18.937 Spring 2006 – Lecture Notes

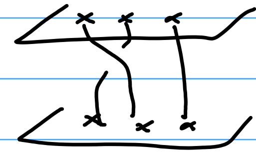
D. Auroux (Mapping class gps & braid gps)

Lecture 1: $\boxed{\text{Overview}}$

∗ **Braid group:** $B_n = \pi_1(\mathcal{C}_n(\mathbb{R}^2))$ ← or another 2-mfld, but classical braids $= \mathbb{R}^2$

(Emil Artin) 1925

$\mathcal{C}_n(\mathbb{R}^2) = \{n \text{ distinct pts} \in \mathbb{R}^2\} = ((\mathbb{R}^2)^n - \Delta)/\mathfrak{S}_n$.

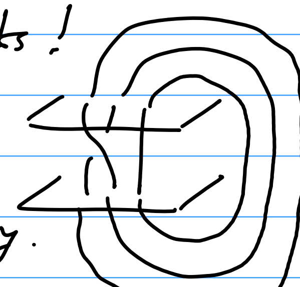


(product = juxtaposition)

Presentation (Artin): $B_n = \langle \sigma_1 \dots \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| \geq 2 \rangle$

∗ **Markov's thm:** theory of braids is related to knots & links!

Fact: every link can be represented by a closed braid



Markov's thm gives a criterion for when 2 braids represent the same link up to isotopy.

⇝ braids are an algebraically easier version of links...

2 moves: conjugation & stab$^{\underline{n}}$ $\cdot \sigma_n^{\pm 1}$

∗ **Word & conjugacy problems:**

There are efficient algorithms for manipulating braids; e.g. word & conj. problems.

(We'll see the classical ones – Garside ← (not so efficient) – but this is an active area of research, due in part to "braid cryptography" – maybe say a little bit)

∗ **Mapping class groups:** = isotopy classes of homeos (or diffeos) of a mfld (esp. dim 2) (if ∂, fix boundary; can also ask that a given finite subset be mapped to itself)

simplest: $\mathrm{Map}_g = \pi_0 \, \underset{\text{Homeo}}{\mathrm{Diff}^+}(\Sigma_g)$ ↳ compact closed genus g

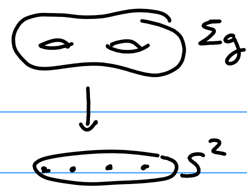
Important in study of Riemann surfaces (e.g. $\mathcal{M}_g = \mathcal{T}_g / \mathrm{Map}_g$)

cx structs — marked cx structures

∗ closely related to braid group: in fact $B_n = \pi_0 \mathrm{Diff}^+_{\text{cpct}}(\mathbb{R}^2, \{p_1 \dots p_n\})$

and $B_n(S^2)$ is closely related to $\mathrm{Map}_{0,\{n\}}(S^2 \to S^2 \text{ preserving } \{n \text{ pts}\})$.

• Lifting:



branched cover (e.g.: double cover: $\{z^2 = P(x;y)\}$ genus g / homogeneous coords on $\mathbb{CP}^1$, deg. $2g+2$, simple roots)

$\Rightarrow$ in favorable cases, can lift a homeo of S^2 fixing the branch pts to a homeo of Σ_g.
(also for surfaces w/ boundary).

• can use this & other techniques to get presentations of Map_g.
(Generators = Dehn twists) $\rightsquigarrow$ explain

Geometric applications: monodromy

• E.g. dim 3: Thm: every 3-mfld $\xrightarrow[\text{branched cover / link}]{} S^3$

Put link in braid form $\to$ get "open book" structure, with monodromy determined by the braid

• dim 4: – braid monodromy of a $\mathbb{C}$ plane curve (alg., or a bit more general) $\pi_1(-) \to B_n$
– Lefschetz pencils on alg. $\mathbb{C}$ surface $\subset \mathbb{CP}^N$
vs proj. to $\mathbb{CP}^2$, braid monodromy of branch curve

} extends to sympl. world

$\to$ Moishezon–Teicher approach to $\mathbb{C}$ surfaces, sympl. extension, ----