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Note

On characters of Weyl groups

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Abstract

In this note a combinatorial character formula related to the symmetric group is generalized to an arbitrary finite Weyl group.

1. The case of the symmetric group

The length $\ell(\pi)$ of a permutation $\pi \in S_n$ is the number of inversions of π, i.e., the number of pairs (i,j) with $1 \leqslant i < j \leqslant n$ and $\pi(i) > \pi(j)$.

For any permutation $\pi \in S_n$ let $m(\pi)$ be defined as

$$m(\pi) := \begin{cases} (-1)^m & \text{if there exists } 0 \leqslant m < n \text{ so that} \\ & \quad \pi(1) > \pi(2) > \cdots > \pi(m+1) < \cdots < \pi(n); \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Let $\mu = (\mu_1, \ldots, \mu_t)$ be a partition of n, and let $S_\mu = S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_t}$ be the corresponding Young subgroup of S_n. For any permutation $\pi = r \cdot (\pi_1 \times \cdots \times \pi_t)$, where $\pi_i \in S_{\mu_i}$ $(1 \leqslant i \leqslant t)$ and r is a representative of minimal length of a left coset of S_μ in S_n, define

$$\text{weight}_\mu(\pi) := \prod_{i=1}^{t} m(\pi_i), \tag{2}$$

where $m(\pi_i)$ is defined in S_{μ_i} by (1).

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Denote by χ_μ^k the value, at a conjugacy class of type μ, of the character of the natural S_n-action on the kth homogeneous component of the coinvariant algebra. The following combinatorial character formula was proved in [4, Theorem 1].

Theorem. With the above notations

$$\chi_\mu^k = \sum_{\{\pi\in S_n:\ell(\pi)=k\}} \mathrm{weight}_\mu(\pi).$$

2. Arbitrary weyl group

Let W be an arbitrary finite Weyl group. Denote the set of positive roots by Φ_+. Let t_α be the reflection corresponding to $\alpha \in \Phi_+$, and let $\check{\alpha}$ be the corresponding coroot. Let α_i be the simple root corresponding to the simple reflection s_i. Denote by $\mathfrak{S}_w$ the Schubert polynomial indexed by $w \in W$.

The following theorem describes the action of the simple reflections on the coinvariant algebra. This theorem is a reformulation of [2, Theorem 3.14 (iii)].

Theorem 1. *For any simple reflection s_i in W and any $w \in W$,*

$$s_i(\mathfrak{S}_w) = \begin{cases} \mathfrak{S}_w & \text{if } \ell(ws_i) > \ell(w), \\ -\mathfrak{S}_w + \displaystyle\sum_{\{\alpha\in\Phi_+ \mid \alpha\neq\alpha_i \wedge \ell(ws_it_\alpha)=\ell(w)\}} \alpha_i(\check{\alpha})\mathfrak{S}_{ws_it_\alpha} & \text{if } \ell(ws_i) < \ell(w). \end{cases}$$

Proof. In the above notations, [2, Theorem 3.14 (iii)] states that

$$s_i(\mathfrak{S}_w) = \begin{cases} \mathfrak{S}_w & \text{if } \ell(ws_i) > \ell(w), \\ \mathfrak{S}_w - \displaystyle\sum_{\{\gamma\in\Phi_+ \mid \ell(t_\gamma ws_i)=\ell(w)\}} w(\alpha_i)(\check{\gamma})\mathfrak{S}_{t_\gamma ws_i} & \text{if } \ell(ws_i) < \ell(w). \end{cases}$$

Obviously, for any $\gamma \in \Phi_+$ there exists a unique $\alpha \in \Phi_+$ such that $t_\gamma ws_i = ws_it_\alpha$. In this case $s_iw^{-1}(\gamma)=\alpha$. If $\ell(ws_i) < \ell(w)$ and $t_\gamma ws_i \neq w$ then the coefficient of $\mathfrak{S}_{ws_it_\alpha} = \mathfrak{S}_{t_\gamma ws_i}$ in $s_i(\mathfrak{S}_w)$ is equal to

$$-w(\alpha_i)(\check{\gamma}) = -\alpha_i(w^{-1}(\check{\gamma})) = -\alpha_i(s_i(\check{\alpha})) = \alpha_i(\check{\alpha}).$$

If $t_\gamma ws_i = w$ then $t_\alpha = s_i$. Hence, the coefficient of $\mathfrak{S}_w$ in $s_i(\mathfrak{S}_w)$ is $1 - w(\alpha_i)(\check{\gamma}) = 1 - \alpha_i(\check{\alpha}) = -1$ if $\ell(s_iw) < \ell(w)$, and 1 otherwise. □

Let $\langle\ ,\ \rangle$ be the inner product on the coinvariant algebra defined by $\langle \mathfrak{S}_v, \mathfrak{S}_w\rangle = \delta_{v,w}$ (the Kronecker delta). Theorem 1 implies

Corollary 2. *Let s_i be a simple reflection in W, and let $z \in W$ such that $\ell(zs_i) < \ell(z)$. Then for any $w \in W$*

$$\langle s_i(\mathfrak{S}_w), \mathfrak{S}_z\rangle = \begin{cases} 0 & \textit{if } z \neq w, \\ -1 & \textit{if } z = w. \end{cases}$$

Proof. For $z = w$ this follows from the second case of Theorem 1. For $z \neq w$ if $\langle s_i(\mathfrak{G}_w), \mathfrak{G}_z\rangle \neq 0$ then (by Theorem 1) $z = ws_it_\alpha$ for some $\alpha \in \Phi_+$ such that $\alpha \neq \alpha_i$ and $\ell(ws_it_\alpha) = \ell(w) > \ell(ws_i)$. Now, for $\alpha \in \Phi_+$, $\ell(ws_it_\alpha) > \ell(ws_i)$ if and only if $ws_i(\alpha) \in \Phi_+$. On the other hand, $\alpha_i \neq \alpha \in \Phi_+ \Rightarrow s_i(\alpha) \in \Phi_+$. Since $ws_i(\alpha) \in \Phi_+$ it follows that $\ell(wt_{s_i(\alpha)}) > \ell(w)$. But $wt_{s_i(\alpha)} = ws_it_\alpha s_i$. Hence, $\ell(zs_i) = \ell(ws_it_\alpha s_i) > \ell(w) = \ell(ws_it_\alpha) = \ell(z)$. □

The following is, surprisingly, an exact Schubert analogue of a useful vanishing condition for Kazhdan–Lusztig coefficients [3, Lemma 4.3].

Corollary 3. *Let s_i, s_j be commuting simple reflections in W, and let $w, z \in W$ such that $\ell(ws_i) > \ell(w)$ and $\ell(zs_i) < \ell(z)$. Then*

$$\langle s_j(\mathfrak{G}_w), \mathfrak{G}_z\rangle = 0.$$

Proof. Obviously, $z \neq w$. If $\ell(ws_j) > \ell(w)$ then our claim is an immediate consequence of Theorem 1. Assume that $\ell(ws_j) < \ell(w)$, and denote $\langle s_j(\mathfrak{G}_w), \mathfrak{G}_z\rangle$ by $b_z^{(j)}(w)$. By Corollary 2

$$s_i(1+s_j)(\mathfrak{G}_w) = s_i\left(\sum_{\ell(zs_j)>\ell(z)} b_z^{(j)}(w)\mathfrak{G}_z\right) = \sum_{\ell(zs_j)>\ell(z)} b_z^{(j)}(w)s_i(\mathfrak{G}_z).$$

On the other hand, by Theorem 1, $\mathfrak{G}_w$ is invariant under s_i. Thus,

$$s_i(1+s_j)(\mathfrak{G}_w) = (1+s_j)s_i(\mathfrak{G}_w) = (1+s_j)(\mathfrak{G}_w) = \sum_{\ell(zs_j)>\ell(z)} b_z^{(j)}(w)\mathfrak{G}_z.$$

We conclude that

$$\sum_{\ell(zs_j)>\ell(z)} b_z^{(j)}(w)(1-s_i)\mathfrak{G}_z = 0.$$

But

$$\begin{aligned}\sum_{\ell(zs_j)>\ell(z)} b_z^{(j)}(w)(1-s_i)(\mathfrak{G}_z) &= \sum_{\ell(zs_j)>\ell(z)\wedge\ell(zs_i)<\ell(z)} b_z^{(j)}(w)(1-s_i)(\mathfrak{G}_z)\\ &= \sum_{\ell(zs_j)>\ell(z)\wedge\ell(zs_i)<\ell(z)} b_z^{(j)}(w)\Big[2\mathfrak{G}_z - \sum_{\ell(ts_i)>\ell(t)} b_t^{(i)}(z)\mathfrak{G}_t\Big].\end{aligned}$$

This sum is equal to zero if and only if $b_z^{(j)}(w) = 0$ for all z with $\ell(zs_j) > \ell(z)$ and $\ell(zs_i) < \ell(z)$.

It remains to check the case in which $\ell(zs_j) < \ell(z)$. By assumption $\ell(zs_i) < \ell(z) \Rightarrow z \neq w$. Corollary 2 completes the proof. □

Let H be a parabolic subgroup of W, which is isomorphic to a direct product of symmetric groups. In the following definition we refer to cycle type and weight$_\mu$ of elements in H under the natural isomorphism, sending simple reflections of H to simple reflections of W.

Definition. Let μ be a cycle type of an element in H. For any element $w = r \cdot \pi \in W$, where $\pi \in H$ and r is the representative of minimal length of the left coset wH in W, define

$$\text{weight}_\mu(w) := \text{weight}_\mu(\pi).$$

Here $\text{weight}_\mu(\pi)$ is defined as in Section 1.

Note that weight_μ is independent of the choice of H, provided that H is isomorhic to a direct product of symmetric groups and that μ is the cycle type of some element in H.

Let R^k be the kth homogeneous component of the coinvariant algebra of W. Denote by χ^k the W-character of R^k. Let $v_\mu \in H$ have cycle type μ. Then,

Theorem 4. With the above notations

$$\chi^k(v_\mu) = \sum_{\{w \in W : \ell(w) = k\}} \text{weight}_\mu(w).$$

Proof. Imitate the proof of [4, Theorem 1]. Here Corollary 2 plays the role of [4, Corollary 3.2] and implies an analogue of [4, Corollary 3.3]. Alternatively, one can prove Theorem 4 by imitating the proof of [3, Theorems 1 and 2], where Corollary 3 plays the role of [3, Lemma 4.3]. □

Note. A formally similar result appears also in Kazhdan–Lusztig theory. The Kazhdan–Lusztig characters of W at v_μ may be represented as sums of exactly the same weights, but over Kazhdan–Lusztig cells instead of Bruhat levels [3, Corollary 3]. This curious analogy seems to deserve further study.For a q-analogue of the result for the symmetric group see [1]. A q-analogue of Theorem 4 is desired.

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