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# TREES, PARKING FUNCTIONS, SYZYGIES, AND DEFORMATIONS OF MONOMIAL IDEALS

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ABSTRACT. For a graph $G$, we construct two algebras whose dimensions are both equal to the number of spanning trees of $G$. One of these algebras is the quotient of the polynomial ring modulo certain monomial ideal, while the other is the quotient of the polynomial ring modulo certain powers of linear forms. We describe the set of monomials that forms a linear basis in each of these two algebras. The basis elements correspond to $G$-parking functions that naturally came up in the abelian sandpile model. These ideals are instances of the general class of monotone monomial ideals and their deformations. We show that the Hilbert series of a monotone monomial ideal is always bounded by the Hilbert series of its deformation. Then we define an even more general class of monomial ideals associated with posets and construct free resolutions for these ideals. In some cases these resolutions coincide with Scarf resolutions. We prove several formulas for Hilbert series of monotone monomial ideals and investigate when they are equal to Hilbert series of deformations. In the appendix we discuss the abelian sandpile model.

## 1. Introduction

The famous formula of Cayley says that the number of trees on $n+1$ labelled vertices equals $(n+1)^{n-1}$. Remarkably, this number has several other interesting combinatorial interpretations. For example, it is equal to the number of parking functions of size $n$.

In this paper we present two algebras $\mathcal{A}_n$ and $\mathcal{B}_n$ of dimension $(n+1)^{n-1}$. The algebra $\mathcal{A}_n$ is a quotient of the polynomial ring modulo a monomial ideal, and the algebra $\mathcal{B}_n$ is a quotient of the polynomial ring modulo some powers of linear forms. It is immediate that the set of monomials $x^b$, where $b$ is a parking function, is the standard monomial basis of the algebra $\mathcal{A}_n$. On the other hand, the same set of monomials forms a basis of the algebra $\mathcal{B}_n$, which is a nontrivial result.

More generally, for any graph $G$, we define two algebras $\mathcal{A}_G$ and $\mathcal{B}_G$ and describe their monomial bases. The basis elements correspond to $G$-parking functions. These functions extend the usual parking functions and are related to the abelian sandpile model; their number equals the number of spanning trees of the graph $G$. This implies that $\dim \mathcal{A}_G = \dim \mathcal{B}_G$ is also the number of spanning trees of $G$.

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All these pairs of algebras are instances of the general class of algebras given by monotone monomial ideals and their deformations. For such an algebra $\mathcal{A}$ and its deformation $\mathcal{B}$, we show that $\dim \mathcal{A} \geq \dim \mathcal{B}$ and the Hilbert series of $\mathcal{B}$ is termwise bounded by the Hilbert series of $\mathcal{A}$. There is a natural correspondence between polynomial generators of the ideal for $\mathcal{B}$ and monomial generators of the ideal for $\mathcal{A}$. However, these monomials are not the initial terms of the polynomial generators for any term order, because they are usually located at the center of the Newton polytope of the corresponding polynomial generators. The standard Gröbner bases technique cannot be applied to this class of algebras.

We also investigate the class of order monomial ideals that extends monotone monomial ideals. These ideals are associated with posets whose elements are marked by monomials. We construct a free resolution for such an ideal as the cellular resolution corresponding to the order complex of the poset. This resolution is minimal if the ideal satisfies some generosity condition. In this case, the numbers of increasing $k$-chains in the poset are exactly the Betti numbers of the ideal. This resolution often coincides with the Scarf resolution.

We discuss some results of our previous works on the algebra generated by the curvature forms on the generalized flag manifold. This algebra extends the cohomology ring of the generalized flag manifold. For type $A_{n-1}$, the dimension of this algebra equals the number of forests on $n+1$ vertices. The algebras generated by the curvature forms are analogous to the algebras that we study in the present paper. This attempt to lift Schubert calculus on the level of differential forms was our original motivation.

The general outline of the paper follows. In Section 2 we define $G$-parking functions for a digraph $G$. We formulate Theorem 2.1 that says that the number of such functions equals the number of oriented spanning trees of $G$. Then we construct the algebra $\mathcal{A}_G$ as the quotient of the polynomial ring modulo a certain monomial ideal. Elements of the standard monomial basis of $\mathcal{A}_G$ correspond to $G$-parking functions. In Section 3 we construct the algebra $\mathcal{B}_G$ as the quotient of the polynomial ring modulo the ideal generated by powers of certain linear forms. Then we formulate Theorem 3.1, which implies that the algebras $\mathcal{A}_G$ and $\mathcal{B}_G$ have the same Hilbert series. In Section 4 we give two examples of these results. For the complete graph $G = K_{n+1}$ we recover the usual parking functions and the algebras $\mathcal{A}_n$ and $\mathcal{B}_n$ of dimension $(n+1)^{n-1}$. For a slightly more general class of graphs we obtain two algebras of dimension $l\,(l+kn)^{n-1}$. Section 5 is devoted to a description of monotone monomial ideals and their deformations. We formulate Theorem 5.2, which implies the inequality for the Hilbert series. In Section 6 we describe a more general class of monomial ideals associated with posets and construct free resolutions for these ideals. Components of the resolution for such an ideal correspond to strictly increasing chains in the poset. In Section 7 we give several examples of minimal free resolutions. In Section 8 we prove general formulas for the Hilbert series and dimension of the algebra given by a monotone monomial ideal. Then we deduce Theorem 2.1. In Section 9 we construct the algebra $\mathcal{C}_G$ and prove Theorem 9.1 that claims that the dimension of this algebra equals the number of spanning trees. Actually, we will see later that $\mathcal{C}_G$ is isomorphic to the algebra $\mathcal{B}_G$. In Section 10 we prove Theorem 5.2. Then we finish the proof of Theorem 3.1, which goes as follows. By Theorem 5.2 and construction of $\mathcal{C}_G$ we know that $\operatorname{Hilb} \mathcal{A}_G \geq \operatorname{Hilb} \mathcal{B}_G \geq \operatorname{Hilb} C_G$ termwise. On the other hand, by

Theorems 2.1 and 9.1, $\dim \mathcal{A}_G = \dim \mathcal{C}_G$ is the number of spanning trees of $G$. Thus the Hilbert series of these three algebras coincide. In Section 11 we discuss some results of our previous works and compare them with results of this paper. We mention a certain algebra, whose dimension equals the number of forests on $n+1$ vertices. This algebra originally appeared in the attempt to lift Schubert calculus of the flag manifold on the level of differential forms. In Section 12 we discuss a special class of monotone monomial ideals and their deformations. We give a minimal free resolution and subtraction-free formula for the Hilbert series of the algebra $\mathcal{A}$ and list several cases when it is equal to the Hilbert series of $\mathcal{B}$. The appendix is devoted to the abelian sandpile model and its links with $G$-parking functions.

## 2. $G$-parking functions

A *parking function* of size $n$ is a sequence $b = (b_1, \ldots, b_n)$ of nonnegative integers such that its increasing rearrangement $c_1 \leq \cdots \leq c_n$ satisfies $c_i < i$. Equivalently, we can formulate this condition as $\#\{i \mid b_i < r\} \geq r$, for $r = 1, \ldots, n$. The parking functions of size $n$ are known to be in bijective correspondence with trees on $n+1$ labelled vertices; see Kreweras [Krew]. Thus, according to Cayley's formula for the number of labelled trees, the total number of parking functions of size $n$ equals $(n+1)^{n-1}$. In this section we extend this statement to a more general class of functions.

A *graph* is given by specifying its set of vertices, set of edges, and a function that associates to each edge an unordered pair of vertices. A *directed graph*, or *digraph*, is given by specifying its set of vertices, set of edges, and a function that associates to each edge an ordered pair of vertices. Thus multiple edges and loops are allowed in graphs and digraphs. A *subgraph* $H$ in a (directed) graph $G$ is a (directed) graph on the same set of vertices whose set of edges is a subset of edges of $G$. We will write $H \subset G$ to denote that $H$ is a subgraph of $G$. For a subgraph $H \subset G$, let $G \setminus H$ denote the *complement subgraph* whose edge set is complementary to that of $H$. We will also write $e \in G$ to show that $e$ is an edge of the graph $G$.

Let $G$ be a digraph on the set of vertices $0, 1, \ldots, n$. The vertex 0 will be the root of $G$. The digraph $G$ is determined by its *adjacency matrix* $A = (a_{ij})_{0 \leq i,j \leq n}$, where $a_{ij}$ is the number of edges from the vertex $i$ to the vertex $j$. We will regard graphs as a special case of digraphs with symmetric adjacency matrix $A$.

An *oriented spanning tree* $T$ of the digraph $G$ is a subgraph $T \subset G$ such that there exists a unique directed path in $T$ from any vertex $i$ to the root 0. The number $N_G$ of such trees is given by the *Matrix-Tree Theorem* (e.g., see [Sta2, Section 5.6]),

$$N_G = \det L_G, \tag{2.1}$$

where $L_G = (l_{ij})_{1 \leq i,j \leq n}$ is the *truncated Laplace matrix*, also known as the *Kirkhoff matrix*, given by

$$l_{ij} = \begin{cases} \displaystyle\sum_{r \in \{0,\ldots,n\} \setminus \{i\}} a_{ir} & \text{for } i = j, \\ -a_{ij} & \text{for } i \neq j. \end{cases} \tag{2.2}$$

If $G$ is a graph, i.e., $A$ is a symmetric matrix, then oriented spanning trees defined above are exactly the usual *spanning trees* of $G$, which are connected subgraphs of $G$ without cycles.

For a subset $I$ in $\{1, \ldots, n\}$ and a vertex $i \in I$, let

$$d_I(i) = \sum_{j \notin I} a_{ij},$$

i.e., $d_I(i)$ is the number of edges from the vertex $i$ to a vertex outside of the subset $I$. Let us say that a sequence $b = (b_1, \ldots, b_n)$ of nonnegative integers is a *$G$-parking function* if, for any nonempty subset $I \subseteq \{1, \ldots, n\}$, there exists $i \in I$ such that $b_i < d_I(i)$.

If $G = K_{n+1}$ is the complete graph on $n+1$ vertices, then $K_{n+1}$-parking functions are the usual parking functions of size $n$ defined in the beginning of this section.

**Theorem 2.1** (cf. [Gab1]). *The number of $G$-parking functions equals the number $N_G = \det L_G$ of oriented spanning trees of the digraph $G$.*

Interestingly, $G$-parking functions are related to the abelian sandpile model introduced by Dhar [Dhar]. In the appendix we will discuss the sandpile model and show that Theorem 2.1 is essentially equivalent to the result of Gabrielov [Gab1, Eq. (21)] on sandpiles. In Section 8 we will prove Theorem 2.1 without using the sandpile model.

We can reformulate the definition of $G$-parking functions in algebraic terms as follows. Throughout this paper we fix a field $\mathbb{K}$. Let $\mathcal{I}_G = \langle m_I \rangle$ be the monomial ideal in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ generated by the monomials

$$m_I = \prod_{i \in I} x_i^{d_I(i)}, \tag{2.3}$$

where $I$ ranges over all nonempty subsets $I \subseteq \{1, \ldots, n\}$. Define the algebra $\mathcal{A}_G$ as the quotient $\mathcal{A}_G = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}_G$.

A nonnegative integer sequence $b = (b_1, \ldots, b_n)$ is a $G$-parking function if and only if the monomial $x^b = x_1^{b_1} \cdots x_n^{b_n}$ is nonvanishing in the algebra $\mathcal{A}_G$.

For a monomial ideal $\mathcal{I}$, the set of all monomials that do not belong to $\mathcal{I}$ is a basis of the quotient of the polynomial ring modulo $\mathcal{I}$, called the *standard monomial basis.* Thus the monomials $x^b$, where $b$ ranges over $G$-parking functions, form the standard monomial basis of the algebra $\mathcal{A}_G$.

**Corollary 2.2.** *The algebra $\mathcal{A}_G$ is finite dimensional as a linear space over $\mathbb{K}$. Its dimension is equal to the number of oriented spanning trees of the digraph $G$:*

$$\dim \mathcal{A}_G = N_G.$$

For an undirected graph, $G$-parking functions and monomials $m_I$ also appeared in a recent paper by Cori, Rossin, and Salvy [CRS].

## 3. Power algebras

Let $G$ be an undirected graph on the set of vertices $0, 1, \ldots, n$. In this case the dimension of the algebra $\mathcal{A}_G$ is equal to the number of usual spanning trees of $G$.

For a nonempty subset $I$ in $\{1, \ldots, n\}$, let $D_I = \sum_{i \in I, j \notin I} a_{ij} = \sum_{i \in I} d_I(i)$ be the total number of edges that join some vertex in $I$ with a vertex outside of $I$. For any nonempty subset $I \subseteq \{1, \ldots, n\}$, let

$$p_I = \left(\sum_{i \in I} x_i\right)^{D_I}. \tag{3.1}$$

Let $\mathcal{J}_G = \langle p_I \rangle$ be the ideal in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ generated by the polynomials $p_I$ for all nonempty subsets $I$. Define the algebra $\mathcal{B}_G$ as the quotient $\mathcal{B}_G = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}_G$.

The algebras $\mathcal{A}_G$ and $\mathcal{B}_G$, as well as all other algebras in this paper, are graded. For a graded algebra $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1 \oplus \mathcal{A}^2 \oplus \cdots$, the *Hilbert series* of $\mathcal{A}$ is the formal power series in $q$ given by

$$\operatorname{Hilb} \mathcal{A} = \sum_{k \geq 0} q^k \ \dim \mathcal{A}^k.$$

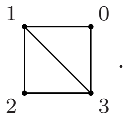
Our first main result is the following statement.

**Theorem 3.1.** *The monomials $x^b$, where $b$ ranges over $G$-parking functions, form a linear basis of the algebra $\mathcal{B}_G$. Thus the Hilbert series of the algebras $\mathcal{A}_G$ and $\mathcal{B}_G$ coincide termwise:* $\operatorname{Hilb} \mathcal{A}_G = \operatorname{Hilb} \mathcal{B}_G$. *In particular, both of these algebras are finite dimensional as linear spaces over $\mathbb{K}$, and*

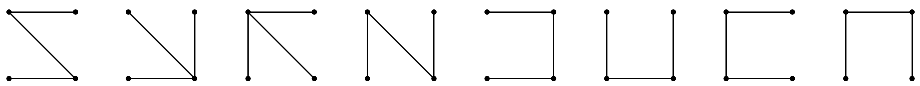
$$\dim \mathcal{A}_G = \dim \mathcal{B}_G = N_G$$

*is the number of spanning trees of the graph $G$.*

**Example 3.2.** Let $n = 3$ and let $G$ be the graph given by

(3.2) $G =$  .

The graph $G$ has 8 spanning trees:



The ideals $\mathcal{I}_G$ and $\mathcal{J}_G$ are given by

$$\mathcal{I}_G = \langle x_1^3, x_2^2, x_3^3, x_1^2 x_2, x_1^2 x_3^2, x_2 x_3^2, x_1 x_2^0 x_3 \rangle,$$

$$\mathcal{J}_G = \langle x_1^3, x_2^2, x_3^3, (x_1 + x_2)^3, (x_1 + x_3)^4, (x_2 + x_3)^3, (x_1 + x_2 + x_3)^2 \rangle.$$

The standard monomial basis of the algebra $\mathcal{A}_G$ is $\{1, x_1, x_2, x_3, x_1^2, x_1 x_2, x_2 x_3, x_3^2\}$. The corresponding $G$-parking functions are

$$(0,0,0), (1,0,0), (0,1,0), (0,0,1), (2,0,0), (1,1,0), (0,1,1), (0,0,2).$$

We have $\dim \mathcal{A}_G = \dim \mathcal{B}_G = 8$ is the number of spanning trees of $G$, and $\operatorname{Hilb} \mathcal{A}_G = \operatorname{Hilb} \mathcal{B}_G = 1 + 3q + 4q^2$.

We thank the referee who suggested that we include the following comment. For an ideal $\mathcal{J}$ and a given term order, the *initial ideal* of $\mathcal{J}$ is the monomial ideal formed by initial terms of elements of $\mathcal{J}$. For the ideal $\mathcal{J}_G$ above, all possible initial ideals (in all terms orders) are

$$\langle x_1^2, x_1 x_2 x_3, x_1 x_3^2, x_2^2, x_2 x_3^2, x_3^3 \rangle, \ \langle x_1^3, x_1^2 x_3, x_1 x_2, x_1 x_3^2, x_2^2, x_2 x_3^2, x_3^3 \rangle,$$
$$\langle x_1^3, x_1^2 x_2, x_1^2 x_3, x_1 x_2 x_3, x_2^2, x_3^2 \rangle, \ \langle x_1^3, x_1^2 x_2, x_1^2 x_3, x_1 x_3^2, x_2^2, x_2 x_3, x_3^3 \rangle.$$

Notice that $\mathcal{I}_G$ *is not* equal to any of these initial ideals for $\mathcal{J}_G$.

We will refine Theorem 3.1 and interpret dimensions of graded components of the algebras $\mathcal{A}_G$ and $\mathcal{B}_G$ in terms of certain statistics on spanning trees. Let us fix a linear ordering of all edges of the graph $G$. For a spanning tree $T$ of $G$, an edge $e \in G \setminus T$ is called *externally active* if there exists a cycle $C$ in the graph $G$

such that $e$ is the minimal edge of $C$ and $(C \setminus \{e\}) \subset T$. The *external activity* of a spanning tree is the number of externally active edges. Let $N_G^k$ denote the number of spanning trees $T \subset G$ of external activity $k$. Even though the notion of external activity depends on a particular choice of ordering of edges, the numbers $N_G^k$ are known to be invariant on the choice of ordering.

Let $\mathcal{A}_G^k$ and $\mathcal{B}_G^k$ be the $k$-th graded components of the algebras $\mathcal{A}_G$ and $\mathcal{B}_G$, correspondingly.

**Theorem 3.3.** *The dimensions of the $k$-th graded components $\mathcal{A}_G^k$ and $\mathcal{B}_G^k$ are equal to*

$$\dim \mathcal{A}_G^k = \dim \mathcal{B}_G^k = N_G^{|G|-n-k},$$

*the number of spanning trees of $G$ of external activity $|G|-n-k$, where $|G|$ denotes the number of edges of $G$.*

## 4. Examples: Tree ideals and their generalizations

4.1. **Two algebras of dimension** $(n+1)^{n-1}$**.** Suppose that $G = K_{n+1}$ is the complete graph on $n+1$ vertices. As we have already mentioned, the $K_{n+1}$-parking functions are the usual parking functions of size $n$ defined in the beginning of Section 2.

Let $\mathcal{I}_n = \langle m_I \rangle$ and $\mathcal{J}_n = \langle p_I \rangle$ be the ideals in the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ generated by the monomials $m_I$ and the polynomials $p_I$, correspondingly, given by

$$m_I = (x_{i_1} \cdots x_{i_r})^{n-r+1},$$

$$p_I = (x_{i_1} + \cdots + x_{i_r})^{r(n-r+1)},$$

where in both cases $I = \{i_1, \dots, i_r\}$ runs over all nonempty subsets of $\{1, \dots, n\}$. Let $\mathcal{A}_n = \mathbb{K}[x_1, \dots, x_n]/\mathcal{I}_n$ and $\mathcal{B}_n = \mathbb{K}[x_1, \dots, x_n]/\mathcal{J}_n$.

**Corollary 4.1.** *The graded algebras $\mathcal{A}_n$ and $\mathcal{B}_n$ have the same Hilbert series. They are finite dimensional, as linear spaces over $\mathbb{K}$. Their dimensions are equal to*

$$\dim \mathcal{A}_n = \dim \mathcal{B}_n = (n+1)^{n-1}.$$

*The images of the monomials $x^b$, where $b$ ranges over parking functions of size $n$, form linear bases in both algebras $\mathcal{A}_n$ and $\mathcal{B}_n$.*

An *inversion* in a tree $T$ on the $n+1$ vertices labelled $0, \dots, n$ is a pair of vertices labelled $i$ and $j$ such that $i > j$ and the vertex $i$ belongs to the shortest path in $T$ that joins the vertex $j$ with the root $0$.

**Corollary 4.2.** *The dimension* $\dim \mathcal{A}_n^k = \dim \mathcal{B}_n^k$ *of the $k$-th graded components of the algebras $\mathcal{A}_n$ and $\mathcal{B}_n$ is equal to*

(A) *the number of parking functions $b$ of size $n$ such that $b_1 + \cdots + b_n = k$;*
(B) *the number of trees on $n+1$ vertices with external activity $\binom{n}{2} - k$;*
(C) *the number of trees on $n+1$ vertices with $\binom{n}{2} - k$ inversions.*

It is well known that the numbers in (A), (B), and (C) are equal; see [Krew]. The *inversion polynomial* is defined as the sum $I_n(q) = \sum_T q^{\#\text{ of inversions in } T}$ over all trees $T$ on $n+1$ labelled vertices. Thus the Hilbert series of the algebras $\mathcal{A}_n$ and $\mathcal{B}_n$ are equal to

$$\mathrm{Hilb}\, \mathcal{A}_n = \mathrm{Hilb}\, \mathcal{B}_n = q^{\binom{n}{2}}\, I_n(q^{-1}).$$

4.2. **Two algebras of dimension $l\,(l+kn)^{n-1}$.** It is possible to extend the previous example as follows. Fix two nonnegative integers $k$ and $l$. Let $G = K_{n+1}^{k,l}$ be the complete graph on the vertices $0, 1, \ldots, n$ with the edges $(i,j)$, $i, j \neq 0$, of multiplicity $k$ and the edges $(0,i)$ of multiplicity $l$. The $K_{n+1}^{k,l}$-parking functions are the nonnegative integer sequences $b = (b_1, \ldots, b_n)$ such that, for $r = 1, \ldots, n$,

$$\#\{i \mid b_i < l + k(r-1)\} \geq r.$$

The definition of these functions can also be formulated as $c_i < l + (i-1)\,k$, where $c_1 \leq \cdots \leq c_n$ is the increasing rearrangement of elements of $b$. Such functions were studied by Pitman and Stanley [PiSt] and then by Yan [Yan]. These authors demonstrated that their number equals $l\,(l+kn)^{n-1}$. One can show, using for example the Matrix-Tree Theorem (2.1), that the number of spanning trees in the graph $K_{n+1}^{k,l}$ equals $l\,(l+kn)^{n-1}$. Thus Theorem 2.1 recovers the above formula for the number of $K_{n+1}^{k,l}$-parking functions.

Let $\mathcal{I}_{n,k,l} = \langle m_I \rangle$ and $\mathcal{J}_{n,k,l} = \langle p_I \rangle$ be the ideals in the ring $\mathbb{K}[x_1, \ldots, x_n]$ generated by the monomials $m_I$ and the polynomials $p_I$, correspondingly, given by

$$m_I = (x_{i_1} \cdots x_{i_r})^{l+k(n-r)},$$

$$p_I = (x_{i_1} + \cdots + x_{i_r})^{r(l+k(n-r))},$$

where in both cases $I = \{i_1, \ldots, i_r\}$ runs over all nonempty subsets of $\{1, \ldots, n\}$. Let $\mathcal{A}_{n,k,l} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}_{n,k,l}$ and $\mathcal{B}_{n,k,l} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}_{n,k,l}$.

**Corollary 4.3.** *The graded algebras $\mathcal{A}_{n,k,l}$ and $\mathcal{B}_{n,k,l}$ have the same Hilbert series. They are finite dimensional, as linear spaces over $\mathbb{K}$. Their dimensions are*

$$\dim \mathcal{A}_{n,k,l} = \dim \mathcal{B}_{n,k,l} = l\,(l+kn)^{n-1}.$$

*The images of the monomials $x^b$, where $b$ ranges over $K_{n+1}^{k,l}$-parking functions, form linear bases in both algebras $\mathcal{A}_{n,k,l}$ and $\mathcal{B}_{n,k,l}$.*

## 5. Monotone monomial ideals and their deformations

A *monotone monomial family* is a collection $\mathcal{M} = \{m_I \mid I \in \Sigma\}$ of monomials in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ labelled by a set $\Sigma$ of nonempty subsets in $\{1, \ldots, n\}$ that satisfies the following three conditions:

(MM1) For $I \in \Sigma$, $m_I$ is a monomial in the variables $x_i$, $i \in I$.
(MM2) For $I, J \in \Sigma$ such that $I \subset J$ and $i \in I$, we have $\deg_{x_i}(m_I) \geq \deg_{x_i}(m_J)$.
(MM3) For $I, J \in \Sigma$, $\mathrm{lcm}(m_I, m_J)$ is divisible by $m_K$ for some $K \supseteq I \cup J$ in $\Sigma$.

The *monotone monomial ideal* $\mathcal{I} = \langle \mathcal{M} \rangle$ associated with a monotone monomial family $\mathcal{M}$ is the ideal in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ generated by the monomials $m_I$ in $\mathcal{M}$.

It follows from (MM1) and (MM2) that condition (MM3) can be replaced by the condition: For $I, J \in \Sigma$ there is $K \supseteq I \cup J$ in $\Sigma$ such that $m_K$ is a monomial in the $x_i$, $i \in I \cup J$. This condition is always satisfied if $I, J \in \Sigma$ implies that $I \cup J \in \Sigma$.

The monomial ideal $\mathcal{I}_G$ constructed in Section 2 for a digraph $G$ is monotone. In this case $\Sigma$ is the set of all nonempty subsets in $\{1, \ldots, n\}$ and $m_I$ is given by (2.3).

Remark that two different monotone monomial families may produce the same monotone monomial ideal. For example, the ideal $\mathcal{I}_G$, for the graph $G$ shown on (3.2), has generator $m_{\{1,3\}} = x_1^2 x_3^2$. This generator is redundant because it is divisible by $m_{\{1,2,3\}} = x_1x_3$. Thus the same ideal corresponds to the monotone monomial family with $\Sigma = \{\{1\},\{2\},\{3\},\{1,2\},\{2,3\},\{1,2,3\}\}$.

Let $I = \{i_1,\dots,i_r\}$. For a monomial $m \in \mathbb{K}[x_{i_1},\dots,x_{i_r}]$, an *$I$-deformation* of $m$ is a homogeneous polynomial $p \in \mathbb{K}[x_{i_1},\dots,x_{i_r}]$ of degree $\deg(p) = \deg(m)$ satisfying the *generosity condition*

$$\mathbb{K}[x_{i_1},\dots,x_{i_r}] = \langle R_m\rangle \oplus (p), \tag{5.1}$$

where $\langle R_m\rangle$ is the linear span of the set $R_m$ of monomials in $\mathbb{K}[x_{i_1},\dots,x_{i_r}]$ which are not divisible by $m$, $(p)$ is the ideal in $\mathbb{K}[x_{i_1},\dots,x_{i_r}]$ generated by $p$, and "$\oplus$" stands for a direct sum of subspaces. Notice that the generosity condition is satisfied for a Zarisky open set of polynomials in $\mathbb{K}[x_{i_1},\dots,x_{i_r}]$ of degree $\deg(m)$. For example, the polynomial $p = a\,x_1 + b\,x_2$ is a $\{1,2\}$-deformation of the monomial $m = x_1$ if and only if $a \neq 0$.

The following lemma describes a class of $I$-deformations of monomials.

**Lemma 5.1.** *Let $I = \{i_1,\dots,i_r\}$, let $m$ be a monomial in $\mathbb{K}[x_{i_1},\dots,x_{i_r}]$, and let $\alpha_1,\dots,\alpha_r \in \mathbb{K}\setminus\{0\}$. Then the polynomial*

$$p = (\alpha_1 x_{i_1} + \cdots + \alpha_r x_{i_r})^{\deg m}$$

*is an $I$-deformation of the monomial $m$.*

*Proof.* Let $m = x_{i_1}^{a_1}\cdots x_{i_r}^{a_r}$. The generosity condition (5.1) is equivalent to the condition that the operator

$$A : \mathbb{K}[x_{i_1},\dots,x_{i_r}] \to \mathbb{K}[x_{i_1},\dots,x_{i_r}],$$
$$A : f \mapsto (\partial/\partial x_{i_1})^{a_1}\cdots(\partial/\partial x_{i_r})^{a_r}(p\cdot f)$$

has zero kernel. Let us change the coordinates to $y_1 = x_{i_1},\dots,\ y_{r-1} = x_{i_{r-1}}$, $y_r = \alpha_1 x_{i_1} + \cdots + \alpha_r x_{i_r}$. The operator $A$ can be written in these coordinates as

$$A(f) = (\tilde\partial_1 + \alpha_1\tilde\partial_r)^{a_1}\cdots(\tilde\partial_{r-1} + \alpha_{r-1}\tilde\partial_r)^{a_{r-1}}(\alpha_r\tilde\partial_r)^{a_r}(y_r^{a_1+\cdots+a_r}\cdot f),$$

where $\tilde\partial_j = \partial/\partial y_j$. Then $A(f) = c\cdot f + g$, where $c$ is a nonzero constant and $\deg_{y_r}(g) < \deg_{y_r}(f)$. Thus, in an appropriate basis, the operator $A$ is given by a triangular matrix with nonzero diagonal elements. This implies that $\operatorname{Ker} A = 0$. $\square$

A *deformation* of a monotone monomial ideal $\mathcal{I} = \langle m_I \mid I \in \Sigma\rangle$ is an ideal $\mathcal{J} = \langle p_I \mid I \in \Sigma\rangle$ generated by polynomials $p_I$ such that $p_I$ is an $I$-deformation of $m_I$ for each $I \in \Sigma$. For example, according to Lemma 5.1, the ideal $\mathcal{J}_G$ given in Section 3 is a deformation of the monotone monomial ideal $\mathcal{I}_G$.

Lemma 5.1 describes a class of deformations $\mathcal{J}$ generated by powers of linear forms. According to results of Emsalem and Iarrobino [EmIa], these ideals are closely related to ideals of fatpoints. Actually, results of [EmIa] imply an alternative simple proof of Lemma 5.1.

**Theorem 5.2.** *Let $\mathcal{I}$ be a monotone monomial ideal, and let $R$ be the standard monomial basis of the algebra $\mathcal{A} = \mathbb{K}[x_1,\dots,x_n]/\mathcal{I}$, i.e., $R$ is the set of monomials that do not belong to $\mathcal{I}$. Let $\mathcal{J}$ be a deformation of the ideal $\mathcal{I}$, and $\mathcal{B} = \mathbb{K}[x_1,\dots,x_n]/\mathcal{J}$.*

*Then the monomials in $R$ linearly span the algebra $\mathcal{B}$.*

Remark that the set of monomials $R$ may or may not be a basis for $\mathcal{B}$.

**Corollary 5.3.** *Let $\mathcal{I}$ be a monotone monomial ideal, $\mathcal{J}$ be a deformation of the ideal $\mathcal{I}$, $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}$, and $\mathcal{B} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}$. Then we have the following termwise inequalities for the Hilbert series:*

$$\operatorname{Hilb}\mathcal{I} \le \operatorname{Hilb}\mathcal{J} \qquad \textit{or, equivalently,} \qquad \operatorname{Hilb}\mathcal{A} \ge \operatorname{Hilb}\mathcal{B}.$$

In some cases the Hilbert series are actually equal to each other. According to Theorem 3.1, $\operatorname{Hilb}\mathcal{A}_G = \operatorname{Hilb}\mathcal{B}_G$, for any graph $G$. However in general the Hilbert series may not be equal to each other. It would be interesting to describe a general class of monotone monomial ideals and their deformations with equal Hilbert series.

There is an obvious correspondence between the generators $m_I$ of a monotone monomial ideal $\mathcal{I}$ and the generators $p_I$ of its deformation $\mathcal{J}$. Notice however that (except for very special cases) the monomial generator $m_I$ does not belong to the boundary of the Newton polytopes of it polynomial deformation $p_I$. Thus the monomial $m_I$ is not the initial term of the polynomial $p_I$ for any term order; cf. Example 3.2. This shows that the above results cannot be tackled by the standard Gröbner bases technique.

Usually, Gröbner bases of deformations $\mathcal{J}$ and corresponding initial ideals have much more complicated structure than ideals $\mathcal{I}$. For example, reduced Gröbner bases (with respect to the lexicographical term order) for ideals $\mathcal{J}_3$ and $\mathcal{J}_4$ from Section 4.1 have 8 and 29 elements, respectively. These numbers exceed $2^n - 1$, which is the number of generators of $\mathcal{I}_n$.

## 6. Syzygies of order monomial ideals

In this section we introduce a class of ideals that extends monotone monomial ideals and construct free resolutions for these ideals. This construction is related to results of [BPS, BaSt, GPW, MSY] on monomial resolutions.

Let $P$ be a finite partially ordered set, or *poset*. Let $\mathcal{M} = \{m_u \mid u \in P\}$ be a collection of monomials in the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ labelled by elements of the poset $P$. Also let $M_u$ denote the set of all monomials divisible by $m_u$. Let us say that $\mathcal{M}$ is an *order monomial family*, and the ideal $\mathcal{I} = \langle \mathcal{M} \rangle$ generated by the monomials $m_u$ is an *order monomial ideal*, if the following condition is satisfied:

(OM) For any pair $u, v \in P$, there exists an upper bound $w \in P$ of $u$ and $v$ such that $M_u \cap M_v \subseteq M_w$, i.e., $m_w$ divides $\operatorname{lcm}(m_u, m_v)$.

Here an upper bound means an element $w$ such that $w \ge u$ and $w \ge v$ in $P$. In particular, this condition implies that the poset $P$ has a unique maximal element.

Every monotone monomial family is an order monomial family labelled by the set $P = \Sigma$ of subsets in $\{1, \ldots, n\}$ partially ordered by inclusion. Indeed, condition (MM3) is equivalent to condition (OM).

Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring. For a nonnegative $n$-vector $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, let $S(-a) = x^a\, S$ denote the free $\mathbb{Z}^n$-graded $S$-submodule in $S$ generated by the monomial $x^a = x_1^{a_1} \cdots x_n^{a_n}$. This submodule is isomorphic to $S$ with the $\mathbb{Z}^n$-grading shifted by the vector $a$. If $a \ge b$ componentwise, then $S(-a)$ is a submodule of $S(-b)$ and we will write $S(-a) \hookrightarrow S(-b)$ to denote the natural multidegree-preserving embedding of $S$-modules.

For an order monomial family $\mathcal{M} = \{m_u \mid u \in P\}$ and a subset $U \subseteq P$ of elements of the poset $P$, let $m_U = \operatorname{lcm}(m_u \mid u \in U)$ be the least common multiple of the monomials $m_u$, $u \in U$. We assume that $m_\emptyset = 1$. Also let $a_U \in \mathbb{Z}^n$ be the exponent vector of the monomial $m_U$.

Let us define the *homological order complex* $C_*(\mathcal{M})$ for an order monomial ideal $\mathcal{I} = \langle \mathcal{M} \rangle$ as the sequence of $\mathbb{Z}^n$-graded $S$-modules

$$\cdots \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 = S \longrightarrow S/\mathcal{I} \to 0 \tag{6.1}$$

whose $k$-th component is

$$C_k = \bigoplus_{u_1 \lneq \cdots \lneq u_k} S(-a_{\{u_1,\dots,u_k\}}),$$

where the direct sum is over strictly increasing $k$-chains $u_1 \lneq \cdots \lneq u_k$ in the poset $P$. The differential $\partial_k : C_k \to C_{k-1}$ is defined on the component $S(-a_{\{u_1,\dots,u_k\}})$ as the alternating sum $\partial_k = \sum_{i=1}^{k} (-1)^i \, E_i$ of the multidegree-preserving embeddings $E_i : S(-a_{\{u_1,\dots,u_k\}}) \hookrightarrow S(-a_{\{u_1,\dots,\widehat{u_i},\dots,u_k\}})$ of $S$-modules, where $u_1, \dots, \widehat{u_i}, \dots, u_k$ denotes the sequence with skipped $i$-th element.

**Theorem 6.1.** *Let $\mathcal{M}$ be an order monomial family. The homological order complex $\mathcal{C}_*(\mathcal{M})$ is a free resolution of the order monomial ideal $\mathcal{I} = \langle \mathcal{M} \rangle$.*

*If $m_{\{u_1,\dots,u_k\}} \neq m_{\{u_1,\dots,\widehat{u_i},\dots,u_k\}}$, for any increasing chain $u_1 \lneq \cdots \lneq u_k$ in the poset $P$ and $i = 1, \dots, k$, then the homological order complex $\mathcal{C}_*(\mathcal{M})$ is a minimal free resolution of the order monomial ideal $\mathcal{I} = \langle \mathcal{M} \rangle$.*

The above construction of the homological order complex $C_*(\mathcal{M})$ is an instance of the general construction of cellular complexes for monomial ideals due to Bayer and Sturmfels [BaSt]. Their cellular complexes are associated with cell complexes,[1] whose faces are marked by certain monomials. In our case the cell complex is the *geometrical order complex* $\Delta = \Delta(P)$ of the poset $P$. It is the simplical complex whose faces correspond to nonempty strictly increasing chains in $P$:

$$\Delta(P) = \{\{u_1, \cdots, u_k\} \subseteq P \mid u_1 \lneq \cdots \lneq u_k, \ k \geq 1\}.$$

For example, if $P$ consists of all nonempty subsets in $\{1, \dots, n\}$ ordered by inclusion, then $\Delta(P)$ is the barycentric subdivision of the $(n-1)$-dimensional simplex. The face $F$ of $\Delta(P)$ given by an increasing chain $u_1 \lneq \cdots \lneq u_k$ is marked by the monomial $m_F = m_{\{u_1,\dots,u_k\}}$. For a monomial $m$, let $\Delta_{\leq m}$ denote the subcomplex of $\Delta(P)$ formed by the faces $F$ whose mark $m_F$ divides $m$:

$$\Delta_{\leq m} = \{F \in \Delta(P) \mid m_F \text{ divides } m\}.$$

The faces of $\Delta(P)$ are partially ordered by containment of closures. More precisely, $F \geq F'$ if the increasing chain for the face $F'$ is a subchain of the chain for $F$.

A result of Bayer and Sturmfels [BaSt, Proposition 1.2] on cellular complexes implies the following statement.

**Lemma 6.2.** *The complex $C_*(\mathcal{M})$ is exact if and only if $\Delta_{\leq m}$ is acyclic over $\mathbb{K}$ for any monomial $m$. If, in addition, $m_F \neq m_{F'}$, for any $F \gneq F'$, then the complex $C_*(\mathcal{M})$ is a minimal free resolution of the ideal $\mathcal{I} = \langle \mathcal{M} \rangle$.*

Actually, the subcomplex $\Delta_{\leq m}$ is not only acyclic but also contractible. This follows from the following result of Narushima.

[1] "Cellular complexes" should not be confused with "cell complexes." The former are homological complexes and the latter are geometrical complexes.

**Lemma 6.3** ([Naru]). *Let $M_u$, $u \in P$, be a finite collection of subsets in some set $\mathfrak{M}$, whose index set is a poset $P$. Assume that, for any $u, v \in P$, $M_u \cap M_v \subseteq M_w$ for some upper bound $w \in P$ of $u$ and $v$. Then, for any $m \in \mathfrak{M}$, the subcomplex of the order complex $\Delta(P)$ of $P$ formed by the following collection of nonempty increasing chains in $P$*

$$\{\{u_1 \lneq \cdots \lneq u_k\} \mid m \in M_{u_1} \cap \cdots \cap M_{u_k},\ k \geq 1\}$$

*is contractible.*

Theorem 6.1 follows easily from Lemmas 6.2 and 6.3.

*Proof of Theorem 6.1.* Let $\mathcal{M} = \{m_u \mid u \in P\}$ be an order monomial family. In view of Lemma 6.2 it is enough to show that the subcomplex $\Delta_{\leq m}$ is contractible for any monomial $m$. According to (OM), the conditions of Lemma 6.3 are satisfied, where $\mathfrak{M}$ is the set of all monomials in $\mathbb{K}[x_1, \ldots, x_n]$. For an increasing chain $u_1 \lneq \cdots \lneq u_k$ in $P$, the intersection $M_{u_1} \cap \cdots \cap M_{u_k}$ is the set of monomials divisible by $m_{\{u_1,\ldots,u_k\}}$. Thus the contractible complex from Lemma 6.3 is exactly the complex $\Delta_{\leq m}$. Lemma 6.2 implies that the homological order complex $C_*(\mathcal{M})$ is exact. $\square$

Let us now assume that $\mathcal{M} = \{m_I \mid I \in \Sigma\}$, $m_I = \prod_{i\in I} x_i^{\nu_I(i)}$, is a monotone monomial family. In this case, for a strictly increasing chain of subsets $I_1 \subsetneq \cdots \subsetneq I_k$, the least common multiple $m_{\{I_1,\ldots,I_k\}} = \mathrm{lcm}(m_{I_1}, \ldots, m_{I_k})$ is given by

$$m_{\{I_1,\ldots,I_k\}} = \prod_{i_1\in I_1} x_{i_1}^{\nu_{I_1}(i_1)} \times \prod_{i_2\in I_2\setminus I_1} x_{i_2}^{\nu_{I_2}(i_2)} \times \cdots \times \prod_{i_k\in I_k\setminus I_{k-1}} x_{i_k}^{\nu_{I_k}(i_k)}. \tag{6.2}$$

In other words, the exponent vector $a_{\{I_1,\ldots,I_k\}} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ of the monomial $m_{\{I_1,\ldots,I_k\}}$ is given by

$$a_i = \begin{cases} \nu_{I_r}(i) & \text{if } i \in I_r \setminus I_{r-1}, \\ 0 & \text{if } i \notin I_k, \end{cases}$$

where we assume that $I_0 = \emptyset$.

Let us say that the monotone monomial family $\mathcal{M}$ and the corresponding monotone monomial ideal $\mathcal{I} = \langle\mathcal{M}\rangle$ are *strictly monotone* if the following additional conditions hold:

(SM1) The ideal $\mathcal{I}$ is *minimally* generated by the set of monomials $\{m_I \mid I \in \Sigma\}$, i.e., there are no two elements $I \neq J$ in $\Sigma$ such that $m_I$ divides $m_J$.

(SM2) For any $I \subsetneq J \subsetneq K$ in $\Sigma$, there exists $i \in J \setminus I$ such that $\nu_J(i) > \nu_K(i)$.

For example, a monotone monomial family such that the inequality in (MM2) is always strict and $\nu_I(i) > 0$, for any $I \in \Sigma$ and $i \in I$, will be strictly monotone.

Conditions (SM1) and (SM2) are equivalent to the statement that $m_{\{I_1,\ldots,I_k\}} \neq m_{\{I_1,\ldots,\widehat{I_i},\ldots,I_k\}}$ for any increasing chain $I_1 \subsetneq \cdots \subsetneq I_k$ in $\Sigma$ and $i = 1, \ldots, k$. Thus Theorem 6.1 specializes to the following statement.

**Corollary 6.4.** *Let $\mathcal{M}$ be a monotone monomial family. Then the homological order complex $\mathcal{C}_*(\mathcal{M})$ is a free resolution of the ideal $\mathcal{I} = \langle\mathcal{M}\rangle$. If $\mathcal{M}$ is a strictly monotone monomial family, then $\mathcal{C}_*(\mathcal{M})$ is a minimal free resolution of the strictly monotone ideal $\mathcal{I}$.*

Homological order complexes are related to Scarf complexes of generic monomial ideals. Let $\mathcal{I} = \langle m_1, \ldots, m_r \rangle$ be an arbitrary ideal in the polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$ minimally generated by monomials $m_1, \ldots, m_r$, and let $m_U = \mathrm{lcm}(m_u \mid u \in U)$ for $U \subseteq \{1, \ldots, r\}$. The *geometrical Scarf complex* of $\mathcal{I}$ was defined by Bayer, Peeva, and Sturmfels [BPS] as the following simplicial complex:

$$\Delta_{\mathcal{I}}^{\mathrm{scarf}} = \{U \subseteq \{1, \ldots, r\} \mid m_U \neq m_V \text{ for all subsets } V \neq U, \text{ and } |U| \geq 1\}.$$

The corresponding cellular complex is called the *homological Scarf complex.*

**Lemma 6.5.** *Let $\mathcal{I} = \langle m_u \mid u \in P \rangle$ be an order monomial ideal. Then the geometrical Scarf complex $\Delta_{\mathcal{I}}^{scarf}$ is a subcomplex of the geometrical order complex $\Delta(P)$.*

*Proof.* Let $U$ be a subset of elements in $P$. Suppose that $U$ contains two incomparable elements $u$ and $v$. Let us pick an upper bound $w$ of $u$ and $v$ provided by condition (OM). Let $V = U \cup \{w\}$ if $w \notin U$, or $V = U \setminus \{w\}$ if $w \in U$. Then, according to (OM), $m_U = m_V$. Thus $V$ does not belong to the geometrical Scarf complex.

This implies that, for any $U \in \Delta_{\mathcal{I}}^{\mathrm{scarf}}$, all elements of $U$ are comparable with each other, i.e., $U$ is an increasing chain in the poset $P$. Thus $U \in \Delta(P)$. $\square$

Let us say that a monomial $m$ *strictly divides* a monomial $m'$, if $m$ divides $m'$ and $\deg_{x_i}(m'/m) \neq 0$ whenever $\deg_{x_i}(m') \neq 0$. According to Miller, Sturmfels, and Yanagawa [MSY], the ideal $\mathcal{I} = \langle m_1, \ldots, m_r \rangle$ is called a *generic monomial ideal* if the following condition holds:

(GM) If two distinct minimal generators $m_u$ and $m_v$ have the same positive degree in some variable $x_i$, there is a third generator $m_w$ which strictly divides $\mathrm{lcm}(m_u, m_v)$.

The general, the Scarf complex may not be acyclic, but, for generic monomial ideals, Miller, Sturmfels, and Yanagawa [MSY] proved the following result.

**Proposition 6.6** ([MSY, Corollary 1.8])**.** *If $\mathcal{I}$ is a generic monomial ideal, then the homological Scarf complex is a minimal free resolution of $\mathcal{I}$.*

We will see in Section 7 that there are strictly monotone monomial ideals that are not generic and there are generic monomial ideals that are not strictly monotone. The following claim shows that these two classes of ideals have an interesting intersection.

**Proposition 6.7.** *Let $\mathcal{I} = \langle m_I \mid I \in \Sigma \rangle$ be a monotone monomial ideal such that the inequality in* (MM2) *is always strict, and $\nu_I(i) > 0$, for any $I \in \Sigma$ and $i \in I$. Then the monomial ideal $\mathcal{I}$ is both generic and strictly monotone. In this case, the geometrical/homological order complex for $\mathcal{I}$ coincides with geometrical/homological Scarf complex for $\mathcal{I}$.*

*Proof.* If monomials $m_I$ and $m_J$ have the same positive degree in some variable, then $I$ and $J$ are incomparable in $\Sigma$: $I \not\subseteq J$ and $J \not\subseteq I$. By (MM3) there exists $K \supseteq I \cup J$ such that $m_K$ divides $\mathrm{lcm}(m_I, m_J)$. Then $K \neq I, J$. Since we assume that the inequality in (MM2) is strict, $m_K$ strictly divides $\mathrm{lcm}(m_I, m_J)$. It follows that the ideal $\mathcal{I}$ is generic.

According to Lemma 6.5, the geometrical Scarf complex $\Delta_{\mathcal{I}}^{\mathrm{scarf}}$ is a subcomplex of the geometrical order complex $\Delta(\Sigma)$. Let us prove that $\Delta_{\mathcal{I}}^{\mathrm{scarf}} = \Delta(\Sigma)$. We need to show that, for any increasing chain $I_1 \subsetneq \cdots \subsetneq I_k$ in $\Sigma$, we have $m_{\{I_1,\dots,I_k\}} \neq m_R$, where $R$ is any subset of $\Sigma$ such that $R \neq \{I_1, \dots, I_k\}$. This is clear if $R$ is a subchain in $I_1 \subsetneq \cdots \subsetneq I_k$. Otherwise, suppose that $m_{\{I_1,\dots,I_k\}} = m_R$ and $R$ contains an element $J \notin \{I_1, \dots, I_k\}$. Then $m_J$ divides $m_{\{I_1,\dots,I_k\}}$. According to conditions of the proposition, the monomial $m_J$ depends nontrivially on all $x_i$, $i \in J$. Thus $J \subseteq I_k$. Then $J \subseteq I_r$ and $J \not\subseteq I_{r-1}$ for some $r \in \{1, \dots, k\}$. (We assume that $I_0 = \emptyset$.) Pick an element $i \in J \setminus I_{r-1}$. Then $\nu_J(i) \leq \deg_{x_i}(m_{\{I_1,\dots,I_k\}}) = \nu_{I_r}(i)$ because $m_J$ divides $m_{\{I_1,\dots,I_k\}}$. Also we have $\nu_J(i) > \nu_{I_r}(i)$ because $J \subsetneq I_r$. Contradiction. □

The $k$-th *Betti number* $\beta_k(\mathcal{I})$ of an ideal $\mathcal{I}$ is the rank of the $k$-th term in a minimal free resolution of $\mathcal{I}$. The *graded Betti number* $\beta_{k,d}(\mathcal{I})$ of a graded ideal $\mathcal{I}$ is the number of direct summands in the $k$-th term of a minimal free resolution of $\mathcal{I}$ with generator of degree $d$. Then $\beta_k(\mathcal{I}) = \sum_d \beta_{k,d}(\mathcal{I})$.

Let $d(I_1, \dots, I_k)$ be the degree of the monomial $m_{\{I_1,\dots,I_k\}}$ given by

$$d(I_1, \dots, I_k) = \sum_{i_1 \in I_1} \nu_{I_1}(i_1) + \sum_{i_2 \in I_2 \setminus I_1} \nu_{I_2}(i_2) + \cdots + \sum_{i_k \in I_k \setminus I_{k-1}} \nu_{I_k}(i_k). \tag{6.3}$$

**Corollary 6.8.** *Let $\mathcal{I} = \langle m_I \mid I \in \Sigma \rangle$ be a strictly monotone monomial ideal. The $k$-th Betti number $\beta_k(\mathcal{I})$ of $\mathcal{I}$ is equal to the number of strictly increasing $k$-chains in the poset $\Sigma$. Moreover, the graded Betti number $\beta_{k,d}(\mathcal{I})$ of $\mathcal{I}$ is equal to the number of strictly increasing $k$-chains $I_1 \subsetneq \cdots \subsetneq I_k$ in $\Sigma$ such that $d(I_1, \dots, I_k) = d$.*

*In particular, if $\Sigma$ is the set of all nonempty subsets in $\{1, \dots, n\}$, then*

$$\beta_k(\mathcal{I}) = k!\, S_{n+1,k+1}, \tag{6.4}$$

*where $S_{n+1,k+1}$ is the Stirling number of the second kind, i.e., the number of partitions of the set $\{0, \dots, n\}$ into $k+1$ nonempty blocks.*

The last claim is obtained by counting strictly increasing $k$-chains of nonempty subsets $I_1 \subsetneq \cdots \subsetneq I_k$ in $\{1, \dots, n\}$. Indeed, these chains are in one-to-one correspondence with partitions of the set $\{0, \dots, n\}$ into a linearly ordered family of $k+1$ nonempty blocks $(I_1, I_2 \setminus I_1, \cdots, I_k \setminus I_{k-1}, \{0, \dots, n\} \setminus I_k)$ such that the last block contains 0. There are $k!$ ways to pick such a linear ordering of blocks.

Let us say that a (directed) graph is *saturated* if all off-diagonal entries of the adjacency matrix $A = (a_{ij})$ are nonzero: $a_{ij} \geq 1$ for $i \neq j$. If $G$ is a saturated digraph, then the monotone monomial ideal $\mathcal{I}_G$ constructed in Section 2 satisfies the conditions of Proposition 6.7.

**Corollary 6.9.** *The monotone monomial ideal $\mathcal{I}_G$, for a saturated digraph $G$, is both strictly monotone and generic. In this case $\Sigma$ is the poset of all nonempty subsets in $\{1, \dots, n\}$. The homological order complex $C_*(\mathcal{M})$, which coincides with the homological Scarf complex, gives a minimal free resolution of the ideal $\mathcal{I}_G$. Its Betti numbers are given by* (6.4).

It would be interesting to find a minimal free resolution of the ideal $\mathcal{I}_G$ for any nonsaturated digraph $G$. More generally, it would be interesting to find a minimal free resolution for any monotone monomial ideal.

Computer experiments suggest the following conjecture on Betti numbers of deformations of monotone monomial ideals.

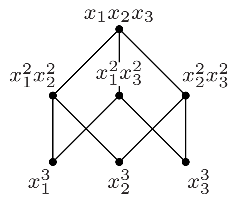
**Conjecture 6.10.** *Let $\mathcal{J}$ be a deformation of a monotone monomial ideal $\mathcal{I}$ such that $\dim_{\mathbb{K}} \mathbb{K}[x_1, \dots, x_n]/\mathcal{I} = \dim_{\mathbb{K}} \mathbb{K}[x_1, \dots, x_n]/\mathcal{J}$. Then all graded Betti numbers* *of the ideals $\mathcal{I}$ and $\mathcal{J}$ coincide: $\beta_{k,d}(\mathcal{I}) = \beta_{k,d}(\mathcal{J})$. In particular, for a graph $G$, the ideals $\mathcal{I}_G$ and $\mathcal{J}_G$ have the same graded Betti numbers.*

## 7. Examples

Let us illustrate Corollaries 6.4, 6.8, and 6.9 and Proposition 6.7 by several examples. In all examples $n = 3$, $S = \mathbb{K}[x_1, x_2, x_3]$, and $S(-d)$ denotes the $\mathbb{Z}$-graded $S$-module isomorphic to $S$ with grading shifted by integer $d$, so that the generator has degree $d$.

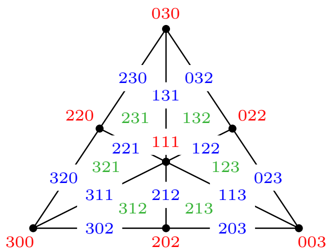
**Example 7.1.** Let $G = K_4$ be the complete graph on 4 vertices. This graph is saturated. Thus the monomial ideal $\mathcal{I}_G$ is both strictly monotone and generic. The poset $\Sigma$ consists of all nonempty subsets in $\{1,2,3\}$. The Hasse diagram of $\Sigma$ marked by the monomials $m_I$ is given by

$\Sigma =$




The poset $\Sigma$ has seven 1-chains, twelve 2-chains, and six 3-chains. In this case the geometrical order complex $\Delta = \Delta(\Sigma)$ is the barycentric subdivision of a triangle. The following figure shows the complex $\Delta$ with faces marked by vectors $a_{\{I_1,\dots,I_k\}}$:

$\Delta =$



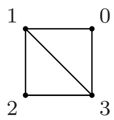

The Betti numbers $(\beta_0, \beta_1, \beta_2, \beta_3) = (1, 7, 12, 6)$ of the ideal $\mathcal{I} = \mathcal{I}_{K_4}$, which are also the $f$-numbers of the order complex $\Delta$, can be expressed in terms of the Stirling numbers by formula (6.4). The graded Betti numbers of this ideal are indicated on the following minimal free resolution:

$$0 \longrightarrow S(-6)^6 \longrightarrow S(-5)^{12} \longrightarrow S(-3)^4 \oplus S(-4)^3 \longrightarrow S \longrightarrow S/\mathcal{I} \longrightarrow 0.$$

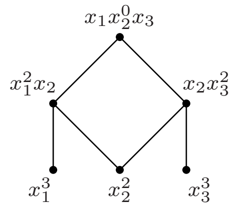
This resolution is the homological order complex and also the homological Scarf complex of $\mathcal{I}$.

Similarly, a minimal free resolution of the ideal $\mathcal{I}_n = \mathcal{I}_{K_{n+1}}$ associated with the complete graph $K_{n+1}$ is given by the cellular complex corresponding to the simplicial complex $\Delta(\Sigma) = \Delta^{\mathrm{scarf}}_{\mathcal{I}_n}$, which is the barycentric subdivision of the $(n-1)$-dimensional simplex; cf. [MSY, Example 1.2].

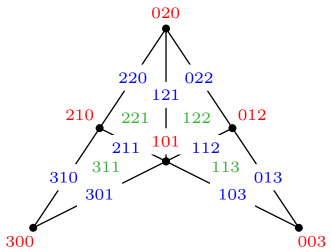
**Example 7.2.** Let $G$ be the graph given by

$G =$  .

This graph is not saturated and the monotone monomial family that generates the ideal $\mathcal{I} = \mathcal{I}_G = \langle x_1^3, x_2^2, x_3^3, x_1^2x_2, x_1^2x_3^2, x_2x_3^2, x_1x_2^0x_3\rangle$ will not be strictly monotone if we assume that the labelling set $\Sigma$ consists of all nonempty subsets in $\{1,2,3\}$. As we mentioned before, the generator $m_{\{1,3\}} = x_1^2x_3^2$ is redundant. The same monomial ideal $\mathcal{I}$ is minimally generated by the *strictly* monotone monomial family $\{m_I \mid I \in \Sigma\}$ with $\Sigma = \{\{1\},\{2\},\{3\},\{1,2\},\{2,3\},\{1,2,3\}\}$. The Hasse diagram of this poset $\Sigma$ marked by the monomials $m_I$ is given by

$\Sigma =$ 


This poset has six 1-chains, nine 2-chains, and four 3-chains. Its geometrical order complex $\Delta = \Delta(\Sigma)$ with faces marked by vectors $a_{I_1,\ldots,I_k}$ is shown in the following figure:

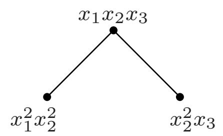
$\Delta =$ 


The homological order complex gives a minimal free resolution of the ideal $\mathcal{I}$:

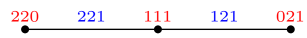
$$0 \longrightarrow S(-5)^4 \longrightarrow S(-4)^9 \longrightarrow S(-2)^2 \oplus S(-3)^4 \longrightarrow S \longrightarrow S/\mathcal{I} \longrightarrow 0.$$

In this case the ideal $\mathcal{I}$ is also generic and the above resolution is the homological Scarf complex.

**Example 7.3.** Let $\mathcal{I} = \langle x_1^2x_2^2, x_2^2x_3, x_1x_2x_3\rangle$ be the monotone monomial ideal, whose poset $\Sigma$ marked by the monomials is given by

$\Sigma =$ 


The ideal $\mathcal{I}$ is strictly monotone but *is not* generic. The geometrical order complex with faces marked by vectors $a_{\{I_1,\ldots,I_k\}}$ is given by

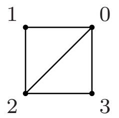
$\Delta =$ 


It produces the following minimal free resolution of the ideal $\mathcal{I}$:

$$0 \longrightarrow S(-4) \oplus S(-5) \longrightarrow S(-3)^2 \oplus S(-4) \longrightarrow S \longrightarrow S/\mathcal{I} \longrightarrow 0.$$

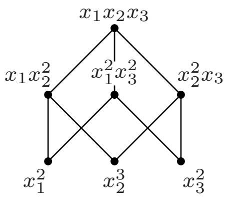
On the other hand, the geometrical Scarf complex in this case is disconnected and does not give a resolution for $\mathcal{I}$.

**Example 7.4.** Let $G$ be the graph given by

$G =$ 

.

Then $\mathcal{I} = \mathcal{I}_G = \langle x_1^2, x_2^3, x_3^2, x_1x_2^2, x_1^2x_3, x_1^2x_3^2, x_1x_2x_3\rangle$. The Hasse diagram of $\Sigma$ marked by the monomials $m_I$ is given by

$\Sigma =$ 


The corresponding homological order complex $C_*(\mathcal{M})$ gives a free resolution, which is not minimal. In this case the monomial generator $m_{\{1,3\}} = x_1^2x_3^2$ is redundant because it is divisible by $m_{\{1\}} = x_1^2$. However, the family $\mathcal{M} \setminus \{m_{\{1,3\}}\}$ *is not* a monotone monomial family because condition (MM3) no longer holds. On the other hand, $\mathcal{I}$ is a generic ideal and its Scarf complex gives a minimal free resolution:

$$0 \longrightarrow S(-5)^4 \longrightarrow S(-4)^9 \longrightarrow S(-2)^2 \oplus S(-3)^4 \longrightarrow S \longrightarrow S/\mathcal{I} \longrightarrow 0.$$

In Examples 7.1, 7.2, and 7.4 above, the graded Betti numbers of the deformed ideal $\mathcal{J}_G$ coincide with graded Betti numbers of $\mathcal{I}_G$.

## 8. Hilbert series and dimensions of monotone monomial ideals

In this section we give formulas for the Hilbert series and dimensions of monotone monomial ideals. Then we prove Theorem 2.1.

Let $\{m_I \mid I \in \Sigma\}$, $m_I = \prod_{i\in I} x_i^{\nu_I(i)}$, be a monotone monomial family, let $\mathcal{I} = \langle m_I\rangle$ be the corresponding ideal in the polynomial ring $\mathbb{K}[x_1,\dots,x_n]$, and let $\mathcal{A} = \mathbb{K}[x_1,\dots,x_n]/\mathcal{I}$.

**Proposition 8.1.** *The Hilbert series of the algebra $\mathcal{A}$ equals*

$$\operatorname{Hilb}\mathcal{A} = \frac{1 + \sum_{k\geq 1}(-1)^k \sum_{I_1 \subsetneq \cdots \subsetneq I_k} q^{d(I_1,\dots,I_k)}}{(1-q)^n},$$

*where the sum is over all strictly increasing chains in $\Sigma$ and the number $d(I_1,\dots,I_k)$ is given by* (6.3).

*First proof of Proposition 8.1.* According to Corollary 6.4, the homological order complex (6.1) is a free resolution of $\mathcal{I}$. Thus

$$\operatorname{Hilb}\mathcal{A} = \sum_{k\geq 0}(-1)^k \operatorname{Hilb} C_k = \sum_{k\geq 0}(-1)^k \sum_{I_1 \subsetneq \cdots \subsetneq I_k} \operatorname{Hilb} S(-a_{\{I_1,\dots,I_k\}}).$$

Since $\operatorname{Hilb} S(-a_{\{I_1,\dots,I_k\}}) = q^{d(I_1,\dots,I_k)}/(1-q)^n$, the proposition follows. $\square$

Let us give another more expanded proof of Proposition 8.1. We will need the *improved inclusion-exclusion formula* due to Narushima [Naru]. For a subset $M$ in some set $\mathfrak{M}$, let $\chi(M)$ denote the characteristic function of $M$:

$$\chi(M) : a \longmapsto \begin{cases} 1 & \text{if } a \in M, \\ 0 & \text{if } a \in \mathfrak{M} \setminus M. \end{cases}$$

**Lemma 8.2** ([Naru]). *Let $M_u$, $u \in P$, be a finite collection of subsets in some set $\mathfrak{M}$, whose index set is a poset $P$, such that, for any $u, v \in P$, $M_u \cap M_v \subseteq M_w$ for some upper bound $w$ of $u$ and $v$. Then we have*

$$\chi(\mathfrak{M} \setminus \bigcup_{u \in P} M_u) = \chi(\mathfrak{M}) + \sum_{k \geq 1} (-1)^k \sum_{u_1 < \cdots < u_k} \chi(M_{u_1} \cap \cdots \cap M_{u_k}),$$

*where the second sum is over all strictly increasing chains $u_1 < \cdots < u_k$ in the poset $P$.*

*Proof.* According to the usual *inclusion-exclusion principle* (see [Sta1, Section 2.1]), we have

$$\chi(\mathfrak{M} \setminus \bigcup_{u \in P} M_u) = \chi(\mathfrak{M}) - \sum_u \chi(M_u) + \sum_{u,v} \chi(M_u \cap M_v) - \cdots .$$

The general summand in this expression is $s_U = (-1)^k \chi(M_{u_1} \cap \cdots \cap M_{u_k})$, where $U = \{u_1, \ldots, u_k\}$ is an unordered $k$-element subset in $P$. We argue that if we take the summation only over *increasing chains* $u_1 < \cdots < u_k$ in $P$ we get exactly the same answer. Indeed, let us show that the contribution of all other subsets $U$ is zero. We will use the *involution principle*; see [Sta1, Section 2.6]. Let us construct an involution $\iota$ on the set of all subsets $U \subseteq P$ of all possible sizes $k \geq 0$ such that the elements of $U$ cannot be arranged in an increasing chain. Let us fix a linear order on elements of $P$. Find the lexicographically minimal pair of of incomparable elements $u$ and $v$ in $U$, i.e., $u \not\leq v$ and $v \not\leq u$. Let $w \in P$ be the minimal (with respect to the linear order) upper bound of $u$ and $v$ such that $M_u \cap M_v \subseteq M_w$. Define the map $\iota$ as follows:

$$\iota : U \longmapsto \begin{cases} U \cup \{w\}, & \text{if } w \notin U, \\ U \setminus \{w\}, & \text{if } w \in U. \end{cases}$$

Then $\iota$ is an involution such that $|\iota(U)| = |U| \pm 1$. Conditions of the lemma imply that $s_U = -s_{\iota(U)}$. Thus all summands $s_U$ corresponding to nonchains cancel each other. ∎

*Second proof of Proposition 8.1.* Let $\mathfrak{M}$ be the set of monomials in $\mathbb{K}[x_1, \ldots, x_n]$, and, for $I \in \Sigma$, let $M_I \subset \mathfrak{M}$ denote the set of monomials in $\mathbb{K}[x_1, \ldots, x_n]$ divisible by $m_I$. For a subset of monomials $M \subset \mathfrak{M}$, let

$$[M] = \sum_{m \in M} q^{\deg(m)} = \sum_{m \in \mathfrak{M}} q^{\deg(m)} \chi(M)(m).$$

Then $\operatorname{Hilb} \mathcal{A} = [\mathfrak{M} \setminus \bigcup M_I]$ and $[\mathfrak{M}] = (1-q)^{-n}$. All conditions of Lemma 8.2 are satisfied, where $P = \Sigma$. For an increasing chain $I_1 \subsetneq \cdots \subsetneq I_k$, the least common multiple (6.2) of the monomials $m_{I_1}, \ldots, m_{I_k}$ has degree $d(I_1, \ldots, I_k)$. Thus $[M_{I_1} \cap \cdots \cap M_{I_k}] = q^{d(I_1, \ldots, I_k)}(1-q)^{-n}$. Proposition 8.1 follows from Lemma 8.2. ∎

**Lemma 8.3.** *The algebra $\mathcal{A}$ is finite dimensional as a linear space over $\mathbb{K}$ if and only if $\Sigma$ contains all one-element subsets in $\{1,\dots,n\}$.*

*Proof.* If there is $i \in \{1,\dots,n\}$ such that $\{i\} \notin \Sigma$, then the powers $x_i^c$ form an infinite linearly independent subset in $\mathcal{A}$. Thus $\mathcal{A}$ is infinite dimensional. Otherwise, if $\Sigma$ contains all one-element subsets, then the algebra $\mathcal{A}$ is finite dimensional. Indeed, a monomial $x_1^{a_1}\dots x_n^{a_n}$ vanishes in $\mathcal{A}$ unless $a_1 < \nu_{\{1\}}(1),\dots,a_n < \nu_{\{n\}}(n)$. $\square$

**Proposition 8.4.** *Assume that $\Sigma$ contains all one-element subsets in $\{1,\dots,n\}$, and let $\nu(i) = \nu_{\{i\}}(i)$. The dimension of the algebra $\mathcal{A}$ is given by the following polynomial in the variables $\{\nu_I(i) \mid I \in \Sigma, i \in I\}$:*

$$\begin{aligned}\dim\mathcal{A} = \sum_{I_1\subsetneq\cdots\subsetneq I_k} (-1)^k \prod_{i_1\in I_1}(\nu(i_1)-\nu_{I_1}(i_1)) \times \prod_{i_2\in I_2\setminus I_1}(\nu(i_2)-\nu_{I_2}(i_2))\\ \times\cdots\times \prod_{i_k\in I_k\setminus I_{k-1}}(\nu(i_k)-\nu_{I_k}(i_k)) \times \prod_{i_{k+1}\notin I_k}\nu(i_{k+1}),\end{aligned} \tag{8.1}$$

*where the sum is over all strictly increasing chains $I_1 \subsetneq \cdots \subsetneq I_k$ of nonempty subsets in $\{1,\dots,n\}$ of all sizes $k \geq 0$, including the empty chain of size $k=0$.*

*Proof.* Let $\tilde{\mathfrak{M}}$ be the set of monomials $x_1^{a_1}\cdots x_n^{a_n}$ such that $a_1 < \nu(1),\dots,a_n < \nu(n)$. A monomial $x^a$ vanishes in the algebra $\mathcal{A}$ unless $x^a \in \tilde{\mathfrak{M}}$. Let $\tilde M_I = M_I \cap \tilde{\mathfrak{M}}$. Lemma 8.2 for $\tilde{\mathfrak{M}}$ implies that

$$\dim\mathcal{A} = |\tilde{\mathfrak{M}}| + \sum_{k\geq 1}(-1)^k \sum_{I_1\subsetneq\cdots\subsetneq I_k} |\tilde M_{I_1}\cap\cdots\cap\tilde M_{I_k}|.$$

The intersection $\tilde M_{I_1}\cap\cdots\cap\tilde M_{I_k}$ is the set of all monomials in $\tilde{\mathfrak{M}}$ divisible by the monomial $m_{I_1,\dots,I_k}$ given by (6.2). Thus $(-1)^k\,|\tilde M_{I_1}\cap\cdots\cap\tilde M_{I_k}|$ is equal to the summand in (8.1). $\square$

Remark that if $I_1,\dots,I_k$ is not a chain of subsets, then $|\tilde M_{I_1}\cap\cdots\cap\tilde M_{I_k}|$ may not be a polynomial in the $\nu_I(i)$. It may include expressions like $\min(\nu_I(i),\nu_J(i))$. Thus the inclusion-exclusion principle does not immediately produce a polynomial expression for $\dim\mathcal{A}$. Miraculously, all nonpolynomial terms cancel each other.

We can now prove Theorem 2.1, which claims that the dimension of the algebra $\mathcal{A}_G$ equals the number of oriented spanning trees of $G$.

*Proof of Theorem 2.1* (cf. [Gab1, Appendix E]). Let $G$ be a digraph on the vertices $0,\dots,n$, and let $A=(a_{ij})$ be its adjacency matrix. Specializing Proposition 8.4, we obtain the following polynomial formula for the dimension of the algebra $\mathcal{A}_G$:

$$\begin{aligned}\dim\mathcal{A}_G = \sum_{I_1\subsetneq\cdots\subsetneq I_k}(-1)^k \prod_{i_1\in I_1}\left(\sum_{j_1\in I_1} a_{i_1j_1}\right) \times \prod_{i_2\in I_2\setminus I_1}\left(\sum_{j_2\in I_2} a_{i_2j_2}\right)\\ \times\cdots\times \prod_{i_k\in I_k\setminus I_{k-1}}\left(\sum_{j_k\in I_k} a_{i_kj_k}\right) \times \prod_{i_{k+1}\in\{1,\dots,n\}\setminus I_k}\left(\sum_{0\leq j_{k+1}\leq n} a_{i_{k+1}j_{k+1}}\right),\end{aligned} \tag{8.2}$$

where the sum is over all strictly increasing chains $I_1 \subsetneq \cdots \subsetneq I_k$ of nonempty subsets in $\{1,\dots,n\}$ of all sizes $k\geq 0$. In this formula, we assume that $a_{ii}=0$.

Let us show that the expression (8.2) for $\dim \mathcal{A}_G$ is equal to the number of oriented spanning trees of $G$. We will use the involution principle again.

Let us first give a combinatorial interpretation of the right-hand side of (8.2). The summand that corresponds to an increasing chain $I_* = I_1 \subsetneq \cdots \subsetneq I_k$ is equal to $(-1)^k$ times the number of subgraphs $H$ of $G$ such that

(a) $H$ contains exactly $n$ directed edges $(i, f(i))$ for $i = 1, \ldots, n$.
(b) If $i \in I_r \setminus I_{r-1}$, then $f(i) \in I_r$. (We assume that $I_0 = \emptyset$.)

For such a subgraph $H$, let $J_H \subseteq \{1, \ldots, n\}$ be the set of vertices $i$ such that $f^p(i) = 0$ for some power $p$, i.e., $J_H$ is the set of vertices $i$ such that there is a directed path in $H$ from $i$ to the root 0. Notice that if $i \in \bigcup I_r$, then $f^p(i) \in \bigcup I_r$, thus $f^p(i) \neq 0$, for any $p$. Thus $I_1, \ldots, I_k \subseteq \overline{J_H} = \{1, \ldots, n\} \setminus J_H$. Also notice that $H$ is an oriented spanning tree of $G$ if and only if $J_H = \{1, \ldots, n\}$.

Let us now construct an involution $\kappa$ on the set of pairs $(I_*, H)$ such that $H$ is not an oriented spanning tree. In other words, the involution $\kappa$ acts on the set of pairs $(I_*, H)$ with nonempty $\overline{J_H}$. It is given by

$$\kappa : (I_*, H) \longmapsto \begin{cases} (I_1 \subsetneq \cdots \subsetneq I_{k-1}, H) & \text{if } I_k = \overline{J_H}\,, \\ (I_1 \subsetneq \cdots \subsetneq I_k \subsetneq \overline{J_H}, H) & \text{if } I_k \neq \overline{J_H}\,. \end{cases}$$

The contribution of the pair $(I_*, H)$ to the right-hand side of (8.2) is opposite to the contribution of $\kappa(I_*, H)$. Thus the contributions of all subgraphs $H$ which are not oriented spanning trees cancel each other. This implies that $\dim \mathcal{A}_G$ is the number of oriented spanning trees. $\square$

It would be interesting to find a combinatorial proof of Theorem 2.1. In other words, one would like to present a bijection between $G$-parking functions and oriented spanning trees of $G$. There are several known bijections between the usual parking functions and trees. One such bijection is relatively easy to construct. There is a more elaborate bijection that maps parking functions $b$ with $b_1 + \cdots + b_n = k$ to trees with $\binom{n}{2} - k$ inversions; see [Krew].

## 9. Square-free algebra

Let $G$ be a graph on the set of vertices $0, \ldots, n$. We will say that a subgraph $H \subset G$ of the graph $G$ is *slim* if the complement subgraph $G \setminus H$ is connected. Let us associate commuting variables $\phi_e$, $e \in G$, with edges of the graph $G$, and let $\Phi_G$ be the algebra over $\mathbb{K}$ generated by the $\phi_e$ with the defining relations:

$$(\phi_e)^2 = 0, \quad \text{for any edge } e;$$

$$\prod_{e \in H} \phi_e = 0, \quad \text{for any nonslim subgraph } H \subset G.$$

Clearly, the square-free monomials $\phi_H = \prod_{e \in H} \phi_e$, where $H$ ranges over all slim subgraphs in $G$, form a linear basis of the algebra $\Phi_G$. Thus the dimension of $\Phi_G$ is equal to the number of connected subgraphs in $G$.

For $i = 1, \ldots, n$, let

$$X_i = \sum_{e \in G} c_{i,e}\, \phi_e,$$

where

$$c_{i,e} = \begin{cases} 1 & \text{if } e = (i,j),\ i < j; \\ -1 & \text{if } e = (i,j),\ i > j; \\ 0 & \text{otherwise.} \end{cases}$$

Define $\mathcal{C}_G$ as the subalgebra in $\Phi_G$ generated by the elements $X_1, \dots, X_n$.

Fix a linear ordering of edges of the graph $G$. Recall that $N_G^k$ denotes the number of spanning trees of $G$ with external activity $k$; see Section 3.

**Theorem 9.1.** (1) *The dimension of the algebra $\mathcal{C}_G$ as a linear space over $\mathbb{K}$ equals the number of spanning trees in the graph $G$.*

(2) *The dimension of the $k$-th graded component $\mathcal{C}_G^k$ of the algebra $\mathcal{C}_G$ equals the number $N_G^{|G|-n-k}$ of spanning trees of $G$ with external activity $|G| - n - k$.*

Recall that, for a nonempty subset $I \subset \{1, \dots, n\}$, $D_I = \sum_{i \in I, j \notin I} a_{ij}$ is the number of edges in $G$ that connect a vertex inside $I$ with a vertex outside of $I$; see Section 3.

**Lemma 9.2.** *For any nonempty subset $I \subset \{1, \dots, n\}$, the following relation holds in the algebra $\mathcal{C}_G$:*

$$\left(\sum_{i \in I} X_i\right)^{D_I} = 0.$$

This lemma shows that the algebra $\mathcal{C}_G$ is a quotient of the algebra $\mathcal{B}_G$. We will eventually see that $\mathcal{B}_G = \mathcal{C}_G$, but let us pretend that we do not know this yet.

*Proof.* Let $H_I \subset G$ be the subgraph that consists of all edges that connect a vertex in $I$ with a vertex outside of $I$. We have $\sum_{i \in I} X_i = \sum_{e \in H_I} \pm\phi_e$. Thus $\left(\sum_{i \in I} X_i\right)^{D_I} = \pm \prod_{e \in H_I} \phi_e = 0$, because $H_I$ is not a slim subgraph of $G$. □

Let $\mathcal{S}_G$ be the subspace in $\mathbb{K}[y_1, \dots, y_n]$ spanned by the elements

$$\alpha_H = \prod_{e \in H} \alpha_e,$$

for all slim subgraphs $H \subset G$, where $\alpha_e = y_i - y_j$, for an edge $e = (i,j)$, $i < j$. Let $\mathcal{S}_G^k$ denote the $k$-th graded component of the space $\mathcal{S}_G$.

**Lemma 9.3.** *For any graph $G$ and any $k$, we have* $\dim \mathcal{C}_G^k = \dim \mathcal{S}_G^k$.

*Proof.* Let $b_{H,a}$ be the coefficient of $\prod_{e \in H} \phi_e$ in the expansion of $X_1^{a_1} \cdots X_n^{a_n}$, where $a = (a_1, \dots, a_n)$. Then $\dim \mathcal{C}_G^k$ is equal to the rank of the matrix $B = (b_{H,a})$, where $H$ ranges over all slim subgraphs in $G$ with $k$ edges and $a$ ranges over nonnegative integer $n$-element sequences with $a_1 + \cdots + a_n = k$. On the other hand, $b_{H,a}$ is also equal to the coefficient of $y_1^{a_1} \cdots y_n^{a_n}$ in the expansion of $\alpha_H$. Thus $\dim \mathcal{S}_G^k$ equals the rank of the same matrix $B = (b_{H,a})$. □

For a spanning tree $T$ in $G$, let $T^+$ denote the graph obtained from $T$ by adding all externally active edges. By virtue of Lemma 9.3, the following claim implies Theorem 9.1.

**Proposition 9.4.** *The collection of elements $\alpha_{G \setminus T^+}$, where $T$ ranges over all spanning trees of $G$, forms a linear basis of the space $\mathcal{S}_G$.*

Let us first prove a weaker version of this claim.

**Lemma 9.5.** *The elements $\alpha_{G\setminus T^+}$, where $T$ ranges over all spanning trees of $G$, spans the space $\mathcal{S}_G$.*

*Proof.* Suppose not. Let $H$ be the lexicographically maximal slim subgraph of $G$ such that $\alpha_H$ cannot be expressed as a linear combination of the $\alpha_{G\setminus T^+}$. Then there exists a cycle $C = \{e_1, \ldots, e_l\} \subset G$ with the minimal element $e_1$ such that $H \cap C = \{e_1\}$. Then $\alpha_{e_1}$ is a linear combination of $\alpha_{e_2}, \ldots, \alpha_{e_l}$. Let $H_i$ be the graph obtained from $H$ by replacing the edge $e_1$ with $e_i$. For $i = 2, \ldots, l$, the graph $H_i$ is a slim subgraph of $G$, which is lexicographically greater than $H_1$. Then $\alpha_H$ can be expressed as a linear combination of $\alpha_{H_2}, \ldots, \alpha_{H_l}$. Contradiction. □

*Proof of Proposition 9.4.* Recall that $N_G$ denotes the number of spanning trees in the graph $G$. In view of Lemma 9.5 it is enough to show that $\dim \mathcal{S}_G = N_G$. We will prove this statement by induction on the number of edges in $G$.

If $G$ is a disconnected graph, then it has no slim subgraphs and $\dim \mathcal{S}_G = N_G = 0$. If $G$ is a tree, then $\dim \mathcal{S}_G = N_G = 1$. This establishes the base of induction.

Suppose that $G$ is a graph with at least one edge. Pick an edge $e$ of $G$. Let $G\setminus e$ be the graph obtained from $G$ by removing the edge $e$, also let $G/e$ be the graph obtained from $G$ by contracting the edge $e$. Then $N_G = N_{G\setminus e} + N_{G/e}$. Indeed, for a spanning tree $T$ in $G$, we have either $e \notin T$ or $e \in T$. The former trees are exactly the spanning trees of $G\setminus e$. The latter trees are in a bijective correspondence with spanning trees of $G/e$. This correspondence is given by contracting the edge $e$. Assume by induction that the statement is true for both graphs $G\setminus e$ and $G/e$.

Let $\mathcal{S}'_G \subset \mathcal{S}_G$ be the span of the $\alpha_{H'}$ with slim subgraphs $H' \subset G$ such that $e \in H'$ and let $\mathcal{S}''_G \subset S_G$ be the span of the $\alpha_{H''}$ with slim subgraphs $H'' \subset G$ such that $e \notin H''$. Then the space $\mathcal{S}_G$ is spanned by $\mathcal{S}'_G$ and $\mathcal{S}''_G$. Thus

$$\dim \mathcal{S}_G = \dim \mathcal{S}'_G + \dim \mathcal{S}''_G - \dim(\mathcal{S}'_G \cap \mathcal{S}''_G). \tag{9.1}$$

We have $\mathcal{S}'_G = (y_i - y_j)\,\mathcal{S}_{G\setminus e}$, where $e = (i, j)$. Thus $\dim \mathcal{S}'_G = \dim \mathcal{S}_{G\setminus e}$. Let $p : f(y_1, \ldots, y_n) \mapsto f(y_1, \ldots, y_n) \bmod (y_i - y_j)$ be the natural projection. Then $p(\mathcal{S}''_G) = \mathcal{S}_{G/e}$ and $\mathcal{S}'_G \cap \mathcal{S}''_G \subset \mathrm{Ker}(p)$. Thus

$$\dim \mathcal{S}''_G = \dim \mathcal{S}_{G/e} + \dim \mathrm{Ker}(p) \geq \dim \mathcal{S}_{G/e} + \dim(\mathcal{S}'_G \cap \mathcal{S}''_G). \tag{9.2}$$

Combining (9.1) and (9.2), we get

$$\dim \mathcal{S}_G \geq \mathcal{S}_{G\setminus e} + \dim \mathcal{S}_{G/e}.$$

By the induction hypothesis, the right-hand side of this expression equals $N_{G\setminus e} + N_{G/e} = N_G$. Thus $\dim \mathcal{S}_G \geq N_G$. On the other hand, Lemma 9.5 implies that $\dim \mathcal{S}_G \leq N_G$. Thus $\dim \mathcal{S}_G = N_G$, as needed. □

## 10. Proof of Theorems 3.1, 3.3, and 5.2

Let $\{m_I \mid I \in \Sigma\}$ be a monotone monomial family, and let $\mathcal{I} = \langle m_I \mid I \in \Sigma\rangle$ be the corresponding monomial ideal in $\mathbb{K}[x_1, \ldots, x_n]$.

For a subset $I = \{i_1, \ldots, i_r\} \in \{1, \ldots, n\}$, let $\mathbb{K}[x_I] = \mathbb{K}[x_{i_1}, \ldots, x_{i_k}]$, and let $\mathfrak{M}_I$ denote the set of all monomials in the variables $x_i$, $i \in I$. Also let $\mathfrak{M} = \mathfrak{M}_{\{1,\ldots,n\}}$. For $I \in \Sigma$, let $M_I = m_I \cdot \mathfrak{M}$ be the set of all monomials in $\mathbb{K}[x_1, \ldots, x_n]$ divisible by $m_I$. The standard monomial basis $R$ of the algebra $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}$ is the set of monomials that survive in the algebra $\mathcal{A}$:

$$R = \mathfrak{M} \setminus \bigcup_{I \in \Sigma} M_I.$$

For $I, J \in \Sigma$, denote by $m_{J/I}$ the monomial obtained from $m_J$ by removing all $x_i$'s with $i \in I$, and let $M_{J/I} = m_{J/I} \cdot \mathfrak{M}_{\overline{I}}$, where $\overline{I} = \{1, \ldots, n\} \setminus I$. Let $\mathcal{I}_I$ be the monomial ideal in the polynomial ring $\mathbb{K}[x_{\overline{I}}]$ generated by the monomials $\{m_{J/I} \mid J \in \Sigma, J \not\subset I\}$. It follows from the monotonicity condition (MM2) that the ideal $\mathcal{I}_I$ is also generated by the set of monomials $\{m_{J/I} \mid J \in \Sigma, J \supsetneq I\}$. Notice that $\mathcal{I}_I$ is also a monotone monomial ideal. Let $R_I$ be the standard monomial basis of the algebra $\mathcal{A}_I = \mathbb{K}[x_{\overline{I}}]/\mathcal{I}_I$:

$$R_I = \mathfrak{M}_{\overline{I}} \setminus \bigcup_{J \supsetneq I} M_{J/I}.$$

**Proposition 10.1.** *The polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ decomposes into the direct sum of subspaces:*

$$\mathbb{K}[x_1, \ldots, x_n] = \langle R \rangle \oplus \bigoplus_{I \in \Sigma} m_I \, \mathbb{K}[x_I] \, \langle R_I \rangle,$$

*where $\langle R \rangle$ and $\langle R_I \rangle$ denote the linear spans of monomials in $R$ and $R_I$, respectively.*

**Lemma 10.2.** *For any monomial $x^a = x_1^{a_1} \cdots x_n^{a_n}$ in the ideal $\mathcal{I}$ there is a unique maximal by inclusion subset $J \in \Sigma$ such that $x^a \in M_I$.*

*Proof.* Let $\Sigma' = \{I \in \Sigma \mid x^a \in M_I\}$. If $I, J \in \Sigma'$, then, according to condition (MM3), there is an upper bound $K \in \Sigma$ of $I$ and $J$ such that $M_I \cap M_J \subseteq M_K$. Thus $M_K \in \Sigma'$. This implies that $\Sigma'$ has a unique maximal element. □

*Proof of Proposition 10.1.* For $I \in \Sigma$, let $M_I^{\max}$ be the following set of monomials:

$$M_I^{\max} = M_I \setminus \bigcup_{J \supsetneq I} M_J,$$

i.e., $M_I^{\max}$ is the set of monomials $x^a \in \mathfrak{M}$ such that $I$ is the maximal by inclusion subset $I \in \Sigma$ with $x^a \in M_I$; see Lemma 10.2. Thus the set of all monomials in $\mathbb{K}[x_1, \ldots, x_n]$ decomposes into the disjoint union

$$\mathfrak{M} = R \,\dot\cup\, \dot{\bigcup_{I \in \Sigma}} M_I^{\max}. \tag{10.1}$$

Using monotonicity condition (MM2), we obtain, for $I \subsetneq J$,

$$M_I \setminus M_J = m_I \times \mathfrak{M}_I \times (\mathfrak{M}_{\overline{I}} \setminus M_{J/I}),$$

where the notation "$\times$" means that every monomial in the left-hand side decomposes uniquely into the product of monomials. Thus we have

$$M_I^{\max} = \bigcap_{J \supsetneq I} (M_I \setminus M_J) = m_I \times \mathfrak{M}_I \times \bigcap_{J \supsetneq I} (\mathfrak{M}_{\overline{I}} \setminus M_{J/I}) = m_I \times \mathfrak{M}_I \times R_I. \tag{10.2}$$

Formulas (10.1) and (10.2) imply the required statement. □

Let $p_I$, $I \in \Sigma$, be a collection of polynomials such that $p_I$ is an $I$-deformation of the monomial $m_I$. Remarkably, a similar statement is valid for the polynomials $p_I$.

**Proposition 10.3.** *The polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ decomposes into the direct sum of subspaces:*

$$\mathbb{K}[x_1, \ldots, x_n] = \langle R \rangle \oplus \bigoplus_{I \in \Sigma} p_I \, \mathbb{K}[x_I] \, \langle R_I \rangle.$$

Proposition 10.3 immediately implies Theorem 5.2, which says that the monomials in $R$ linearly span the algebra $\mathcal{B} = \mathbb{K}[x_1,\dots,x_n]/\langle p_I \mid I \in \Sigma\rangle$.

**Lemma 10.4.** *Suppose that a polynomial $p \in \mathbb{K}[x_I]$ is an $I$-deformation of a monomial $m \in \mathbb{K}[x_I]$; see (5.1). Then for any polynomial $f \in \mathbb{K}[x_I]$ there exists a unique polynomial $g \in \mathbb{K}[x_I]$ such that the difference $m \cdot f - p \cdot g$ contains no monomials divisible by $m$. The map $f \mapsto g$ is one-to-one.*

*Proof.* According to the generosity condition (5.1) the polynomial $m \cdot f$, as well as any other polynomial in $\mathbb{K}[x_I]$, can be written uniquely in the form $m \cdot f = p \cdot g + r$, where $g \in \mathbb{K}[x_I]$ and $r$ is in the linear span $\langle R_m\rangle$ of monomials in $\mathbb{K}[x_I]$ not divisible by $m$. This defines the map $f \mapsto g$.

On the other hand, for any $g \in \mathbb{K}[x_I]$ there exist unique $f \in \mathbb{K}[x_I]$ and $r \in \langle R_m\rangle$ such that $p \cdot g = m \cdot f - r$. Thus the map $f \mapsto g$ is invertible. The statement of the lemma follows. □

*Proof of Proposition 10.3.* Pick any linear ordering $I_1,\dots,I_N$ of the set $\Sigma$ compatible with the inclusion relation, i.e, the inclusion $I_s \subset I_t$ implies that $s \le t$. Let $\Sigma_{(s)} = \{I_1,\dots,I_s\}$ and $\Sigma^{(s)} = \{I_s,\dots,I_N\}$ be initial and terminal intervals of $\Sigma$.

We will prove by induction on $N - s$ that the polynomial ring $\mathbb{K}[x_1,\dots,x_n]$ decomposes into the direct sum of subspaces

$$\mathbb{K}[x_1,\dots,x_n] = \langle R\rangle \oplus \bigoplus_{I' \in \Sigma_{(s)}} m_{I'}\,\mathbb{K}[x_{I'}]\,\langle R_{I'}\rangle \oplus \bigoplus_{I'' \in \Sigma^{(s+1)}} p_{I''}\,\mathbb{K}[x_{I''}]\,\langle R_{I''}\rangle. \tag{10.3}$$

If $s = N$, then (10.3) is true according to Proposition 10.1. This gives the base of induction.

Assume by the induction hypothesis that (10.3) holds for some $s$ and derive the same statement for $s-1$. Let $I = I_s$. For a polynomial $\phi \in \mathbb{K}[x_1,\dots,x_n]$, write its unique presentation

$$\phi = r + \sum_{I' \in \Sigma_{s-1}} m_{I'} \cdot f_{I'} \cdot r_{I'} + m_I \cdot f_I \cdot r_I + \sum_{I'' \in \Sigma^{s+1}} p_{I''} \cdot f_{I''} \cdot r_{I''}, \tag{10.4}$$

where $r \in \langle R\rangle$ and $f_J \in \mathbb{K}[x_J]$ and $r_J \in R_J$, for any $J \in \Sigma$.

Let $\tilde f_I \in \mathbb{K}[x_I]$ be the unique polynomial, provided by Lemma 10.4, such that the difference $d = m_I \cdot f_I - p_I \cdot \tilde f \in \mathbb{K}[x_I]$ contains no monomials divisible by $m_I$. Let $\psi \in \mathbb{K}[x_1,\dots,x_n]$ be the polynomial obtained from $\phi$ by keeping all terms in (10.4) except for $m_I \cdot f_I \cdot r_I$ which we substitute by the term $p_I \cdot \tilde f_I \cdot r_I$. Then $\phi - \psi = d \cdot r_I$. Pick any monomial $e$ in $d$. Recall that, according to (10.2), $M_J^{\max}$ is the set of all monomials in $m_J\,\mathbb{K}[x_J]\,\langle R_J\rangle$. If $e \cdot r_I \notin M_J^{\max}$ for all $J \in \Sigma$, then $e \cdot r_I \in \langle R\rangle$. Otherwise, suppose that $e \cdot r_I \in M_J^{\max}$ for some $J$. If $J \not\subset I$, then $e \cdot r_I \in M_J^{\max} \subset M_J$ implies that $r_I \in M_{J/I}$, which is impossible. Thus $J \subset I$. Also $J \ne I$ because $e$ is not divisible by $m_I$. This shows that

$$\phi - \psi \in \langle R\rangle \oplus \bigoplus_{J \subsetneq I} \langle M_J^{\max}\rangle.$$

Therefore, $\phi$ can be written as

$$\phi = \tilde r + \sum_{I' \in \Sigma_{s-1}} m_{I'} \cdot \tilde f_{I'} \cdot \tilde r_{I'} + p_I \cdot \tilde f_I \cdot r_I + \sum_{I'' \in \Sigma^{s+1}} p_{I''} \cdot f_{I''} \cdot r_{I''}, \tag{10.5}$$

where $\tilde r \in \langle R\rangle$, $\tilde f_{I'} \in \mathbb{K}[x_{I'}]$, $\tilde r_{I'} \in R_{I'}$, and $f_{I''}$ and $r_{I''}$ are the same as before.

Notice that all steps in the transformation of the presentation (10.4) to the presentation (10.5) are invertible. Also if $p_I \cdot \tilde{f}_I \cdot r_I = 0$, then all summands in (10.4) and (10.5) coincide. So, if at least the one of the summands in the presentation (10.5) of $\phi = 0$ is nonzero, then we can also find a nonzero presentation of the form (10.4) for $\phi = 0$, which is impossible by the induction hypothesis. This shows that the presentation (10.5) of $\phi$ is unique.

This proves (10.3). For $s = 0$ we obtain the claim of Proposition 10.3. ☐

Finally we can put everything together and prove Theorems 3.1 and 3.3.

*Proof of Theorems 3.1 and 3.3.* For a graph $G$, let $\mathcal{A}_G$, $\mathcal{B}_G$, and $\mathcal{C}_G$ be the algebras defined in Sections 2, 3, and 9. Then we have the following termwise inequalities of Hilbert series:

$$\mathrm{Hilb}\,\mathcal{A}_G \geq \mathrm{Hilb}\,\mathcal{B}_G \geq \mathrm{Hilb}\,\mathcal{C}_G. \tag{10.6}$$

The first inequality follows from Theorem 5.2 because $\mathcal{I}_G$ is a monotone monomial ideal and, by Lemma 5.1, $\mathcal{J}_G$ is its deformation. The second inequality follows from Lemma 9.2, which says that $\mathcal{C}_G$ is a quotient of $\mathcal{B}_G$. Theorem 2.1 claims that $\dim \mathcal{A}_G = N_G$ is the number of spanning trees of the graph $G$. On the other hand, by Theorem 9.1, $\dim \mathcal{C}_G = N_G$. Thus all inequalities in (10.6) are actually equalities. Moreover, by Theorem 9.1, the dimensions of $k$-th graded components are equal to

$$\dim \mathcal{A}_G^k = \dim \mathcal{B}_G^k = \dim \mathcal{C}_G^k = N_G^{|G|-n-k},$$

the number of spanning trees of $G$ with external activity $|G| - n - k$. ☐

**Corollary 10.5.** *The algebras $\mathcal{B}_G$ and $\mathcal{C}_G$ are isomorphic.*

## 11. Algebras related to forests

Definitions of the algebras $\mathcal{B}_G$ and $\mathcal{C}_G$ and the proof of Theorem 9.1 are similar to constructions from [PSS1]. Let us briefly review some results from [PSS1].

Let $G$ be a graph on the vertices $0, \ldots, n$. Let $\hat{\mathcal{J}}_G$ be the ideal in $\mathbb{K}[x_1, \ldots, x_n]$ generated by the polynomials

$$\hat{p}_I = \left(\sum_{i \in I} x_i\right)^{D_I + 1},$$

where $I$ ranges over all nonempty subsets in $\{1, \ldots, n\}$ and the number $D_I$ is the same as in Section 3; cf. (3.1). Let $\hat{\mathcal{B}}_G = \mathbb{K}[x_1, \ldots, x_n]/\hat{\mathcal{J}}_G$.

Let $\hat{\Phi}_G$ be the commutative algebra generated by the variables $\hat{\phi}_e$, $e \in G$, with the defining relations:

$$(\hat{\phi}_e)^2 = 0, \qquad \text{for any edge } e.$$

Also, let $\hat{\mathcal{C}}_G$ be the subalgebra of $\hat{\Phi}_G$ generated by the elements

$$\hat{X}_i = \sum_{e \in G} e_{i,e} \hat{\phi}_e,$$

for $i = 1, \ldots, n$; cf. Section 9.

A *forest* is a graph without cycles. The connected components of a forest are trees. A *subforest* in a graph $G$ is a subgraph $F \subset G$ without cycles. Fix a linear order of edges of $G$. An edge $e \in G \setminus F$ is called *externally active* for a forest $F$ if there exists a cycle $C$ in $G$ such that $e$ is the minimal element of $C$ and

$(C \setminus \{e\}) \subset F$. The *external activity* of $F$ is the number of externally active edges for $F$.

**Theorem 11.1** ([PSS1]). *The algebras $\hat{\mathcal{B}}_G$ and $\hat{\mathcal{C}}_G$ are isomorphic to each other. Their dimension is equal to the number of subforests in the graph $G$.*

*The dimension $\dim \hat{\mathcal{B}}_G^k$ of the $k$-th graded component of the algebra $\hat{\mathcal{B}}_G$ equals the number of subforests $F$ of $G$ with external activity $|G|-|F|-k$.*

In [PSS2] we investigated the algebra $\hat{\mathcal{B}}_G$ for the graph $G = K_{n+1}$. Let $\hat{\mathcal{I}}_n = \langle \hat{m}_I \rangle$ and $\hat{\mathcal{J}}_n = \langle \hat{p}_I \rangle$ be two ideals in the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ generated by the monomials $\hat{m}_I$ and the polynomials $\hat{p}_I$, correspondingly, given by

$$\hat{m}_I = (x_{i_1} \cdots x_{i_r})^{n-r+1} x_{i_1},$$

$$\hat{p}_I = (x_{i_1} + \cdots + x_{i_r})^{r(n-r+1)+1},$$

where $I = \{i_1 < \cdots < i_r\}$ ranges over nonempty subsets of $\{1, \dots, n\}$; cf. Subsection 4.1. Notice that $\hat{\mathcal{I}}_n$ is a monotone monomial ideal and $\hat{\mathcal{J}}_n$ is its deformation. Let $\hat{\mathcal{A}}_n = \mathbb{K}[x_1, \dots, x_n]/\hat{\mathcal{I}}_n$ and $\hat{\mathcal{B}}_n = \mathbb{K}[x_1, \dots, x_n]/\hat{\mathcal{J}}_n$.

Let us say that a nonnegative integer sequence $b = (b_1, \dots, b_n)$ is an *almost parking function* of size $n$ if the monomial $x^b = x_1^{b_1} \cdots x_n^{b_n}$ does not belong to the ideal $\hat{\mathcal{I}}_n$. Clearly the class of almost parking functions includes usual parking functions.

For a forest $F$ on the vertices $0, \dots, n$, an *inversion* is a pair of vertices labelled $i$ and $j$ such that $i > j$ and the vertex $i$ belong to the path in $F$ that joins the vertex $j$ with the minimal vertex in its connected component.

**Theorem 11.2** ([PSS1, PSS2]). *The algebras $\hat{\mathcal{A}}_n$ and $\hat{\mathcal{B}}_n$ have the same Hilbert series. The dimension of these algebras is equal to the number of forests on $n+1$ vertices.*

*Moreover, the dimension $\dim \hat{\mathcal{A}}_n^k = \dim \hat{\mathcal{B}}_n^k$ of the $k$-th graded components of the algebras $\hat{\mathcal{A}}_n$ and $\hat{\mathcal{B}}_n$ is equal to*

(A) *the number of almost parking functions $b$ of size $n$ such that $\sum_{i=1}^n b_i = k$;*
(B) *the number of forests on $n+1$ vertices with external activity $\binom{n}{2} - k$;*
(C) *the number of forests on $n+1$ vertices with $\binom{n}{2} - k$ inversions.*

*The images of the monomials $x^b$, where $b$ ranges over almost parking functions of size $n$, form linear bases in both algebras $\hat{\mathcal{A}}_n$ and $\hat{\mathcal{B}}_n$.*

Theorem 11.2, first stated in [PSS2], follows from results of [PSS1]. The algebra $\hat{\mathcal{B}}_n$ is the algebra generated by curvature forms on the complete flag manifold. It was introduced in an attempt to lift Schubert calculus on the level of differential forms; see [PSS1, PSS2, ShSh]. This example related to Schubert calculus was our original motivation.

## 12. $\rho$-ALGEBRAS AND $\rho$-PARKING FUNCTIONS

We conclude the paper with a discussion of a special class of monotone monomial ideals and their deformations.

Let $\rho = (\rho_1, \dots, \rho_n)$ be a weakly decreasing sequence of nonnegative integers, called a *degree function*. Let $\mathcal{I}_\rho = \langle m_I \rangle$ and $\mathcal{J}_\rho = \langle p_I \rangle$ be the ideals the ring

$\mathbb{K}[x_1, \ldots, x_n]$ generated by the monomials $m_I$ and the polynomials $p_I$, correspondingly, given by

$$m_I = (x_{i_1} \cdots x_{i_r})^{\rho_r},$$

$$p_I = (x_{i_1} + \cdots + x_{i_r})^{r \cdot \rho_r},$$

where in both cases $I = \{i_1, \ldots, i_r\}$ runs over all nonempty subsets of $\{1, \ldots, n\}$. Let $\mathcal{A}_\rho = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}_\rho$ and $\mathcal{B}_\rho = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{J}_\rho$.

Let us say that a nonnegative integer sequence $b = (b_1, \ldots, b_n)$ is a *$\rho$-parking function* if the monomial $x_1^{b_1} \cdots x_n^{b_n}$ does not belong to the ideal $\mathcal{I}_\rho$. More explicitly, this condition can be reformulated as follows. A nonnegative integer sequence $b = (b_1, \ldots, b_n)$ is a $\rho$-parking function if and only if, for $r = 1, \ldots, n$, we have

$$\#\{i \mid b_i < \rho_{n-r+1}\} \geq r.$$

This condition can also be formulated in terms of the increasing rearrangement $c_1 \leq \cdots \leq c_n$ of the elements of $b$ as $c_i < \rho_{n+1-i}$. The $\rho$-parking functions were studied in [PiSt] and in [Yan]. Earlier they appeared under a different name in [PP]. Notice that $(n, \ldots, 1)$-parking functions are exactly the usual parking functions of size $n$.

The monomials $x^b$, where $b$ ranges over $\rho$-parking functions, form the standard monomial basis of the algebra $\mathcal{A}_\rho$. Thus the Hilbert series of the algebra $\mathcal{A}_\rho$ equals

$$\operatorname{Hilb} \mathcal{A}_\rho = \sum_b q^{b_1 + \cdots + b_n},$$

where the sum is over $\rho$-parking functions. The dimension $\dim \mathcal{A}_\rho$ of this algebra is equal to the number of $\rho$-parking functions.

Theorem 5.2 specializes to the following statement.

**Corollary 12.1.** *The monomials $x^b$, where $b$ ranges over $\rho$-parking functions, linearly span the algebra $\mathcal{B}_\rho$. Thus we have the termwise inequality of Hilbert series:*

$$\operatorname{Hilb} \mathcal{A}_\rho \geq \operatorname{Hilb} \mathcal{B}_\rho.$$

It would be interesting to describe the class of degree functions $\rho$ such that $\operatorname{Hilb} \mathcal{A}_\rho = \operatorname{Hilb} \mathcal{B}_\rho$. If $\rho_r = l + k(n-r)$ is a linear degree function, then, according to Corollary 4.3, the Hilbert series of $\mathcal{A}_\rho$ and $\mathcal{B}_\rho$ are equal to each other and

$$\dim \mathcal{A}_\rho = \dim \mathcal{B}_\rho = l\,(l + kn)^{n-1}.$$

For $n = 3$, Schenck [Sche] gave another proof of this fact using ideals of fatpoints.

Let us say that a degree function $\rho$ is *almost linear* if there exists an integer $k$ such that $\rho_i - \rho_{i+1}$ equals either $k$ or $k+1$, for $i = 1, \ldots, n-1$. Computer experiments show that the equality $\operatorname{Hilb} \mathcal{A}_\rho = \operatorname{Hilb} \mathcal{B}_\rho$ often holds for almost linear degree functions $\rho$. The table below lists some almost linear degree functions, for which the equality $\operatorname{Hilb} \mathcal{A}_\rho = \operatorname{Hilb} \mathcal{B}_\rho$ holds.

On the other hand, the equality of Hilbert series fails for the almost linear degree functions $\rho = (9, 6, 3, 1)$ and $\rho = (9, 7, 5, 4, 3)$. We do not know an example when $\operatorname{Hilb} \mathcal{A}_\rho = \operatorname{Hilb} \mathcal{B}_\rho$ and $\rho$ is not almost linear.

The ideal $\mathcal{I}_\rho$ is a *strictly* monotone monomial ideal provided that the degree function is strictly decreasing $\rho_1 > \cdots > \rho_n > 0$. Corollary 6.4 gives a minimal free resolution for this ideal. Recall that $S = \mathbb{K}[x_1, \ldots, x_n]$ and $S(-d)$ is the free $\mathbb{Z}$-graded $S$-module of rank 1 with generator of degree $d$.

| $\rho$ | $\dim \mathcal{A}_\rho$ | $\rho$ | $\dim \mathcal{A}_\rho$ |
|---|---|---|---|
| $(4,2,1)$ | 25 | $(8,5,3)$ | 306 |
| $(8,5,1)$ | 142 | $(8,6,3)$ | 351 |
| $(6,4,3)$ | 153 | $(11,7,2)$ | 506 |
| $(9,5,2)$ | 290 | $(12,8,3)$ | 855 |
| $(6,4,3,2)$ | 632 | $(8,7,5,3)$ | 3021 |
| $(9,6,4,2)$ | 2512 | $(9,7,6,5)$ | 4925 |
| $(8,6,5,3)$ | 2643 | $(11,8,6,3)$ | 7587 |
| $(8,6,5,4)$ | 2832 | $(12,9,7,4)$ | 12460 |
| $(9,8,6,4,2)$ | 31472 | $(10,9,7,5,3)$ | 65718 |

**Corollary 12.2.** *Let $\rho$ be a degree function such that $\rho_1 > \cdots > \rho_n > 0$. The ideal $\mathcal{I}_\rho$ has a minimal free resolution of the form*

$$\cdots \longrightarrow C_3 \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 = S \longrightarrow S/\mathcal{I}_\rho \longrightarrow 0,$$

*with*

$$C_k = \bigoplus_{l_1,\dots,l_k} S(-d(l_1,\dots,l_k))^{\binom{n}{l_1,\cdots,l_k}},$$

*where the direct sum is over $l_1,\dots,l_k \geq 1$ such that $l_1+\cdots+l_k \leq n$,*

$$d(l_1,\dots,l_k) = l_1\,\rho(l_1) + l_2\,\rho(l_1+l_2) + \cdots + l_k\,\rho(l_1+\cdots+l_k),$$

*and $\binom{n}{l_1,\cdots,l_k} = \frac{n!}{l_1!\cdots l_k!(n-l_1-\cdots-l_k)!}$ is the multinomial coefficient.*

Conjecture 6.10 implies that if $\dim \mathcal{A}_\rho = \dim \mathcal{B}_\rho$, then the ideals $\mathcal{I}_\rho$ and $\mathcal{J}_\rho$ have the same graded Betti numbers. It is already a nontrivial open problem to prove (or disprove) that the graded Betti numbers of the ideal $\mathcal{J}_\rho$, for a linear degree function $\rho$, are given by the expression in Corollary 12.2. Schenck's results [Sche] for $n=3$ seem to support this conjecture.

Proposition 8.1 specializes to an expression for the Hilbert series $\operatorname{Hilb}\mathcal{A}_\rho$ with alternating signs. Actually, in this case it is possible to give a simpler subtraction-free expression for the Hilbert series. The following statement is a slight enhancement of a result of Pitman and Stanley, who gave a formula for the number of $\rho$-parking functions.

**Proposition 12.3** (cf. [PiSt, Theorem 11])**.** *The Hilbert series of the algebra $\mathcal{A}_\rho$ equals*

$$\operatorname{Hilb}\mathcal{A}_\rho = \sum_a \prod_{i=1}^n \frac{q^{\rho_{n-a_i+1}} - q^{\rho_{n-a_i}}}{1-q}, \tag{12.1}$$

*where the sum is over $(n+1)^{n-1}$ usual parking functions $a=(a_1,\dots,a_n)$ of size $n$. Here we assume that $\rho_{n+1}=0$. Thus the dimension of $\mathcal{A}_\rho$, which is the number of $\rho$-parking functions, is given by the following polynomial in $\rho_1,\dots,\rho_n$:*

$$\dim \mathcal{A}_\rho = \sum_a \prod_{i=1}^n (\rho_{n-a_i} - \rho_{n-a_i+1}),$$

*where again the sum is over usual parking functions of size $n$.*

*Proof.* For $i=0,\dots,n$, let $Z_i$ be the interval of integers $Z_i = [\rho_{n-i+1}, \rho_{n-i}[$, where we assume that $\rho_0 = +\infty$ and $\rho_{n+1}=0$. Then the set of positive integers is the disjoint union of $Z_0,\dots,Z_n$. Let $f : b \mapsto a$ be the map that sends a positive integer sequence $b=(b_1,\dots,b_n)$ to the sequence $a=(a_1,\dots,a_n)$ such that $b_i \in Z_{a_i}$ for

$i = 1, \dots, n$. Then $b$ is a $\rho$-parking function if and only if $a$ is a usual parking function of size $n$. Fix a parking function $a$ of size $n$. Then

$$\sum_{b\colon f(b)=a} q^{b_1+\cdots+b_n} = \prod_{i=1}^{n} \sum_{b_i \in Z_{a_i}} q^{b_i}$$

is exactly the summand in (12.1). $\square$

For example, the Hilbert series of $\mathcal{A}_\rho$, for $n = 2$ and $n = 3$, are given by

$$\operatorname{Hilb} \mathcal{A}_{(\rho_1,\rho_2)}(q) = [\rho_2]^2 + 2q^{\rho_2}[\rho_1 - \rho_2]\,[\rho_2]\,,$$

$$\begin{aligned}\operatorname{Hilb} \mathcal{A}_{(\rho_1,\rho_2,\rho_3)}(q) &= [\rho_3]^3 + 3q^{\rho_3}[\rho_2 - \rho_3]\,[\rho_3]^2 + 3q^{2\rho_3}[\rho_2 - \rho_3]^2\,[\rho_3] \\ &\quad + 3q^{\rho_2}[\rho_1 - \rho_2]\,[\rho_3]^2 + 6q^{\rho_2+\rho_3}[\rho_1 - \rho_2]\,[\rho_2 - \rho_3]\,[\rho_3]\,,\end{aligned}$$

where $[s] = 1 + q + \cdots + q^{s-1}$ denotes the $q$-analogue of an integer $s$.

Finally, we formulate a theorem that gives a combinatorial interpretation of the value of the Hilbert series $\operatorname{Hilb} \mathcal{A}_\rho$ at $q = -1$. This theorem follows from results of [PP] on $\rho$-parking functions.

**Theorem 12.4** ([PP]). *The number $(-1)^{\rho_1+\cdots+\rho_n-n} \operatorname{Hilb} \mathcal{A}_\rho(-1)$ equals the number of permutations $\sigma_1, \dots, \sigma_n$ of $1, \dots, n$ such that*

$$\sigma_1 \vee^{\rho_1} \sigma_2 \vee^{\rho_2} \cdots \vee^{\rho_{n-1}} \sigma_n \vee^{\rho_n} 0\,,$$

*where the notation $a \vee^k b$ means that $a < b$ for even $k$ and $a > b$ for odd $k$. In particular, $\operatorname{Hilb} \mathcal{A}_\rho(-1)$ is zero if and only if $\rho_n$ is even.*

This theorem basically says that $\operatorname{Hilb} \mathcal{A}_\rho(-1)$ is either 0 or plus/minus the number of permutations with prescribed descent positions.

In the case of usual parking functions of size $n$, i.e., for $\rho = (n, \dots, 1)$, this theorem amounts to the well-known result of Kreweras [Krew] that the value of the inversion polynomial $I_n(-1) = (-1)^{\binom{n}{2}} \operatorname{Hilb} \mathcal{A}_{(n,\dots,1)}(-1)$ is the number of alternating permutations of size $n$.

## 13. Appendix: Abelian sandpile model

In this appendix we discuss the *abelian sandpile model*, also known as the *chip-firing game*. It was introduced by Dhar [Dhar] and was studied by many authors. We review the sandpile model for a class of toppling matrices introduced by Gabrielov [Gab2], which is more general than in [Dhar]. Then we show how $G$-parking functions from Section 2 are related to this model.

Let $\Delta = (\Delta_{ij})_{1\le i,j\le n}$ be an integer $n \times n$-matrix. We say that $\Delta$ is a *toppling matrix* if it satisfies the following two conditions:

(13.1) $\quad \Delta_{ij} \le 0$, for $i \ne j$; and there exists a vector $h > 0$ such that $\Delta\, h > 0$.

Here the notation $h > 0$ means that all coordinates of $h$ are strictly positive. Notice that conditions (13.1) imply that $\Delta_{ii} > 0$ for any $i$. These matrices appeared in [Gab2] under the name *avalanche-finite redistribution matrices*.

Let us list some properties of toppling matrices. Recall that $L_G$ denotes the truncated Laplace matrix that corresponds to a digraph $G$ on the vertices $0, \dots, n$; see (2.2) in Section 2.

**Proposition 13.1** (cf. [Gab2]). **1**. *A matrix* $\Delta$ *is a toppling matrix if and only if its transposed matrix* $\Delta^T$ *is a toppling matrix.*

**2**. *Every integer matrix* $\Delta$ *such that*

$$\Delta_{ij} \leq 0 \text{ for } i \neq j; \qquad \sum_j \Delta_{ij} \geq 0 \text{ for all } i; \qquad \det \Delta \neq 0 \tag{13.2}$$

*is a toppling matrix. Equivalently, the truncated Laplace matrix* $\Delta = L_G$ *corresponding to a digraph* $G$ *with at least one oriented spanning tree is a toppling matrix.*

**3**. *If* $\Delta$ *is a toppling matrix, then all principal minors of* $\Delta$ *are strictly positive.*

**4**. *If* $\Delta$ *is a symmetric integer matrix with nonpositive off-diagonal entries, then* $\Delta$ *is a toppling matrix if and only if it is positive-definite.*

*Proof.* **1.** This claim follows from [Gab2, Theorem 1.5]. It also follows from the result of [Kac, Theorem 4.3], obtained for classification of generalized Cartan matrices.

**2.** Conditions (13.2) are equivalent to the statement that $\Delta = L_G$ is the truncated Laplace matrix for some digraph $G$ with at least one oriented spanning tree; see the Matrix-Tree Theorem, equation (2.1) in Section 2. Let $\mathrm{dist}(i)$ be the length of the shortest directed path in the digraph $G$ from the vertex $i$ to the root 0, and let $h(\epsilon) = (h_1, \ldots, h_n)^T$, where $h_i = 1 - \epsilon^{\mathrm{dist}(i)}$. Then $\Delta\, h(\epsilon) > 0$ for sufficiently small $\epsilon > 0$. Indeed, the $i$-th coordinate of the vector $\Delta\, h(\epsilon)$ is

$$a_{i0}(1 - \epsilon) + \sum_{j \neq 0, i} a_{ij}(\epsilon^{\mathrm{dist}(j)} - \epsilon^{\mathrm{dist}(i)}),$$

where the $a_{ij}$ are the entries the adjacency matrix of $G$. The leading term of this expression has order of $\epsilon^{\mathrm{dist}(i)-1}$ and is strictly positive.

**3.** The fact that $\det \Delta > 0$ is given in [Gab2, Proposition 1.12]. Let us show that it also easily follows from the Matrix-Tree theorem. Let $\Delta$ be a toppling matrix and $h = (h_1, \ldots, h_n)^T > 0$ be an integer vector such that $\Delta\, h > 0$. Then all row sums of the matrix $\tilde{\Delta} = \Delta \cdot \mathrm{diag}(h_1, \ldots, h_n)$ are positive. This means $\tilde{\Delta} = L_G$ is the truncated Laplace matrix for some digraph $G$. The $i$-th row sum of $\tilde{\Delta}$ is the number of edges in $G$ connecting the vertex $i$ with the root 0. According to the Matrix-Tree Theorem, the determinant $\det \tilde{\Delta}$ is the number of oriented spanning trees in the digraph $G$. This number is positive, because each vertex is connected with the root by an edge in $G$. Thus $\det \Delta = (h_1 \cdots h_n)^{-1} \det \tilde{\Delta} > 0$. Any principal minor of $\tilde{\Delta}$ also has positive row sums. The same argument holds for the minors.

**4.** This claim follows from [Kac, Lemma 4.5]. □

Let us now fix a toppling matrix $\Delta$, and let $\Delta_i = (\Delta_{i1}, \ldots, \Delta_{in})$ be the $i$-th row of $\Delta$. A *configuration* $u = (u_1, \ldots, u_n)$ is a vector of nonnegative integers. In the sandpile model, the number $u_i$ is interpreted as the the number of particles, or grains of sand, at site $i = 1, \ldots, n$. A site $i$ is *critical* if $u_i \geq \Delta_{ii}$. A *toppling* at a critical site $i$ consists in subtraction the vector $\Delta_i$ from the vector $u$. In other words, toppling at site $i$ decreases $u_i$ by $\Delta_{ii}$ particles and increases $u_j$ by $-\Delta_{ij}$ particles, for all $j \neq i$. A configuration $u$ is called *stable* if no toppling is possible, i.e., $0 \leq u_i < \Delta_{ii}$ for all sites $i$.

Dhar [Dhar] assumed that the toppling matrix $\Delta$ has nonnegative row sums, i.e., he assumed that $\Delta = L_G$ satisfies conditions (13.2). In this case a toppling cannot increase the total number of particles. When a toppling occurs, some of the particles at site $i$ are distributed among the neighboring sites and some particles are removed from the system. While this condition is important from the physical point of view, it is not really necessary for the following algebraic constructions; cf. Gabrielov [Gab2]. Moreover, there are interesting examples, for which this condition fails. We still attribute the following results to Dhar even though we will not assume that $\Delta$ has nonnegative row sums. The proofs that we include for completeness sake are close to the proofs from [Dhar].

**Lemma 13.2** ([Gab2]; cf. [Dhar]). *Every configuration can be transformed into a stable configuration by a sequence of topplings. This stable configuration does not depend on the order in which topplings are performed.*

*Proof.* Conditions (13.1) for a toppling matrix imply that there exists a vector $h > 0$ such that $(h, \Delta_i) > 0$ for any $i$. For a configuration $u$, the value $h(u)$ is nonnegative and every toppling strictly decreases this value. Thus, after at most $h(u)/\min(h, \Delta_i)$ topplings, the configuration $u$ transforms into a stable configuration.

If an unstable configuration $u$ has two critical sites $i$ and $j$, then $j$ is still a critical site for $u - \Delta_i$. Thus it is possible to perform a toppling at site $i$ followed by a toppling at site $j$ producing the configuration $u - \Delta_i - \Delta_j$. This operation is symmetric in $i$ and $j$. Using this argument repeatedly, we deduce that the final stable configuration does not depend on the order of topplings. ☐

The *avalanche operators* $A_1, \ldots, A_n$ map the set of stable configurations to itself. The operator $A_i$ is given by adding 1 particle at site $i$, i.e., increasing $u_i$ by 1, and then performing a sequence of topplings that lead to a new stable configuration.

**Lemma 13.3** ([Dhar]). *The avalanche operators $A_1, \ldots, A_n$ commute pairwise.*

*Proof.* The stable configuration $A_i A_j u$ is obtained from $u$ by adding a particle at site $j$, then performing a sequence of topplings, and then adding a particle at site $i$ and performing another sequence of topplings. If we first add two particles at sites $i$ and $j$, then all topplings in these two sequences are still possible and lead to the same stable configuration. This shows that $A_i$ and $A_j$ commute. ☐

The *abelian sandpile model* is the random walk on the set of stable configurations that is given by picking a site $i$ at random with some probability $p_i > 0$ and performing the avalanche operator $A_i$. Informally, we can describe it as the model where we drop a grain of sand at a random site and allow the system to settle to a stable configuration.

Dhar described the steady state of this random walk. A stable configuration $u$ is called *recurrent* if there are positive integers $c_i$ such that $A_i^{c_i} u = u$ for all $i$. Let $\mathcal{R}$ denote the set of recurrent configurations. The commutativity of the avalanche operators $A_i$ implies that the set $\mathcal{R}$ is closed under the action of these operators. Moreover, the operators $A_i$ are invertible on the set $\mathcal{R}$. Indeed, $A_i^{-1} u$ can be defined as $a_i^{c_i - 1} u$ for a recurrent configuration $u$. According to the theory of Markov chains all recurrent configurations have the same nonzero probability of occurrence in the steady state and all nonrecurrent configurations have zero probability.

The *sandpile group* $SG$, also known as the *critical group*, is the finite abelian group generated by the avalanche operators $A_1, \ldots, A_n$ acting on the set $\mathcal{R}$.

**Theorem 13.4** ([Dhar]). *The sandpile group is isomorphic to the quotient of the integer lattice $SG \simeq \mathbb{Z}^n/\langle\Delta\rangle$, where $\langle\Delta\rangle = \mathbb{Z}\Delta_1 \oplus \cdots \oplus \mathbb{Z}\Delta_n$ is the sublattice in $\mathbb{Z}^n$ spanned by the vectors $\Delta_i$. The order of this group is equal to the number of recurrent configurations and is given by $|SG| = |\mathcal{R}| = \det\Delta$.*

*Proof.* Since there are finitely many recurrent configurations, we may assume that the numbers $c_i$ are the same for all recurrent configurations. For a recurrent configuration $u \in \mathcal{R}$ and an integer vector $v = (v_1, \ldots, v_n)$, let $A^v u = A_1^{v_1} \cdots A_n^{v_n} u$. Then $u = A^{u-v} v$ for any $u, v \in \mathcal{R}$. Indeed, the configuration $A^{u-v} v$ is given by performing topplings to the configuration $(u - v + Nc) + v = u + Nc$, where $c = (c_1, \ldots, c_n)$ and $N$ an integer large enough to make the vector $u - v + Nc$ positive. The result of these toppling equals $A^{Nc} u = u$. This shows that $SG$ acts transitively on $\mathcal{R}$. If an element of $SG$ stabilizes a configuration $u \in \mathcal{R}$, then, by transitivity, it stabilizes any other element of $\mathcal{R}$ and is the identity in $SG$. Thus the order of the sandpile group $SG$ equals $|\mathcal{R}|$. The bijection between $SG$ and $\mathcal{R}$ is given by $A^v \mapsto A^v \cdot u_*$, where $u_*$ is any fixed element of $\mathcal{R}$.

If we add $\Delta_{ii}$ particles at site $i$ to a configuration $u$ and perform a toppling at the (unstable) site $i$, the result will be the same as adding $-\Delta_{ij}$ particles at all other sites $j \neq i$. Thus

$$A_i^{\Delta_{ii}} = \prod_{j \neq i} A_j^{-\Delta_{ij}}, \text{ or equivalently, } A^{\Delta_i} = 1 \text{ for any } i.$$

On the other hand, $A^v \neq 1$ if $v \notin \langle\Delta\rangle$, since topplings are given by subtraction of the vectors $\Delta_i$ and $A^v u \in u + v + \langle\Delta\rangle$. This shows that $A^v = 1$ if and only if $v \in \langle\Delta\rangle$ and the map $v \mapsto A^v$ is an isomorphism between the sandpile group $SG$ and the quotient $\mathbb{Z}^n/\mathbb{Z}\Delta$.

Finally, the order of $\mathbb{Z}^n/\langle\Delta\rangle$ equals $\det\Delta$. □

Dhar suggested a more explicit characterization of the set $\mathcal{R}$ of recurrent configurations. Let us say that a configuration $u$ is *allowed* if for any nonempty subset $I$ of sites there exists $j \in I$ such that

$$u_j \geq \sum_{i \in I \setminus \{j\}} (-\Delta_{ij}).$$

**Proposition 13.5** ([Dhar]). *Every recurrent configuration is allowed.*

*Proof.* Let $u$ be a recurrent configuration. Then $A^c u = u$, where $c = (c_1, \ldots, c_n)$ and $c_i > 0$ for all $i$. This means that there exists a sequence of sites $i_1, \ldots, i_k$ such that (i) $\Delta_{i_1} + \cdots + \Delta_{i_k} = c$, and (ii) $u + \Delta_{i_1} + \cdots + \Delta_{i_r} \geq 0$ for any $r = 1, \ldots, k$. Since all coordinates of $\Delta_i$, except the $i$-th coordinate, are nonpositive and $c > 0$, condition (i) implies that the sequence $i_1, \ldots, i_k$ contains all sites $1, \ldots, n$ at least once.

Let us say that a configuration $v$ is *$I$-forbidden*, for some subset $I$ of sites, if

$$0 \leq v_j < \sum_{i \in I \setminus \{j\}} (-\Delta_{ij}),$$

for all $j \in I$. If $v$ is $I$-forbidden and $v + \Delta_i \geq 0$, for some site $i$, then the configuration $v + \Delta_i$ is $I \setminus \{i\}$-forbidden. Also notice that there are no $\emptyset$-forbidden configurations.

Suppose that the recurrent configuration $u$ is not allowed. Then $u$ is $I$-forbidden for some subset $I$. We obtain by induction on $r = 0, \ldots, k$ that the configuration

$u_{(r)} = u + \Delta_{i_1} + \cdots + \Delta_{i_r}$ is $I_r$-forbidden, where $I_r = I \setminus \{i_1, \ldots, i_r\}$. In particular, $u_{(k)}$ is $I_k$-forbidden, where $I_k = \emptyset$, which is impossible. This shows that the configuration $u$ is allowed. □

Dhar suggested that a configuration is recurrent if and only if it is stable and allowed. Gabrielov [Gab1, Section 3, Appendix E] showed that this statement is not true in general, and proved the conjecture for a toppling matrix $\Delta$ with nonnegative column sums and, in particular, for a symmetric toppling matrix $\Delta = L_G$ corresponding to an undirected graph $G$. For symmetric $\Delta = L_G$, Dhar's conjecture was also proved by Ivashkevich and Priezzhev [IvPr], and recently by Meester, Redig, Znamenski [MRZ, Theorem 5.4], and by Cori, Rossin, Salvy [CRS, Theorem 15].

The following two claims show how $G$-parking functions from Section 2 are related to the sandpile model. For a vector $u = (u_1, \ldots, u_n)$, let $u^\vee = (u_1^\vee, \ldots, u_n^\vee)$, where $u_i^\vee = \Delta_{ii} - 1 - u_i$.

**Lemma 13.6.** *Let $G$ be a digraph with at least one oriented spanning tree, and let $\Delta = L_G^T$ be the transpose of the truncated Laplace matrix for the digraph $G$. For the sandpile model associated with the toppling matrix $\Delta$, a configuration $u$ is stable and allowed if and only if $u^\vee$ is a $G$-parking function.*

*Proof.* Parts 1 and 2 of Proposition 13.1 imply that $\Delta = L_G^T$ is a toppling matrix. The statement of the lemma is immediate from the definitions of allowed configurations and $G$-parking functions. □

A toppling matrix $\Delta$ is the transpose $L_G^T$ of the truncated Laplace matrix for some digraph $G$ if and only if it has nonnegative column sums

$$\sum_i \Delta_{ij} \geq 0 \text{ for any } j.$$

Theorem 2.1 recovers Gabrielov's result on recurrent configurations.

**Corollary 13.7** ([Gab1, Eq. (21)]). *For a toppling matrix $\Delta$ with nonnegative column sums, a configuration is recurrent if and only if it is stable and allowed.*

*Equivalently, a configuration $u$ is recurrent if and only if $u^\vee$ is a $G$-parking function, for $G$ and $\Delta = L_G^T$ such as in Lemma 13.6.*

*Proof.* Theorem 2.1 and the Matrix-Tree Theorem imply that the number of stable allowed configurations equals $\det L_G = \det \Delta$. According to Theorem 13.4, the number of recurrent configurations is also equal to $\det \Delta$. These facts, together with Proposition 13.5, imply the statement. □

Remark that we need to impose the *transpose* of Dhar's physical conditions (13.2) on the toppling matrix $\Delta$ in Corollary 13.7. The number of recurrent configurations for a toppling matrix $\Delta$ is equal to the number of recurrent configurations for the transposed toppling matrix $\Delta^T$, because $\det \Delta = \det \Delta^T$. It would be interesting to present an explicit bijection between these two sets of configurations.

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