PROBLEM SET 2 (due on Tuesday 04/09/2013)

Solve as many problems as you like. Hand in at least 5 problems.

Problem 1. (a) Prove that the (signless) Stirling numbers of the first kind $c(n,k)$ can be computed using the Stirling triangle of the first kind given by the recurrence relation:

$$c(n+1,k) = n\,c(n,k) + c(n,k-1).$$

(b) Prove that the Stirling numbers of the second kind $S(n,k)$ can be computed using the Stirling triangle of the second kind given by the recurrence relation:

$$S(n+1,k) = k\,S(n,k) + S(n,k-1).$$

Problem 2. The Bell number $B(n)$ is the number of all set-partitions of $[n]$. Prove that the Bell numbers can be calculated using the following Bell triangle:

1				
1	2			
2	3	5		
5	7	10	15	
15	20	27	37	52
...	...	...	...	...

In this triangle, the last entry in a row equals the first entry of the next row, and each entry is the sum of two entries to the left (West) and to the North-West of it. (For example, $27 = 20 + 7$.) The Bell numbers 1, 2, 5, 15, 52, ... appear on the main diagonal.

Can you find a combinatorial interpretation of the other entries of this triangle?

Problem 3. A set-partition π of $[n]$ is *non-crossing* if there are no numbers $i < j < k < l$ such that i,k are in one block of π, and j,l are in another block of π. Prove bijectively that the number is non-crossing set-partitions of $[n]$ equals the Catalan number C_n.

Problem 4. In class we described the following procedure. Pick n integer numbers between 0 and $m-1$, and write them in the first row of an $m \times n$ matrix. Fill in the other entries of the matrix by cyclically shifting the first row modulo m, that is, if a is an entry of the first row, then below a we should write $a+1, a+2, \dots, m-1, 0, 1, \dots, a-1$. Find the median entry in each row of this matrix. Let S be the sum of the median entries. Express S in terms of m and n, and prove this formula.

Problem 5. Check (using the axioms of lattices) that the structure of a lattice is equivalent to a poset P such that, for any two elements $x, y \in P$, there is a unique minimal element u such that $u \geq x$ and $u \geq y$, and there is a unique maximal element v such that $v \leq x$ and $v \leq y$.

Problem 6. Prove Minsky's theorem that says that in any finite poset P the maximal number of elements in a chain equals the minimal number of anti-chains needed to cover all elements of P.

Problem 7. Find an example of graded, rank symmetric, rank unimodular poset that has Sperner property, but does not have a symmetric chain decomposition (SCD).

Problem 8. Prove that the lattice $J(J([2] \times [n]))$ is rank unimodular.

Problem 9. In class we constructed the Fibonacci lattice $\mathbb{F}$. Since $\mathbb{F}$ is a differential poset, we have

$$\sum_{x \text{ element of rank n}} (f_x)^2 = n!,$$

where f_x is the number of saturated chains in $\mathbb{F}$ from $\hat{0}$ to x.

(a) Prove that $\mathbb{F}$ is a lattice.

(b) Find a non-inductive combinatorial construction for $\mathbb{F}$. In other words, label elements of $\mathbb{F}$ by some kind of combinatorial objects and describe the covering relation in terms of these objects.

(c) Find an analog of Schensted correspondence for the Fibonacci lattice, which would provide a combinatorial proof of the above identity.

Problem 10. Let P be a differential poset (such as the Young lattice $\mathbb{Y}$ or the Fibonacci lattice $\mathbb{F}$). Show that the number of all paths of length $2n$ from $\hat{0}$ to $\hat{0}$ that go along the edges of the Hasse diagram of P (up or down in any order) equals $(2n-1)!! = (2n-1)(2n-3)\cdots 3\cdot 1$.

Problem 11. A *perfect matching* on $[2n]$ is a graph with vertices labeled $1, 2, \ldots, 2n$ such that each vertex belongs to exactly one edge.

(a) Show that the number of perfect matchings on $[2n]$ equals $(2n-1)!!$.

(b) Find a bijection between paths in the Young lattice $\mathbb{Y}$ as in the previous problem and perfect matchings on $[2n]$.

Problem 12. Let $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ be two sequences of positive integers such that $a_1 + \cdots + a_k = b_1 + \cdots + b_k = n$.

Let λ be the Young diagram such that the path from the bottom left corner to the upper right corner of λ has the form: a_1 right steps, b_1 up steps, a_2 right steps, b_2 up steps, etc.

(a) Prove that the number of paths in a differential poset P that start and end at $\hat{0}$ and have the following form: a_1 steps up, b_1 steps down, a_2 steps up, b_2 steps down, etc., equals the number of ways to place n non-attacking rook of the board of shape λ.

(b) For the Young lattice $\mathbb{Y}$, construct a bijection between paths and rook placements as in part (a).

(c) Find an explicit formula for the number of rook placements as in part (a).

Problem 13. Let p be a prime number, and let $f(x) \in \mathbb{F}_p[x]$ be an irreducible polynomial of degree n over the finite field $\mathbb{F}_p$. One can reduce any monomial x^i modulo the ideal $\langle f(x) \rangle$ and write it as a polynomial of degree $n-1$. Let d_i be the leading coefficient (the coefficient of x^{n-1}) in the reduction of x^i.

Prove that $d_0, d_1, d_2, \ldots, d_{p^n-1}$ is a (base p) de Bruijn sequence, that is, its cyclically consecutive n-tuples contain all p^n p-ary words of length n.

You can use the following facts about the finite field $\mathbb{F}_q$, where $q = p^n$. Let $\alpha \in \mathbb{F}_q$ be a root of the polynomial $f(x)$. Then

(1) $\mathbb{F}_q = \{c_0 + c_1\alpha + \cdots + c_{n-1}\,\alpha^{n-1} \mid c_i \in \mathbb{F}_p\}$, and

(2) $\mathbb{F}_q \setminus \{0\} = \{1, \alpha, \alpha^2, \ldots, \alpha^{q-2}\}$.

Problem 14. Construct a base $p = 3$ de Bruijn sequence of length 27.

Problem 15. Prove that the binary word obtained by concatenation of all Lyndon words whose length divides n written in the lexicographical order is a binary de Bruijn sequence of length 2^n.

Problem 16. (a) Construct a bijection between partitions of n with odd parts and partitions of n with distinct parts.

(b) Generalize (a) as follows. Prove bijectively that, for any positive integers n and k, the number of partitions of n into parts not divisible by $k+1$ equals the number of partitions of n such that parts can be repeated at most k times.

Problem 17. Prove that the number of self-conjugate partitions of n (that is, partitions such that $\lambda' = \lambda$) equals the number of partitions of n with distinct and odd parts.

Problem 18. Prove the following identity for the q-binomial coefficients

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q = \sum_{k=0}^{\min(m,n)} q^{k^2} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

Problem 19. For a partition λ, let $\ell(\lambda)$ be the total number of parts in λ, and $d(\lambda)$ be the number of distinct parts in λ. For example, for $\lambda = (5,5,5,4,2,2,1)$, $\ell(\lambda) = 7$, and $d(\lambda) = 4$. Let

$$A = \sum_{\lambda \text{ partition}} (-1)^{\ell(\lambda)}\, 2^{d(\lambda)}\, q^{|\lambda|}, \qquad B = \prod_{n\geq 1} \frac{1-q^n}{1+q^n}, \qquad C = \sum_{r=-\infty}^{+\infty} (-1)^r\, q^{r^2}.$$

(a) Prove that $A = B$.
(b) Prove that $B = C$.
(c) Prove that $A = C$ combinatorially using the involution principle.

Problem 20. Consider the directed graph given by the $m \times n$ grid where all edges are directed to the right and up. This graph has a unique source A (the lower left corner) and a unique sink B (the upper right corner). An integer N-*flow* is a way to assign non-negative integer numbers to the edges of the graph (flows over the edges) such that for each vertex, excepts A and B, the total in-flow to the vertex equals the total out-flow from the vertex; the out-flow from the source A equals N, and the in-flow to the sink B equals N.

(a) Show that the number of N-flows for the grid graph is a polynomial in N. What is the degree of this polynomial?

(b) Find the leading coefficient of this polynomials.

You can start solving this problem by considering $m \times n$ grids for small values of m or n.