PROBLEM SET 1 (due on Tuesday, March 5, 2013)

Solve as many problems as you like. Hand in at least 5 problems.

Problem 1. Construct a bijection between Dyck paths with $2n$ steps and binary trees with n nodes.

Problem 2. Construct a bijection between *queue-sortable* permutations of size n and *stack-sortable* permutations of size n.

Problem 3. For positive integers k and n, let $C_{k,n}$ be the number of lattice paths from the point $(0,0)$ to the point (n,kn) with steps of the types $(1,0)$ and $(0,1)$ that always stay below the line $y = kx$. (The paths may touch the line.) For example, $C_{1,n}$ is the Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

Find a closed expression for the number $C_{k,n}$.

Problem 4. The *Motzkin number* M_n is the number of paths in the upper half-plane from $(0,0)$ to $(n,0)$ that have steps of the types $(1,1)$, $(1,-1)$, and $(1,0)$. (These paths are similar to Dyck paths, but they may have not only up and down steps but also horizontal steps.)

Calculate the generating function $M(x) := \sum_{n\geq 0} M_n\, x^n$.

Problem 5. A man randomly walks on a road so that with probability 1/3 he makes a step to the right, with probability 1/3 he makes a step to the left, and with probability 1/3 he stays in the same place. The road ends with a cliff.

Initially, the man stays right at the edge of the cliff. He starts walking randomly, as described above. Calculate the probability that the man will not fall off the edge of the cliff after making arbitrary many steps.

Problem 6. In similar setup as in the previous problem, assume that the man makes a right step with probability 1/4, a left step with probability 1/4, and stays in the same place with probability 1/2. Also assume that, initially, he stays k steps away from the edge of the cliff. Calculate the probability of his survival.

Problem 7. Let w be a permutation, and λ be the shape of the corresponding pair of Young tableaux in the Schensted correspondence. In class, we proved that λ_1 is the maximal size of an increasing subsequences in w. Prove that the number of parts in λ is the maximal size of a decreasing subsequence in w.

Problem 8. Find the number of permutations in S_n which avoid both patterns 123 and 132.

Problem 9. Find the number of permutations of size n which avoid both patterns 1234 and 4321.

Problem 10. For fixed positive integers m and n, show that, for sufficiently large N, any sequence of N real numbers either contains a strictly increasing subsequences of size m or a weakly decreasing subsequence of size n. What is the minimal size N for which the statement is true?

Problem 11. An *increasing labelling* of a rooted tree T is a way to label the nodes of T by the numbers $1, 2, \ldots, n$ such that each number appears exactly once, and the labels increase as we move along the edges of T away from the root.

Prove the "baby hooklength formula" for the number increasing labellings of a rooted tree T.

Problem 12. Prove the hooklength formula for the number of shifted standard Young tableaux. Can you modify the probabilistic "hook walk" method for this case?

Problem 13. An *increasing tree* is a tree on the nodes labelled $1, 2, \ldots, n$ such that the labelling is increasing (assuming that the vertex "1" is the root). Find the number of increasing trees on n nodes.

Problem 14. An *increasing binary tree* is a binary tree together with an increasing labelling of its nodes. Prove that the number of increasing binary trees with n nodes and k left edges equals the number of permutations in S_n with k descents (the *Eulerian number*).

Problem 15. An *semi-increasing binary tree* is a plane binary tree together with a labelling of its nodes by the numbers $1, 2, \ldots, n$ (each label appears once), such that the label of a parent node is less than the label of its *left* child.

Find the number of semi-increasing binary trees with n nodes.

Problem 16. For positive integers $n, n_1, \ldots, n_k$ such that $n = n_1 + \cdots + n_k$, prove that the *q-multinomial coefficient* $\begin{bmatrix} n \\ n_1, n_2, \ldots, n_k \end{bmatrix}_q := \dfrac{[n]_q!}{[n_1]_q! \cdots [n_k]_q!}$ is a polynomial in q with positive integer coefficients. What is the degree of this polynomial?

Problem 17. Find the number of $m \times n$ matrices of rank 1 over a finite field $\mathbb{F}_q$.

Problem 18. Let $\binom{[n]}{i}$ denote the set of all i-element subsets in $\{1, \ldots, n\}$. For positive integers n and i such that $i < n/2$, construct an injective map f from $\binom{[n]}{i}$ to $\binom{[n]}{i+1}$ such that $f(I) \supset I$ for any $I \in \binom{[n]}{i}$.

Problem 19. Let X and D be the operators that act on polynomials $f(x)$ as follows $X : f(x) \mapsto x f(x)$. $D : f(x) \mapsto f'(x)$ and Define the sequence of polynomials $f_n(x)$ by $f_n(x) := (X + D)^n \cdot 1$. For example, $f_1 = 1$, $f_2 = x$, $f_3 = x^2 + 1$, $f_4 = x^3 + 3x$, etc. Calculate the constant term $f_n(0)$ of the polynomial $f_n(x)$.

Problem 20. Show that the number of inversions $inv(w)$ and the major index $maj(w)$ are equidistributed statistics on permutations w in S_n.