeo}^+(\Sigma) = \mathrm{Map}(\Sigma)$, $\pi_0 \widetilde{\mathcal{C}}_n(\Sigma) = 1$

$\delta : P_n(\Sigma) \to \mathrm{Map}_n(\Sigma)$ (for the disc, this was an isom!)

geometric pure braid β = motion of $\{z_i^\circ\}$

start at Id, construct an isotopy so that z_i° moves along strands of braid β, time 1 map gives $\delta(\beta)$.
(by constr isotopic to Id, but not among homeos that fix z_i°!)

Thm: (Birman) $i_* : \mathrm{Map}_n(\Sigma_{g,r}) \to \mathrm{Map}(\Sigma_{g,r})$ is a surjective homomorphism, with $\ker(i_*) = \mathrm{Im}(\delta) = \begin{cases} P_n(\Sigma) & \text{if } g \geq 2 \text{ or } r \geq 1, \text{ any } g. \\ P_n(\Sigma)/\text{center} & \text{if } \Sigma = T^2,\ n \geq 2 \text{ or } \Sigma = S^2,\ n \geq 3 \end{cases}$

The same statement holds for $\dots \to B_n(\Sigma) \xrightarrow{\delta} \mathrm{Map}_{\{n\}}(\Sigma) \xrightarrow{i_*} \mathrm{Map}(\Sigma) \to 1$
($\ker i_* = B_n(\Sigma)$, or $B_n(\Sigma)/\text{center}$)

Most of the thm follows from the LES induced by ev. fibration; we just need to understand $\mathrm{Im}(\delta) \cong P_n(\Sigma)/\ker \delta$.