18.318 — Spring 2010 — Problem Set 1

Problem 1. Let $\mathcal{A}$ be an arrangement of affine hyperplanes $H_i = \{x \in \mathbb{R}^d \mid (x, a_i) = c_i\}$, $i = 1, \ldots, m$. Let $\chi_{\mathcal{A}}(q) := \sum_{z \in L(\mathcal{A})} \mu(\hat{0}, z)\, q^{\dim(z)}$ be its characteristic polynomial. Show that

$$\chi_{\mathcal{A}}(q) = \sum_{I \subset [m]} (-1)^{|I|}\, q^{\dim \bigcap_{i \in I} H_i}.$$

Problem 2. Let $\mathcal{A}$ be an affine hyperplane arrangement, as in Problem 1. Let M be the matroid given by the normal vectors a_i to the hyperplanes. A *balanced circuit* C is a circuit of M such that the intersection of the hyperplanes corresponding to elements of C is nonempty. Fix a total order on the ground set of the matroid M. A *broken balanced circuit* is a balanced circuit without its minimal element. A No-Broken-Balanced-Circuit set (NBBC-set) is an independent set of M that has no broken balanced circuit. Prove that

$$\chi_{\mathcal{A}}(q) = (-1)^d \sum_{A \text{ is an NBBC-set}} (-q)^{d-|A|}.$$

Problem 3. Let $\mathcal{A}$ be an affine arrangement, and let M be the corresponding matroid, as in Problems 1 and 2. Suppose that the c_i are *generic* numbers. Find a bijection between regions of $\mathcal{A}$ and independents sets of the matroid M.

Problem 4. Prove Fulkerson-Gross' theorem that says that a graph is chordal if and only if its vertices can be ordered so that, for any node v, all neighbors of v that preceed v in the order form a clique.

Problem 5. Let $\mathcal{A}$ be the central hyperplane arrangement corresponding to the collection of vectors $X = \{a_1, \ldots, a_m\}$. Let $\tilde{Z} = \sum[-a_i/2, a_i/2]$ be the corresponding zonotope centered at the origin. For the braid arrangement, the zonotope $\tilde{Z}$ is the permutohedron, and every region of the braid arrangement contains exactly one vertex of $\tilde{Z}$. It it true that, for any central arrangment $\mathcal{A}$, every region of $\mathcal{A}$ contains exactly one vertex of $\tilde{Z}$?

Problem 6. Let X_G^* be the cographical collection of vectors associated to a graph G (as defined in the lecture). Show that

(a) The matroid of X_G^* is the cographical matroid M_G^*.

(b) The cographical collection of vectors X_G^* is unimodular.

Problem 7. Let $BC(M)$ be the *broken circuit complex* of a matroid M of rank d. The elements of $BC(M)$ are NBC-sets of M. Show that all maximal faces of $BC(M)$ have the same dimension $d-1$. Equivalently, show that every NBC-set is contained in an NBC-base.

Problem 8. Let $T_n(x,y)$ be the Tutte polynomial of the complete graph K_n. Calculate the exponential generating function $\sum_{n\geq 1} T_n(x,y)\,\frac{t^n}{n!}$

Problem 9. Show that $\{\omega_A \mid A \text{ is an NBC-set of } M\}$ is a linear basis of the Orlik-Solomon algebra of a matroid M.

Problem 10. The *Linial arrangement* is the arrangement of affine hyperplanes in $\mathbb{R}^n$ given by the equations $\{x_i - x_j = 1\}$ for $1 \leq i < j \leq n$. Show that the number of regions of this affine arrangement equals the number of alternating trees on $n+1$ nodes.

Problem 11. Let G be a graph on the nodes $1, 2, \ldots, n$. Let $\hat{G}$ be the graph obtained from G by removing the node n. Let $d_i = \text{outdegree}_G(i) - 1$. Show that, for $a_1, \ldots, a_{n-1} \geq 0$ and $a_n = -\sum_{i=1}^{n-1} a_i$, the Kostant's partition function of G equals

$$K_G(a_1, \ldots, a_n) = \sum_{\nu_1,\ldots,\nu_{n-1}\geq 0} K_{\hat{G}}(\nu_1 - d_1, \ldots, \nu_{n-1} - d_{n-1}) \prod_{i=1}^{n-1} \binom{a_i + d_i}{\nu_i},$$

where the sum is over nonnegative integers $\nu_1, \ldots, \nu_{n-1}$.

Problem 12. (a) Describe the combinatorial structure of the Chan-Robbins-Yuen polytope CRY_n. Calculate its f-polynomial $f(q) = \sum f_i\, q^i$, where f_i is the number of i-dimensional faces of CRY_n.

(b) Calculate the volume of the Chan-Robbins-Yuen polytope CRY_n.