PROBLEM SET 1

The problems worth 10 points each. The score will be taken modulo 30, that is the score 30 will give you 30 points, but the score 31 will give you 1 point. A single exception from this rule is that the score 0 will be counted as 0 points, not as 30 points. So you can get the maximal score if you correctly solve exactly 3, 6, or 9 problems. Try to give complete and rigorous proofs. Don't write down solutions if you are not sure that they are correct. You can use any literature, but phrases like "this is proved in such and such book and such and such place" will not be accepted. You'll need to write solutions yourself.

Problem 1 The *polar dual* of subset $P \subset \mathbb{R}^d$ is

$$P^* := \{y \in \mathbb{R}^d \mid (x, y) \leq 1 \text{ for any } x \in P\}.$$

Prove that, for d-dimensional polytope P containing the origin in its interior, the following claims are true: (1) P^* is a polytope. (2) $(P^*)^* = P$. (3) The vertices of P are in bijection with the facets of P^*, and vise versa. More generally, the i-dimensional faces of P are in bijection with the $(d-i)$-dimensional faces of P^*.

Problem 2 For $0 \leq k \leq d$, let $C_{k,d}$ be the polytope in $\mathbb{R}^d$ whose vertices are all vectors $(a_1, \dots, a_d)$ such that $a_i \in \{0, 1, -1\}$ with exactly k nonzero a_i's. For example, $C_{1,d}$ is the crosspolytope, and $C_{d,d}$ is the hypercube.

Calculate the volume of the polytope $C_{k,d}$.

Problem 3 Let $T_{m,n} \subset \mathbb{R}^{mn}$ be the polytope of all real $m \times n$-matrices $X = (x_{ij})$ such that (1) $x_{ij} \geq 0$; (2) for any i, $\sum_j x_{ij} = n$ (the row sums); (3) for any j, $\sum_i x_{ij} = m$ (the column sums). Describe the vertices of the polytope $T_{m,m+1}$. In particular, give a closed formula for the number of vertices.

What can you say about the vertices of $T_{m,m+2}$ and about the vertices of an arbitarary $T_{m,n}$?

Problem 4 Let $G = (V, E)$ be a graph with the vertex set V and the edge set E. Define two polytopes P_G^1 and P_G^2 in the space $\mathbb{R}^{|E|}$ of vectors $(x_e)_{e\in E}$. The polytope P_G^1 is the convex hull of all 01-vectors $(x_e) \in \mathbb{R}^{|E|}$ such that $x_{e_1}x_{e_2} = 0$ whenever edges e_1 and e_2 are adjacent. The polytope P_G^2 is given by the following inequalities $x_e \geq 0$; $\sum_{e\in E(v)} x_e \leq 1$, for any vertex $v \in V$, where $E(v)$ is the set of edges incident to v. Find the necessary and sufficient conditions for the equality $P_G^1 = P_G^2$, and prove these conditions.

Problem 5 Prove that there is a well-defined a bilinear *convolution multiplication* $*$ on the algebra $\mathcal{A}$ of rational polyhedra such that, for any two rational polyhedra P and Q, we have $\chi_P * \chi_Q = \chi_{P+Q}$, where $P + Q := \{x + y \mid x \in P,\, y \in Q\}$ is the *Minkowskii sum* of P and Q.

Problem 6 Prove that there is a well-defined linear automorphism $R : \mathcal{A} \to \mathcal{A}$, where $\mathcal{A}$ is the algebra of rational polyhedra, such that, for any closed i-dimensional polyhedron P and its open part $P^o = P \setminus \partial P$, we have $R : \chi_P \mapsto (-1)^i \chi_{P^o}$.

Problem 7 (1) The *Bernoulli numbers* B_n are defined by $x/(e^x - 1) = \sum_{n=0}^{\infty} B_n x^n/n!$. The numbers of *alternating permutations* of length n are $A_n := \#\{w \in S_n \mid w_1 < w_2 > w_3 < w_4 > \cdots\}$. Prove that $A_{2k-1} = (-1)^{k-1} B_{2k}\, 4^k(4^k - 1)/(2k)$, for $k \geq 1$.

(2) The *descent set* of a permutation $w \in S_n$ is $Des(w) := \{i \in [n-1] \mid w_i > w_{i+1}\}$. For $I \subset [n-1]$, let $D_{n,I}$ be the number of permutations $w \in S_n$ such that $Des(w) = I$. In particular, $A_n = D_{n,\{2,4,6,\ldots\}}$. Prove that $A_n = \max_I D_{n,I}$.

Problem 8 Prove that the definition of matroids in terms of the exchange axiom is equivalent to the definition of matroids in terms of uniqueness of a minimal element in the Bruhat order.

Problem 9 Let $\mu : Gr_{kn} \to \mathbb{R}^n$ be the *moment map* given by

$$\mu(V) = (\mu_1, \ldots, \mu_n), \qquad \mu_i = \frac{\displaystyle\sum_{I\,:\,i\in I} |\Delta_I|^2}{\displaystyle\sum_{I} |\Delta_I|^2}$$

Prove that the image $\mu(\overline{T \cdot V})$ of the closure a torus orbit in the Grassmannian is a convex polytope.