On a Quantum Version of Pieri's Formula

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March 23, 1997

Abstract

We give an algebro-combinatorial proof of a general version of Pieri's formula following the approach developed by Fomin and Kirillov in the paper "Quadratic algebras, Dunkl elements, and Schubert calculus." We prove several conjectures posed in their paper. As a consequence, a new proof of classical Pieri's formula for cohomology of complex flag manifolds, and that of its analogue for quantum cohomology is obtained in this paper.

1 Introduction

The purpose of this paper is to investigate several consequences and generalizations of quantum Monk's formula from [FGP]. In our approach we follow Fomin and Kirillov [FK], who constructed a certain quadratic algebra $\mathcal{E}_n$ equipped with a family of pairwise commuting "Dunkl elements," which generate a subalgebra canonically isomorphic to the cohomology ring of complex flag manifold. They observed that Pieri's formula for the cohomology of flag manifolds can be deduced from a certain identity, which conjecturally holds in $\mathcal{E}_n$. In this paper a more general Pieri-type formula is proved, which specializes to their conjecture in a particular case.

A quantum deformation of the algebra $\mathcal{E}_n$ was also given in [FK], along with the conjecture that its subalgebra generated by the Dunkl elements is canonically isomorphic to the (small) quantum cohomology of the flag manifold. This statement is also a special case of our result.

Pieri's formula for multiplication in the quantum cohomology ring of flag manifolds can also be obtained from our formula (but the opposite is not true). Its classical counterpart is the rule that was formulated by Lascoux and Schützenberger [LS] and proved geometrically by Sottile [S] (see also [W] for a combinatorial proof). Pieri's formula for the quantum cohomology was recently proved by Ciocan-Fontanine [C2], using nontrivial algebro-geometric techniques. By contrast, our proof is combinatorial, and does not rely upon geometry at all—once (quantum) Monk's formula is given. Our proof seems to be new even in the classical case.

Key words and phrases. Flag manifold, Monk's formula, Pieri's formula, quantum cohomology.

The rest of Introduction is devoted to a brief account of main notions and results related to the classical as well as the quantum cohomology rings of complex flag manifolds. For a more complete story, see [BGG], [FGP], [FP], [Ma], and bibliography therein. Although many of the constructions below can be carried out in a more general setup of an arbitrary semisimple Lie group, only the case of type A_{n-1} is considered in this paper.

Let Fl_n denote the manifold of complete flags of subspaces in $\mathbb{C}^n$. According to classical Ehresmann's result [E], the Schubert classes σ_w, indexed by the elements w of the symmetric group S_n, form an additive $\mathbb{Z}$-basis of the cohomology ring $\mathrm{H}^*(Fl_n, \mathbb{Z})$ of the flag manifold.

The multiplicative structure of $\mathrm{H}^*(Fl_n, \mathbb{Z})$ can be recovered from Borel's theorem [B]. Let s_{ij} be the element of S_n that transposes i and j. Also let $s_i = s_{i\,i+1}$, $1 \le i \le n-1$, be the Coxeter generators of S_n. Borel's theorem says that $\mathrm{H}^*(Fl_n, \mathbb{Z})$ is canonically isomorphic, as a graded algebra, to the quotient

$$\mathbb{Z}[x_1, x_2, \ldots, x_n] \,/\, \langle e_1, e_2, \ldots, e_n \rangle \,, \tag{1}$$

where $e_k = e_k(x_1, x_2, \ldots, x_n)$ denotes the kth elementary symmetric polynomial, for $k = 1, 2, \ldots, n$; and $\langle e_1, \ldots, e_n \rangle$ is the ideal generated by the e_k. The isomorphism is given by explicitly specifying

$$x_1 + x_2 + \cdots + x_m \longmapsto \sigma_{s_m} \,, \quad m = 1, 2, \ldots, n-1 \,.$$

A way to relate these two descriptions of the cohomology ring $\mathrm{H}^*(Fl_n, \mathbb{Z})$ was found by Bernstein, Gelfand, and Gelfand [BGG] and Demazure [D]. Lascoux and Schützenberger [LS] then constructed the Schubert polynomials, whose images in the quotient (1) represent the Schubert classes σ_w.

Recently, attention has been drawn to the (small) quantum cohomology ring $\mathrm{QH}^*(Fl_n, \mathbb{Z})$ of the flag manifold. We will not give here the definition of quantum cohomology (see e.g. [FP]), but we mention that structure constants of quantum cohomology are 3-point Gromov-Witten invariants, which count the numbers of certain rational curves and play a role in enumerative algebraic geometry.

As a vector space, the quantum cohomology of Fl_n is essentially the same as the usual cohomology. More precisely,

$$\mathrm{QH}^*(Fl_n, \mathbb{Z}) \cong \mathrm{H}^*(Fl_n, \mathbb{Z}) \otimes \mathbb{Z}[q_1, \ldots, q_{n-1}].$$

However, the multiplicative structure in $\mathrm{QH}^*(Fl_n, \mathbb{Z})$ is different.

A quantum analogue of Borel's theorem was suggested by Givental and Kim [GK], and then justified by Kim [K] and Ciocan-Fontanine [C1]. Let $E_1, E_2, \ldots, E_n \in \mathbb{Z}[x_1, \ldots, x_n; q_1, \ldots, q_{n-1}]$ be the nonidentity coefficients of the characteristic polynomial of the matrix

$$\begin{pmatrix} x_1 & q_1 & 0 & \cdots & 0 \\ -1 & x_2 & q_2 & \cdots & 0 \\ 0 & -1 & x_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_n \end{pmatrix} . \tag{2}$$

The E_k are certain q-deformations of the elementary symmetric polynomials e_k. If $q_1 = \cdots = q_{n-1} = 0$ then E_k specializes to e_k.

Givental, Kim, and Ciocan-Fontanine showed that the quantum cohomology ring $\mathrm{QH}^*(Fl_n, \mathbb{Z})$ is canonically isomorphic to the quotient

$$\mathbb{Z}[x_1, \ldots, x_n; q_1, \ldots, q_{n-1}] \,/\, \langle E_1, E_2, \ldots, E_n \rangle \,. \tag{3}$$

Just as in the classical case, the isomorphism is given by specifying

$$x_1 + x_2 + \cdots + x_m \longmapsto \sigma_{s_m} \,, \quad m = 1, 2, \ldots, n-1 \,. \tag{4}$$

An important problem is to find the expansion of the quantum product $\sigma_u * \sigma_w$ of two Schubert classes in the basis of Schubert classes, where "$*$" denotes the multiplication in the quantum cohomology ring.

This problem was solved, or at least reduced to combinatorics, in [FGP]. In that paper we gave a quantum analogue of the Bernstein-Gelfand-Gelfand theorem and the corresponding deformation of Schubert polynomials of Lascoux and Schützenberger. We also proved there a quantum Monk's formula, which generalizes the classical Monk's result [Mo].

Let us denote $q_{ij} = q_i q_{i+1} \cdots q_{j-1}$, for $i < j$.

Theorem 1.1 (Quantum Monk's formula) [FGP, Theorem 1.3] *For $w \in S_n$ and $1 \le m < n$, the quantum product of Schubert classes σ_{s_m} and σ_w is given by*

$$\sigma_{s_m} * \sigma_w = \sum \sigma_{ws_{ab}} + \sum q_{cd}\, \sigma_{ws_{cd}} \,, \tag{5}$$

where the first sum is over all transpositions s_{ab} such that $a \le m < b$ and $\ell(ws_{ab}) = \ell(w) + 1$, and the second sum is over all transpositions s_{cd} such that $c \le m < d$ and $\ell(ws_{cd}) = \ell(w) - \ell(s_{cd}) = \ell(w) - 2(d-c) + 1$.

The formula (5) unambiguously determines the multiplicative structure of the quantum cohomology ring $\mathrm{QH}^*(Fl_n, \mathbb{Z})$ with respect to the basis of Schubert classes, since this ring is generated by the 2-dimensional classes σ_{s_r}.

As an example, we deduce a rule for the quantum product of any class σ_w with the class $\sigma_{c(k,m)}$, where $c(k, m) = s_{m-k+1} s_{m-k+2} \cdots s_m$ (Corollary 4.3). The main result of our paper is even more general statement (Theorem 3.1) that we call "quantum Pieri's formula." It is formulated in the language of the construction for the cohomology of Fl_n highlighted by Fomin and Kirillov. We were able to extend some of their results and provide a proof to the following conjectures from [FK]: Conjecture 11.1, Conjecture 13.4, and Conjecture 15.1.

Acknowledgments. I am grateful to Sergey Fomin for introducing me to his paper with Anatol Kirillov [FK].

2 Definitions

Let us recall the definition of the quadratic algebra $\mathcal{E}_n^p$ given by Fomin and Kirillov [FK, Section 15]. (Their notation is slightly different from ours.) The algebra $\mathcal{E}_n^p$ is generated over $\mathbb{Z}$ by the elements τ_{ij} and p_{ij}, $i, j \in \{1, 2, \ldots, n\}$, subject to the following relations:

$$\tau_{ij} = -\tau_{ji}\,, \quad \tau_{ii} = 0\,, \tag{6}$$

$$\tau_{ij}^2 = p_{ij}\,, \tag{7}$$

$$\tau_{ij}\tau_{jk} + \tau_{jk}\tau_{ki} + \tau_{ki}\tau_{ij} = 0\,, \tag{8}$$

$$[p_{ij}, p_{kl}] = [p_{ij}, \tau_{kl}] = 0\,, \quad \text{for any } i, j, k, \text{ and } l\,, \tag{9}$$

$$[\tau_{ij}, \tau_{kl}] = 0\,, \quad \text{for any distinct } i, j, k, \text{ and } l\,. \tag{10}$$

Here $[a, b] = ab - ba$ is the usual commutator. Remark that the generator τ_{ij} was denoted by $[ij]$ in [FK]. It follows from (6) and (7) that $p_{ij} = p_{ji}$ and $p_{ii} = 0$.

The commuting elements p_{ij} can be viewed as formal parameters. The quotient $\mathcal{E}_n$ of the algebra $\mathcal{E}_n^p$ modulo the ideal generated by the p_{ij} was the main object of study in [FK]. Also an algebra $\mathcal{E}_n^q$ was introduced in that paper. It can be defined as the quotient of $\mathcal{E}_n^p$ by the ideal generated by the p_{ij} with $|i-j| \geq 2$. The image of $p_{i\,i+1}$ in $\mathcal{E}_n^q$ is denoted q_i.

Following [FK, Section 5], define the *"Dunkl elements"* θ_i, $i = 1, \ldots, n$, in the algebra $\mathcal{E}_n^p$ by

$$\theta_i = \sum_{j=1}^{n} \tau_{ij}. \tag{11}$$

The following important property of these elements is not hard to deduce from the relations (6)–(10).

Lemma 2.1 [FK, Corollary 5.2 and Section 15] *The elements $\theta_1, \theta_2, \ldots, \theta_n$ commute pairwise.*

Let $x_1, x_2, \ldots, x_n$ be a set of commuting variables, and let p be a shorthand for the collection of p_{ij}'s. For a subset $I = \{i_1, \ldots, i_m\}$ in $\{1, 2 \ldots, n\}$, we denote by x_I the collection of variables $x_{i_1}, \ldots, x_{i_m}$. Define the *quantum elementary symmetric polynomial* (cf. [FGP, Section 3.2] or [FK, Section 15])

$$E_k(x_I; p) = E_k(x_{i_1}, x_{i_2}, \ldots, x_{i_m}; p) \tag{12}$$

by the following recursive formulas:

$$E_0(x_{i_1}, x_{i_2}, \ldots, x_{i_m}; p) = 1\,, \tag{13}$$

$$\begin{aligned} E_k(x_{i_1}, x_{i_2}, \ldots, x_{i_m}; p) &= E_k(x_{i_1}, x_{i_2}, \ldots, x_{i_{m-1}}; p) \\ &+ \; E_{k-1}(x_{i_1}, x_{i_2}, \ldots, x_{i_{m-1}}; p)\, x_{i_m} \\ &+ \; \sum_{r=1}^{m-1} E_{k-2}(x_{i_1}, \ldots, \widehat{x_{i_r}}, \ldots, x_{i_{m-1}}; p)\, p_{i_r\, i_m}\,, \end{aligned} \tag{14}$$

where the notation $\widehat{x_{i_r}}$ means that the corresponding term is omitted.

The polynomial $E_k(x_I;p)$ is symmetric in the sense that it is invariant under the simultaneous action of S_m on the variables x_{i_a} and the $p_{i_a\, i_b}$. One can directly verify from (13) and (14) that

$$E_1(x_{i_1}, x_{i_2}, \ldots, x_{i_m}; p) = x_{i_1} + x_{i_2} + \cdots + x_{i_m}\,,$$

$$E_2(x_{i_1}, x_{i_2}, \ldots, x_{i_m}; p) = \sum_{1\le a<b\le m} (x_{i_a}x_{i_b} + p_{i_a\, i_b})\,.$$

The polynomials $E_k(x_I;p)$ have the following elementary monomer-dimer interpretation (cf. [FGP, Section 3.2]). A *partial matching* on the vertex set I is a unordered collection of "dimers" $\{a_1,b_1\},\{a_2,b_2\},\ldots$ and "monomers" $\{c_1\},\{c_2\},\ldots$ such that all a_i,b_j,c_k are distinct elements in I. The *weight* of a matching is the product $p_{a_1\,b_1}p_{a_2\,b_2}\cdots x_{c_1}x_{c_2}\cdots$. Then $E_k(x_I;p)$ is the sum of weights of all matchings which cover exactly k vertices of I.

For example, we have

$$\begin{aligned}E_3(x_1,x_2,x_3,x_4;p) &= x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4\\ &\quad + p_{12}\,(x_3+x_4) + p_{13}\,(x_2+x_4) + p_{14}\,(x_2+x_3)\\ &\quad + p_{23}\,(x_1+x_4) + p_{24}\,(x_1+x_3) + p_{34}\,(x_1+x_2)\,.\end{aligned}$$

Specializing $p_{ij}=0$, one obtains $E_k(x_I;0)=e_k(x_I)$, the usual elementary symmetric polynomial. Assume that $p_{i\,i+1}=q_i$, $i=1,2,\ldots,n-1$, and $p_{ij}=0$, for $|i-j|\ge 2$. Then the polynomial $E_k(x_1,\ldots,x_n;q)$ is the quantum elementary polynomial E_k, which is a coefficient of the characteristic polynomial of the matrix (2). Here and below the letter q stands for the collection of $q_1,q_2,\ldots,q_{n-1}$.

3 Main result

For a subset $I=\{i_1,\ldots,i_m\}$ in $\{1,2,\ldots,n\}$, let θ_I denote the collection of the elements $\theta_{i_1},\ldots,\theta_{i_m}$, and let $E_k(\theta_I;p)=E_k(\theta_{i_1},\ldots,\theta_{i_m};p)$ denote the result of substituting the Dunkl elements (11) in place of the corresponding x_i in (12). This substitution is well defined, due to Lemma 2.1. Our main result can be stated as follows:

Theorem 3.1 (Quantum Pieri's formula) *Let I be a subset in $\{1,2,\ldots,n\}$, and let $J=\{1,2,\ldots,n\}\setminus I$. Then, for $k\ge 1$, we have in the algebra $\mathcal{E}_n^p$:*

$$E_k(\theta_I;p) = \sum \tau_{a_1\,b_1}\tau_{a_2\,b_2}\cdots\tau_{a_k b_k}\,, \tag{15}$$

where the sum is over all sequences $a_1,\ldots,a_k,b_1,\ldots,b_k$ such that (i) $a_j\in I$, $b_j\in J$, *for $j=1,\ldots,k$;* (ii) *the $a_1,\ldots,a_k$ are distinct;* (iii) $b_1\le\cdots\le b_k$.

Note that all terms in (15) involving the p_{ij} cancel each other.

The proof of Theorem 3.1 will be given in Section 5. In the rest of this section we summarize several corollaries of Theorem 3.1.

First of all, let us note that specializing $p_{ij} = 0$ in Theorem 3.1 results in Conjecture 11.1 from [FK]. Conjecture 15.1 is also a consequence of our result.

Corollary 3.2 [FK, Conjecture 15.1] *For $k = 1, 2, \ldots, n$, the following relation in the algebra $\mathcal{E}_n^p$ holds*

$$E_k(\theta_1, \theta_2, \ldots, \theta_n; p) = 0\,.$$

Proof. In this case, the sum in (15) is over the empty set. □

Define a $\mathbb{Z}[p]$-linear homomorphism π by

$$\pi : \mathbb{Z}[x_1, x_2, \ldots, x_n; p] \longrightarrow \mathcal{E}_n^p$$
$$\pi : x_i \longmapsto \theta_i\,.$$

Corollary 3.3 *The kernel of π is generated over $\mathbb{Z}[p]$ by*

$$E_k(x_1, x_2, \ldots, x_n; p)\,, \quad k = 1, 2, \ldots, n\,. \tag{16}$$

Proof. All elements (16) map to zero, due to Corollary 3.2. The statement now follows from a dimension argument (cf. [FK, Section 7]). □

In particular, we can define a homomorphism $\bar{\pi}$ by

$$\bar{\pi} : \mathbb{Z}[x_1, \ldots, x_n] \longrightarrow \mathcal{E}_n\,,$$
$$\bar{\pi} : x_i \longmapsto \bar{\theta}_i\,,$$

where $\bar{\theta}_i$ is the image in $\mathcal{E}_n$ of the element θ_i.

Corollary 3.4 [FK, Theorem 7.1] *The kernel of $\bar{\pi}$ is generated by the elementary symmetric polynomials*

$$e_k(x_1, x_2, \ldots, x_n)\,, \quad k = 1, 2, \ldots, n\,.$$

Thus the subalgebra in $\mathcal{E}_n$ generated by the $\bar{\theta}_i$ is canonically isomorphic to the cohomology of Fl_n, which is isomorphic to the quotient (1).

Likewise, let $\hat{\theta}_i$ be the image in $\mathcal{E}_n^q$ of the element θ_i, and let $\hat{\pi}$ be the $\mathbb{Z}[q]$-linear homomorphism defined by

$$\hat{\pi} : \mathbb{Z}[x_1, \ldots, x_n; q] \longrightarrow \mathcal{E}_n^q\,,$$
$$\hat{\pi} : x_i \longmapsto \hat{\theta}_i\,.$$

Corollary 3.5 [FK, Conjecture 13.4] *The kernel of the homomorphism $\hat{\pi}$ is generated over $\mathbb{Z}[q]$ by*

$$E_k(x_1, x_2, \ldots, x_n; q)\,, \quad k = 1, 2, \ldots, n\,.$$

Thus the subalgebra in $\mathcal{E}_n^q$ generated over $\mathbb{Z}[q]$ by the $\hat{\theta}_i$ is canonically isomorphic to the quantum cohomology of Fl_n, the latter being isomorphic to the quotient (3).

4 Action on the quantum cohomology

Recall that s_{ij} is the transposition of i and j in S_n, $s_i = s_{i\,i+1}$ is a Coxeter generator, and $q_{ij} = q_i q_{i+1} \cdots q_{j-1}$, for $i < j$.

Let us define the $\mathbb{Z}[q]$-linear operators t_{ij}, $1 \le i < j \le n$, acting on the quantum cohomology ring $\mathrm{QH}^*(Fl_n, \mathbb{Z})$ by

$$t_{ij}(\sigma_w) = \begin{cases} \sigma_{ws_{ij}} & \text{if } \ell(ws_{ij}) = \ell(w) + 1\,, \\ q_{ij}\,\sigma_{ws_{ij}} & \text{if } \ell(ws_{ij}) = \ell(w) - 2(j-i) + 1\,, \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

By convention, $t_{ij} = -t_{ji}$, for $i > j$, and $t_{ii} = 0$.

Quantum Monk's formula (Theorem 1.1) can be stated as saying that the quantum product of σ_{s_m} and σ_w is equal to

$$\sigma_{s_m} * \sigma_w = \sum_{a \le m < b} t_{ab}(\sigma_w)\,.$$

The relation between the algebra $\mathcal{E}_n^q$ and quantum cohomology of Fl_n is justified by the following lemma, which is proved by a direct verification.

Lemma 4.1 [FK, Proposition 12.3] *The operators t_{ij} given by* (17) *satisfy the relations* (6)–(10) *with τ_{ij} replaced by t_{ij}, $p_{i\,i+1} = q_i$, and $p_{ij} = 0$, for $|i-j| \ge 2$,*

Thus the algebra $\mathcal{E}_n^q$ acts on $\mathrm{QH}^*(Fl_n, \mathbb{Z})$ by $\mathbb{Z}[q]$-linear transformations

$$\tau_{ij} : \sigma_w \longmapsto t_{ij}(\sigma_w)\,.$$

Monk's formula is also equivalent to the claim that the Dunkl element $\hat{\theta}_i$ acts on the quantum cohomology of Fl_n as the operator of multiplication by x_i, the latter is defined via the isomorphism (4).

Let us denote $c(k,m) = s_{m-k+1} s_{m-k+2} \cdots s_m$ and $r(k,m) = s_{m+k-1} s_{m+k-1} \cdots s_m$. These are two cyclic permutations such that $c(k,m) = (m-k+1, m-k+2, \ldots, m+1)$ and $r(k,m) = (m+k, m+k-1, \ldots, m)$.

The following statement was geometrically proved in [C1] (cf. also [FGP]). For the reader's convenience and for consistency we show how to deduce it directly from Monk's formula.

Lemma 4.2 *The coset of the polynomial $E_k(x_1, \ldots, x_m; q)$ in the quotient ring* (3) *corresponds to the Schubert class $\sigma_{c(k,m)}$ under the isomorphism* (4). *Analogously, the coset of the polynomial $E_k(x_{m+1}, x_{m+2}, \ldots, x_n)$ corresponds to the class $\sigma_{r(k,m)}$.*

Proof. By (4) and (14), it is enough to check that

$$\sigma_{c(k,m)} = \sigma_{c(k,m-1)} + (\sigma_{s_m} - \sigma_{s_{m-1}}) * \sigma_{c(k-1,m-1)} + q_{m-1}\sigma_{c(k-2,m-2)}.$$

This identity immediately follows from Monk's formula:

$$(\sigma_{s_m} - \sigma_{s_{m-1}}) * \sigma_{c(k-1,m-1)} = (\sum_{b>m} t_{mb} - \sum_{a<m} t_{am})(\sigma_{c(k-1,m-1)})\,.$$

The claim about $\sigma_{r(k,m)}$ can be proved using a symmetric argument. $\square$

It is clear now that Theorem 3.1 implies the following statement. This statement, though in a different form, was proved in [C2].

Corollary 4.3 (Quantum Pieri's formulas: QH^* version) *For $w \in S_n$ and $0 \le k \le m < n$, the product of Schubert classes $\sigma_{c(k,m)}$ and σ_w in the quantum cohomology ring $\mathrm{QH}^*(Fl_n, \mathbb{Z})$ is given by the formula*

$$\sigma_{c(k,m)} * \sigma_w = \sum t_{a_1\,b_1} t_{a_2\,b_2} \cdots t_{a_k\,b_m}(\sigma_w), \tag{18}$$

where the sum is over $a_1, \ldots, a_k, b_1, \ldots, b_k$ such that (i) $1 \le a_j \le m < b_j < n$ *for* $j = 1, \ldots, k$; (ii) *the $a_1, \ldots, a_k$ are distinct;* (iii) $b_1 \le \cdots \le b_k$.

Likewise, the quantum product of Schubert classes $\sigma_{r(k,m)}$ and σ_w is given by the formula

$$\sigma_{r(k,m)} * \sigma_w = \sum t_{c_1\,d_1} t_{c_2\,d_2} \cdots t_{c_k\,d_k}(\sigma_w), \tag{19}$$

where the sum is over $c_1, \ldots, c_k, b_1, \ldots, d_k$ such that (i) $1 \le c_j \le m < d_j < n$ *for* $j = 1, \ldots, k$; (ii) $c_1 \le \cdots \le c_k$; (iii) *the $d_1, \ldots, d_k$ are distinct.*

Note that Corollary 4.3 does not imply Theorem 3.1 (or even its weaker form for $\mathcal{E}_n^q$), since the representation $\tau_{ij} \mapsto t_{ij}$ of $\mathcal{E}_n^q$ in the quantum cohomology is not exact.

5 Proof of Theorem 3.1

For a subset I in $\{1, 2, \ldots, n\}$, let $\widetilde{E}_k(I)$ denote the expression in the right-hand side of (15). By convention, $\widetilde{E}_0(I) = 1$. For $k = 1$, Theorem 3.1 says that

$$\widetilde{E}_1(I) = \sum_{i \in I} \sum_{j \notin I} \tau_{ij} = \sum_{i \in I} \sum_{j=1}^{n} \tau_{ij} = E_1(\theta_I; p)\,,$$

which is obvious by (6).

It suffices to verify that the $\widetilde{E}_k(I)$ satisfy the defining relation (14). Then the claim $E_k(\theta_I; p) = \widetilde{E}_k(I)$ will follow by induction on k. Specifically, we have to demonstrate that

$$\widetilde{E}_k(I \cup \{j\}) = \widetilde{E}_k(I) + \widetilde{E}_{k-1}(I)\,\theta_j + \sum_{i \in I} \widetilde{E}_{k-2}(I \setminus \{i\})\,p_{ij}\,, \tag{20}$$

where $I \subset \{1, 2, \ldots, n\}$ and $j \notin I$. To do this we need some extra notation. For a subset $L = \{l_1, l_2, \ldots, l_m\}$ and $r \notin L$, denote

$$\langle\!\langle L \mid r \rangle\!\rangle = \sum \tau_{u_1\,r} \tau_{u_2\,r} \cdots \tau_{u_m\,r}\,,$$

where the sum is over all permutations $u_1, u_2, \ldots, u_m$ of $l_1, l_2, \ldots, l_m$.

For I and j as in (20), let $J = \{1, 2, \dots, n\} \setminus I = \{j_1, j_2, \dots, j_d\}$ with $j_1 = j$. Then the first term in the right-hand side of (20) can be written in the form

$$\widetilde{E}_k(I) = \sum_{I_1 \dots I_d \subset_k I} \langle\!\langle I_1 \mid j_1 \rangle\!\rangle \, \langle\!\langle I_2 \mid j_2 \rangle\!\rangle \cdots \langle\!\langle I_d \mid j_d \rangle\!\rangle \,, \tag{21}$$

where the notation $I_1 \dots I_d \subset_k I$ means that the sum is over all pairwise disjoint (possibly empty) subsets $I_1, I_2, \dots, I_d$ of I such that $\sum_s |I_s| = k$. Let

$$\widetilde{E}_k(I) = A_1 + A_2 \,, \tag{22}$$

where A_1 is the sum of terms in (21) with $I_1 = \emptyset$ and A_2 is the sum of terms with $I_1 \neq \emptyset$. Likewise, we can split the left-hand side of (20) into two parts:

$$\begin{aligned} \widetilde{E}_k(I \cup \{j\}) &= \sum_{I'_2 \cdots I'_d \subset_k I \cup \{j\}} \langle\!\langle I'_2 \mid j_2 \rangle\!\rangle \, \langle\!\langle I'_3 \mid j_3 \rangle\!\rangle \cdots \langle\!\langle I'_d \mid j_d \rangle\!\rangle \\ &= B_1 + B_2 \,, \end{aligned} \tag{23}$$

where B_1 is the sum of the terms such that $j \notin I'_2 \cup \dots \cup I'_d$, and B_2 is the sum of terms with $j \in I'_2 \cup \dots \cup I'_d$. We also split the second term in the right-hand side of (20) into 3 summands:

$$\begin{aligned} \widetilde{E}_{k-1}(I) \, \theta_j &= \sum_{I''_1 \dots I''_d \subset_{k-1} I} \langle\!\langle I''_1 \mid j_1 \rangle\!\rangle \cdots \langle\!\langle I''_d \mid j_d \rangle\!\rangle \sum_{s \neq j} \tau_{js} \\ &= C_1 + C_2 + C_3 \,, \end{aligned} \tag{24}$$

where C_1 is the sum of terms with $s \in I \setminus (I''_1 \cup I''_2 \cup \dots \cup I''_d)$; C_2 is the sum of terms with $s \in I''_2 \cup I''_3 \cup \dots \cup I''_d \cup J$; and C_3 is the sum of terms with $s \in I''_1$.

It is immediate from the definitions that $A_1 = B_1$. It is also not hard to verify that $A_2 + C_1 = 0$, since for $I_1 \neq \emptyset$

$$\langle\!\langle I_1 \mid j_1 \rangle\!\rangle = \sum_{i \in I_1} \langle\!\langle I_1 \setminus \{i\} \mid j_1 \rangle\!\rangle \, \tau_{i\, j_1} \,.$$

To prove the identity (20), it thus suffices to demonstrate that

$$B_2 = C_2 \,, \tag{25}$$

$$C_3 + \sum_{i \in I} \widetilde{E}_{k-2}(I \setminus \{i\}) \, p_{ij} = 0 \,. \tag{26}$$

The following lemma implies the formula (25).

Lemma 5.1 *For any subset K in $\{1, 2, \dots, n\}$ and $j, l \notin K$, we have*

$$\langle\!\langle K \cup \{j\} \mid l \rangle\!\rangle = \sum_{L \subset K} \langle\!\langle L \mid l \rangle\!\rangle \, \langle\!\langle K \setminus L \mid j \rangle\!\rangle \sum_{s \in L \cup \{l\}} \tau_{js} \,. \tag{27}$$

Indeed, let $T = \langle\!\langle I'_2 \mid j_2 \rangle\!\rangle \cdots \langle\!\langle I'_d \mid j_d \rangle\!\rangle$ be a term of B_2. Then $j \in J'_r$ for some r. By Lemma 5.1, T is equal the sum of all terms $\langle\!\langle I''_1 \mid j_1 \rangle\!\rangle \cdots \langle\!\langle I''_d \mid j_d \rangle\!\rangle\, \tau_{js}$ in C_2 with fixed $I''_u = I'_u$ for all $u \neq r$ such that $s \in I''_r \cup \{j_r\}$ and the subsets $I''_1 \cup I''_r = I'_r \setminus \{j\}$. Thus $B_2 = C_2$.

Proof of Lemma 5.1. Induction on $|K|$. For $K = \emptyset$, the both sides of (27) are equal to τ_{jl}. For $|K| \geq 1$, the right-hand side of (27) is equal

$$\begin{aligned}
&\sum_{L \subset K} \langle\!\langle L \mid l \rangle\!\rangle\, \langle\!\langle K \setminus L \mid j \rangle\!\rangle \sum_{s \in L \cup \{l\}} \tau_{js} \\
&= \sum_{L \subsetneq K} \left(\sum_{i \in K \setminus L} \langle\!\langle L \mid l \rangle\!\rangle\, \tau_{ij}\, \langle\!\langle K \setminus L \setminus \{i\} \mid j \rangle\!\rangle \sum_{s \in L \cup \{l\}} \tau_{js} \right) + \langle\!\langle K \mid l \rangle\!\rangle \sum_{s \in K \cup \{l\}} \tau_{js} \\
&= \sum_{i \in K} \tau_{ij}\, \langle\!\langle (K \setminus \{i\}) \cup \{j\} \mid l \rangle\!\rangle + \langle\!\langle K \mid l \rangle\!\rangle \sum_{s \in K \cup \{l\}} \tau_{js} \\
&= \langle\!\langle K \cup \{j\} \mid l \rangle\!\rangle\,.
\end{aligned}$$

The second equality is valid by induction hypothesis; the remaining equalities follow from (8) and (10). □

Using a similar argument to the one after Lemma 5.1, one can derive the formula (26) from the following lemma:

Lemma 5.2 *For any subset K in $\{1, 2, \ldots, n\}$ and $j \notin K$, we have*

$$\sum_{s \in K} \left(\langle\!\langle K \mid j \rangle\!\rangle\, \tau_{js} + \sum_{L \subset K \setminus \{s\}} \langle\!\langle L \mid s \rangle\!\rangle\, \langle\!\langle K \setminus L \setminus \{s\} \mid j \rangle\!\rangle\, p_{js} \right) = 0\,.$$

This statement, in turn, is obtained from the following "quantum analogue" of Lemma 7.2 from [FK]. Its proof is a straightforward extension.

Lemma 5.3 *For $i, u_1, u_2, \ldots, u_m \in \{1, \ldots, n\}$, we have in the algebra $\mathcal{E}^p_n$*

$$\begin{aligned}
&\sum_{r=1}^{m} \tau_{i\,u_r} \tau_{i\,u_{r+1}} \cdots \tau_{i\,u_m} \tau_{i\,u_1} \tau_{i\,u_2} \cdots \tau_{i\,u_r} \\
&= \sum_{r=1}^{m} p_{i\,u_r}\, \tau_{u_r\,u_{r+1}} \tau_{u_r\,u_{r+2}} \cdots \tau_{u_r\,u_m} \tau_{u_r\,u_1} \tau_{u_r\,u_2} \cdots \tau_{u_r\,u_{r-1}}\,,
\end{aligned} \tag{28}$$

where, by convention, the index u_{m+1} is identified with u_1.

Proof. Induction on m. The base of induction, for $m = 1$, is easily established by (7): $\tau_{i\,u_1} \tau_{i\,u_1} = p_{i\,u_1}$. Assume that $m > 1$. Applying (8) and (10) to the left-hand side of (28), we obtain:

$$\begin{aligned}
&\sum_{r=1}^{m} \tau_{i\,u_r}\tau_{i\,u_{r+1}}\cdots\tau_{i\,u_{m-1}}\left(\tau_{i\,u_m}\tau_{i\,u_1}\right)\tau_{i\,i_2}\cdots\tau_{i\,u_r}\\
=\;& \sum_{r=1}^{m} \tau_{i\,u_r}\tau_{i\,u_{r+1}}\cdots\tau_{i\,u_{m-1}}\left(\tau_{i\,u_1}\tau_{u_1\,u_m}+\tau_{u_m\,u_1}\tau_{i\,u_m}\right)\tau_{i\,u_2}\cdots\tau_{i\,u_r}\\
=\;& \left(\sum_{r=1}^{m-1} \tau_{i\,u_r}\tau_{i\,u_{r+1}}\cdots\tau_{i\,u_{m-1}}\tau_{i\,u_1}\tau_{i\,u_2}\cdots\tau_{i\,u_r}\right)\tau_{u_1\,u_m}\\
&+\;\tau_{u_m\,u_1}\left(\sum_{r=2}^{m} \tau_{i\,u_r}\tau_{i\,u_{r+1}}\cdots\tau_{i\,u_m}\tau_{i\,u_2}\tau_{i\,i_3}\cdots\tau_{i\,u_r}\right).
\end{aligned}$$

By induction hypothesis, this expression is equal to

$$\begin{aligned}
&\left(\sum_{r=1}^{m-1} p_{i\,u_r}\,\tau_{u_r\,u_{r+1}}\tau_{u_r\,u_{r+2}}\cdots\tau_{u_r\,u_{m-1}}\tau_{u_r\,u_1}\tau_{u_r\,u_2}\cdots\tau_{u_r\,u_{r-1}}\right)\tau_{u_1\,u_m}\\
&+\;\tau_{u_m\,u_1}\left(\sum_{r=2}^{m} p_{i\,u_r}\,\tau_{u_r\,u_{r+1}}\tau_{u_r\,u_{r+2}}\cdots\tau_{u_r\,u_m}\tau_{u_r\,u_2}\tau_{u_r\,u_3}\cdots\tau_{u_r\,u_{r-1}}\right)\\
=\;& p_{i\,u_1}\,\tau_{u_1\,u_2}\tau_{u_1\,u_3}\cdots\tau_{u_1\,u_m}+p_{i\,u_m}\,\tau_{u_m\,u_1}\tau_{u_m\,u_2}\cdots\tau_{u_m\,u_{m-1}}\\
&+\;\sum_{r=2}^{m-1} p_{i\,u_r}\,\tau_{u_r\,u_{r+1}}\cdots\tau_{u_r\,u_{m-1}}\left(\tau_{u_r\,u_1}\tau_{u_1\,u_m}+\tau_{u_m\,u_1}\tau_{u_r\,u_m}\right)\tau_{u_r\,u_2}\cdots\tau_{u_r\,u_{r-1}}.
\end{aligned}$$

The latter expression coincides with the right-hand side of (28). □

This completes the proof of Theorem 3.1.

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