PROBLEM SET 2

The problems worth 10 points each.

Problem 1 For $n \geq 2$ and $k \geq 1$, calculate the number of regions of the *generalized Shi arrangement* $\mathcal{A}_{nk}$ in $\mathbb{R}^n$ that has the following hyperplanes

$$x_i - x_j = -k+1, -k+2, \ldots, 0, 1, \ldots, k, \qquad \text{for } 1 \leq i < j \leq n.$$

(For example, $\mathcal{A}_{n1}$ is the usual Shi arrangement of type A.)

Problem 2 Calculate the number of regions of the Shi arrangement $\{(\alpha, x) = 0, 1 \mid \alpha \in \Phi^+\}$ for a root system Φ of type B_n, C_n, and D_n.

Find a generalization of parking functions for types B or C and find a natural correspondence between these B- or C-parking functions and regions of the Shi arrangement.

Problem 3 A *semiorder* P is a poset such that there exists a collection of intervals I_x, for $x \in P$, of length 1 such that $x > y$ iff all points in I_x are greater than all points of I_y. Prove that a poset P is a semiorder if and only if P does not have induced subposets on 4 elements a, b, c, d of the following 2 types: $\{a < b$ and $c < d$, all other pairs of elements are incomparable$\}$, $\{a < b < c$, and d is incomparable with $a, b, c\}$.

Problem 4 Let $I_n(q) := \sum_{(a_1,\ldots,a_n)} q^{\binom{n}{2} - a_1 - \cdots - a_n}$, where the sum is over parking functions $(a_1, \ldots, a_n)$. Give combinatorial interpretations (with proofs) for the following values of this polynomial $I_n(-1), I_n(0), I_n(1), I_n(2)$.

Define a k-parking function $(a_1, \ldots, a_n)$ as a sequence of positive integers whose increasing rearrangements $c_1 \leq c_2 \leq \cdots \leq c_n$ satisfies $c_1 \leq 1$, $c_2 \leq 1+k$, $c_3 \leq 1+2k$, $c_4 \leq 1+3k$, etc. Let $I_{n,k}(q) := \sum_{(a_1,\ldots,a_n)} q^{((n-1)k+2)n/2 - a_1 - \cdots - a_n}$, where the sum is over k-parking functions $(a_1, \ldots, a_n)$. Extend the results from the first part to the polynomials $I_{n,k}(q)$.

Problem 5 Describe all connected simple graphs G (without multiple edges) such that the vertices v of G can be labelled by positive integer numbers c_v such that:

1. For any vertex v we have $c_v = \frac{1}{2}\sum_{u\sim v} c_u$, where $u \sim v$ means that the vertices u and v and connected by an edge.

2. There exists at least one vertex v such that $c_v = 1$.

Show that, for a Weyl group W of types A_n, D_n, E_6, E_7, E_8, one can attach an additional vertex to the Dynkin diagram of W so that the resulting graph satisfies the above conditions.

Problem 6 Let W be a Weyl group W of types A_n, D_n, E_6, E_7, E_8 and let G be its Dynkin diagram with an additional vertex attached and vertices labelled by positive integer numbers $c_0, c_1, \ldots, c_r$ as in the previous problem.

Show that the order of the Weyl group W equals $|W| = r!\, f\, c_0\, c_1 \cdots c_r$, where $f = \#\{i \mid c_i = 1\}$.

Problem 7 Let W be the Weyl group of type B_n or D_n. Calculate the number of decompositions $c = r_1\, r_2 \cdots r_l$ of a Coxeter element c in W into a minimal number of reflections $r_1, \ldots, r_l$. (The r_i are not necessarily simple reflections.)

Problem 8 Let W be the Weyl group of types A_n, B_n, C_n, or D_n. Find bijections between the following sets:

1. Generalized Dyck paths: Order-ideals in the root poset.

2. Generalized non-crossing partitions: Elements below a Coxeter element c in the $\prec$ order on W. Here $\prec$ is the partial order on W such that $v \prec w$ iff $w = uv$ and $L(w) = L(u) + L(v)$, where $L(w)$ is the minimal number of reflections (not necessarily simple) needed to express w.

3. W-orbits in $Q/(h+1)Q$, where Q is the lattice generated by the roots and h is the Coxeter number (the order of the Coxeter element).

Find an expression for the number of such objects (together with a combinatorial proof).

Problem 9 Let W be a Weyl group. Let us subdivide the vertices of the corresponding Dynkin diagram into two disjoint subsets I_1 and I_2 so that a vertex in I_1 is connected only with vertices in I_2. (In other words, color this bipartite graph into two colors.) The *Springer number* $S(W)$ is defined as the number of elements w in W whose descent set $Des(w)$ equals to I_1. (This is a generalization of numbers of alternating permutations.)

(A) Find the exponential generating functions for Springer numbers of types A, B, and D.

(B) For any subset I of vertices of the Dynkin diagram, show that the number of $w \in W$ such that $Des(w) = I$ is less than or equal to the Springer number $S(W)$.

Problem 10 For a root system Φ with $N = |\Phi^+|$ positive roots, let C be the $N \times N$-matrix whose entries are $c_{\alpha,\beta} = (\alpha, \beta^\vee)$, for $\alpha, \beta \in \Phi^+$. (The Cartan matrix of Φ is a certain $r \times r$-submatrix of C.) Show that the permanent of the matrix C is equal to

$$per(C) = |W| \prod_{\alpha\in\Phi^+} ht(\alpha) = \prod_{\alpha\in\Phi^+} (1 + ht(\alpha))$$

where the height of α is $ht(\alpha) = c_1 + \cdots + c_r$ for $\alpha = c_1\alpha_1 + \cdots + c_r\alpha_r$.

You can first try to prove this formula for type A, then for B, C, D.

Problem 11 Let R be a regular $(n+2)$-gon with vertices labelled by $1, \ldots, n+2$. For a triangulation T of R, let $v_T = (v_1, \ldots, v_{n+2})$ be the $(n+2)$-vector where v_i is the total area of triangles containing the vertex i. Show that the polytope $conv(v_T)$ is a polytopal realization of the assocahedron.

Problem 12 For a simple polytope P, let G_P be the 1-skeleton of P and Δ_P be the corresponding simplicial complex. Show that the Hilbert series (the dimensions of the graded components) $H_T^*(G_P)$ coincides with the Hilbert series of the Stanley-Reisner ring $\mathbb{R}[\Delta_P]$.