JOURNAL OF COMBINATORIAL THEORY, Series A **79**, 360–366 (1997)
ARTICLE NO. TA962735

NOTE

Intransitive Trees

Alexander Postnikov*

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Communicated by the Managing Editors

Received January 11, 1996

We study the class of trees T on the set $\{1, 2, ..., n\}$ such that for any $1 \leqslant i < j < k \leqslant n$ the pairs $\{i, j\}$ and $\{j, k\}$ cannot both be edges in T. We derive a formula for the number of such trees. We also give a functional equation and a differential equation for the generating function. We mention some additional combinatorial interpretations of these numbers. © 1997 Academic Press

1. MAIN THEOREMS

DEFINITION. An *intransitive tree* or *alternating tree* T on the set of vertices $[n] := \{1, 2, ..., n\}$ is a tree satisfying the following condition: if $1 \leqslant i < j < k \leqslant n$ then $\{i, j\}$ and $\{j, k\}$ cannot both be edges in T. In other words, for every path $i_1, i_2, i_3, i_4, ...$ in T we have $i_1 < i_2 > i_3 < i_4 > \cdots$ or $i_1 > i_2 < i_3 > i_4 < \cdots$.

These trees first appear in the work [1] on hypergeometric functions. Recently, there were found interesting connections between these trees and hyperplane arrangements (see [4, 5] and Section 4.2 below).

Let F_n denote the number of intransitive trees on the set of vertices $[n]$.

THEOREM 1. *The number F_n of intransitive trees on the set $[n]$ is equal to*

$$F_n = \frac{1}{n \cdot 2^{n-1}} \sum_{k=1}^{n} \binom{n}{k} k^{n-1}.$$

The first few numbers F_n are given below.

* E-mail: apost@math.mit.edu.

0097-3165/97 $25.00

n	1	2	3	4	5	6	7	8
F_n	1	1	2	7	36	246	2104	21652

n	9	10	11	12
F_n	260720	3598120	56010096	971055240

In Section 4 we outline two other approaches to the numbers F_n. There exists a combinatorial interpretation of F_n as the number of certain binary trees. On the other hand, the number F_n is equal to the number of regions of a certain hyperplane arrangement. These results will appear elsewhere in more detail (see [4, 5]).

Consider the generating function

$$F(t) := \sum_{n \geqslant 0} F_{n+1} \frac{t^n}{n!}.$$

THEOREM 2. *The function* $F = F(t)$ *satisfies the following functional equation*:

$$F = e^{(t/2)(F+1)}.$$

Remark. Let T_n denote the number of all trees on $[n]$ and $T(t) = \sum_{n \geqslant 1} T_n(t^n/(n-1)!)$. Then $T_n = n^{n-2}$ (Cayley's formula) and $T = T(t)$ satisfies the functional equation $T = te^T$ (e.g., see [3]). Theorems 1 and 2 can be viewed as analogs of these classical results.

In Section 2 we prove Theorem 2 and deduce Theorem 1 from it. An interesting open problem is to find a direct combinatorial proof of Theorem 1.

Let $n \geqslant 2$ and T be an intransitive tree on $[n]$. Two vertices in T are called *adjacent* if they are connected by an edge in T. All vertices in T are of two types: *left* vertices and *right* vertices. The set L of left vertices consists of all vertices i such that all vertices adjacent to i are greater than i and the set R of right vertices consists of all vertices j such that all vertices adjacent to j are less than j.

For $n \geqslant 1$ let f_{nk} denote the number of all intransitive trees on the set $[n+1]$ with k right vertices. Set $f_{00} = 1$ and $f_{01} = 0$.

It is clear that for $n \geqslant 1$ the numbers f_{nk} are symmetric in k: $f_{nk} = f_{n,\,n-k+1}$. Let

$$f_n(x) = \sum_k f_{nk} x^k.$$

Consider the generating function for the polynomials $f_n(x)$:

$$F(x, t) := \sum_{n \geqslant 0} f_n(x) \frac{t^n}{n!}.$$

Clearly, $f_n(1) = F_{n+1}$ and $F(1, t) = F(t)$.

THEOREM 3. *The function $F = F(x, t)$ satisfies the following functional equation*:

$$F \cdot (F + x - 1) = xe^{t(F+x)}.$$

The first few polynomials $f_n(x)$ are

$$f_0 = 1;$$
$$f_1 = x;$$
$$f_2 = x + x^2;$$
$$f_3 = x + 5x^2 + x^3;$$
$$f_4 = x + 17x^2 + 17x^3 + x^4;$$
$$f_5 = x + 49x^2 + 146x^3 + 49x^4 + x^5;$$
$$\ldots$$

2. DECOMPOSITION OF INTRANSITIVE TREES

In this section we give a combinatorial proof of Theorem 2 and deduce Theorem 1 from it.

An intransitive tree T with a chosen vertex (root) is called a *rooted* intransitive tree. If the root is a left vertex we call such a tree *left-rooted*.

Let L_n denote the number of all left-rooted intransitive trees on the set $[n]$.

It is clear that for $n \geqslant 2$ the number L_n is half of the number of all rooted intransitive trees on the set $[n]$ which is equal to n times the number F_n of all intransitive trees:

$$L_n = \frac{n}{2} F_n, \qquad L_1 = \frac{1}{2}(F_1 + 1) = 1.$$

Let $L(t)$ be the generating function for left-rooted intransitive trees:

$$L(t) := \sum_{n \geqslant 1} L_n \frac{t^n}{n!}.$$

We have

$$L(t) = \frac{t}{2}(F(t)+1).$$

Now to get an intransitive tree on the vertex set $[n+1]$ take a forest of left-rooted trees on the vertices $[n]$ and connect $n+1$ to each root. By the exponential formula (e.g., see [3, p. 166]), we get

$$F(t) = e^{L(t)}.$$

This completes the proof of Theorem 2.

Now we can deduce Theorem 1. Let $D(t) = t(F(t)+1)$. Then by Theorem 2

$$D = t(1 + e^{D/2}).$$

Apply Lagrange's inversion theorem (see [3, p. 17]) to get

$$\begin{aligned}
[t^n]\,D &= \frac{1}{n}[\lambda^{n-1}]\,(1+e^{\lambda/2})^n \\
&= \frac{1}{n}[\lambda^{n-1}]\left(\sum_{k=0}^{n}\binom{n}{k} e^{\lambda\kappa/2}\right) \\
&= \frac{1}{n}\sum_{k=0}^{n}\binom{n}{k}\frac{(k/2)^{n-1}}{(n-1)!},
\end{aligned}$$

where $[t^n]\,D$ denotes the coefficient of t^n in D.

On the other hand, $[t^n]\,D = F_n/(n-1)!$ for $n \geqslant 2$, so we get

$$F_n = \frac{1}{n \cdot 2^{n-1}} \sum_{k=1}^{n} \binom{n}{k} k^{n-1}.$$

3. RECURRENCE RELATIONS AND A DIFFERENTIAL EQUATION

In this section we obtain recurrence relations for the numbers f_{nk} and prove Theorem 3.

Let T be an intransitive tree on $[n+1]$ with k right vertices. Delete all edges going from vertex 1. Then T falls into several branches. Let T' be the branch which contains the vertex $n+1$, and let T'' be the tree obtained from T by deleting the branch T' and the edge connecting vertex 1 with T'.

Consider two cases:

1. T' is the tree with one vertex $n+1$. Then T'' is an intransitive tree on the set $[n]$ with $k-1$ right vertices. The number of such trees is equal to $f_{n-1,k-1}$.

2. T' is a tree with $l+1$ vertices, $l \geqslant 1$, and with s right vertices. Then T'' has $n-l$ vertices and $k-s$ right vertices. The number of intransitive trees T with such T' and T'' is equal to $\binom{n-1}{l} \cdot s \cdot f_{ls} \cdot f_{n-l-1,k-s}$.

We get

$$f_{nk} = f_{n-1,k-1} + \sum_{l,s} \binom{n-1}{l} \cdot s \cdot f_{ls} \cdot f_{n-l-1,k-s},$$

where the sum is over $1 \leqslant l \leqslant n-1$ and $1 \leqslant s \leqslant l$.

Now we have the following recurrence for the polynomials $f_n(x)$:

$$f_n(x) = x \left(f_{n-1}(x) + \sum_{1 \leqslant l \leqslant n-1} \binom{n-1}{l} f_l'(x) f_{n-l-1}(x) \right).$$

We get the following differential equation for $F(x,t) = \sum_{n \geqslant 0} f_n(x)\, t^n/n!$:

$$\frac{\partial F}{\partial t} = x \cdot F \cdot \left(1 + \frac{\partial F}{\partial x}\right), \qquad F(x,0) = 1.$$

It is not difficult to check that the function $F(x,t)$ defined by the functional equation $F \cdot (F + x - 1) = x e^{t(F+x)}$ satisfies this differential equation. Thus we have proved Theorem 3.

4. OTHER INTERPRETATIONS OF THE NUMBERS F_n

4.1. *Local Binary Search Trees*

A *local binary search* (*LBS*) *tree* is a labeled binary tree such that every left child has a smaller label than its parent, and every right child has a larger label than its parent. LBS trees were first considered by Gessel [2].

THEOREM. *For $n \geqslant 1$ the number of LBS trees on the set $[n-1]$ is equal to the number F_n of intransitive trees on the set $[n]$.*

Proof. Let $\mathscr{I}_n$ be the set of intransitive trees on $[n]$ with a chosen root. Let $\mathscr{B}_n$ be the set of LBS trees on $[n]$ such that the root has only one child (left or right).

Clearly, $|\mathscr{I}_n| = nF_n$. On the other hand, $|\mathscr{B}_n|$ is n times the number of LBS trees on $[n-1]$. Indeed, for a LBS tree $B \in \mathscr{B}_n$ the root r of B can be any number $r \in [n]$. If we delete the root r we get a LBS tree T' on $[n] \backslash \{r\}$. Conversely, we can always reconstruct T from T'. If r' is the root of T' then we set r' to be the left child of r for $r' < r$ and the right child of r for $r' > r$.

Now construct a bijection $\phi: \mathscr{I}_n \to \mathscr{B}_n$. Let T be a rooted intransitive tree $T \in \mathscr{I}_n$. Construct $B = \phi(T) \in \mathscr{B}_n$ using the following procedure:

1. Orient the tree T from the root (e.g. all vertices adjacent to the root are children of the root).

2. If v is a left vertex in T and $i_1 < i_2 < \cdots < i_k$ are all children of v in T, then set i_1 to be the right child of v in B; i_2 to be the right child of i_1 in B; i_3 to be the right child of i_2 in B; etc.

3. If v is a right vertex in T and $i_1 > i_2 > \cdots > i_k$ are all children of v in T, then set i_1 to be the left child of v in B; i_2 to be the left child of i_1 in B; i_3 to be the left child of i_2 in B; etc.

The construction of the inverse map is clear.

4.2 *Deformed Coxeter Hyperplane Arrangements*

Consider the arrangement $\mathscr{A}_n$ of hyperplanes in $\mathbb{R}^n$ given by

$$x_i - x_j = 1, \quad i < j,$$

where $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$. Denote by R_n the number of regions of $\mathscr{A}_n$, i.e., the number of connected components of the space $R^n - \bigcup_{H \in \mathscr{A}_n} H$.

This hyperplane arrangement was considered by Nati Linial and Shmulik Ravid. They calculated the numbers R_n for $n \leqslant 9$.

THEOREM. *For all $n \geqslant 1$, the number R_n is equal to the number of intransitive trees F_{n+1}.*

This statement was conjectured by Richard Stanley on the basis of the data provided by Linial and Ravid. The proof of this theorem will appear elsewhere. For further results and conjectures on this and related hyperplane arrangements see [4, 5].

I am grateful to my advisor Richard Stanley for interest in this work and helpful suggestions.

REFERENCES

1. I. M. Gelfand, M. I. Graev, and A. Postnikov, Combinatorics of hypergeometric functions associated with positive roots, *in* "Arnold–Gelfand Math. Seminars: Geometry and Singularity Theory, Birkhäuser, Basel, 1997.
2. I. Gessel, private communication.
3. I. P. Goulden and D. M. Jackson, "Combinatorial Enumeration," Wiley, New York, 1983.
4. A. Postnikov and R. Stanley, Deformations of Coxeter hyperplane arrangements, in preparation..
5. R. Stanley, Hyperplane arrangements, interval orders, and trees, *Proc. Nat. Acad. Sci. U.S.A.*, to appear.