18.212 Problem Set 1 (due Monday, February 28, 2022)

Solve 6 or more problems.

**Problem 1.** Construct a bijection between Dyck paths with $2n$ steps and triangulations of an $(n+2)$-gon.

**Problem 2.** Show that, for any pattern $\sigma \in S_3$, the number of $\sigma$-avoiding permutations in $S_n$ equals the Catalan number $C_n$.

**Problem 3.** Prove that stack-sortable permutations are exactly the 231-avoiding permutations. (Stack-sortable permutations are defined on pages 6-7 of these lecture notes.)

**Problem 4.** Calculate the number of permutations in $S_n$ which are both 123-avoiding and 2143-avoiding.

For example, for $n = 4$, there are 13 such permutations because among $C_4 = 14$ permutations in $S_4$ which are 123-avoiding exactly one permutation contains pattern 2143.

**Problem 5.** In class, we discussed plane binary trees; see page 11 of these notes. Plane binary trees have unlabelled vertices. Recall that the number of such trees on $n$ vertices equals the Catalan number $C_n$.

Define an *increasing binary tree* as a plane binary tree with vertices labelled by $1, 2, \ldots, n$ so that the root is labelled 1 and, if vertex $u$ belongs to the shortest path between vertex $v$ and the root, then that lebel of $u$ is less than the label of $v$. In other words, the labels of vertices should increase as we go away from the root. For example, for $n = 3$, there are 6 increasing binary trees.

Find the number of increasing binary trees on $n$ vertices.

**Problem 6.** Fix two integers $n$ and $r < n$. Let $B_{n,r}$ be the number of set partitions of $[n]$ such that, for any pair of entries $i \neq j$ that belong to the same block of $\pi$, we have $|i - j| \geq r$. For example, $B_{n,1}$ is the usual Bell number $B(n)$.

Prove that $B_{n,r}$ equals the Bell number $B(n - r + 1)$.

**Problem 7.** Define a *k-colored set partition* as a set partition $\pi$ of $[n]$ with all elements of $[n]$ colored in $k$ colors such that, if two elements $i \neq j$ belong to the same block of $\pi$, then $i$ and $j$ are colored in different colors. Let $a_{n,k}$ be the number of $k$-colored set patitions of $[n]$.

For example, for $n = 3$ and $k = 2$, we have $a_{3,2} = 20$, because there are 20 colored set partitions in this case: (12|3), (12|3), (12|3), (12|3), (13|2), (13|2), (13|2), (13|2), (23|1), (23|1), (23|1), (23|1), (1|2|3), (1|2|3), (1|2|3), (1|2|3), (1|2|3), (1|2|3), (1|2|3), (1|2|3).

Find the exponential generating function for the number of $k$-colored set partitions

$$f_k(x) = \sum_{n \geq 0} a_{n,k} \frac{x^n}{n!}$$

**Problem 8.** Let $b_{k,l}$ be the number of lattice paths $P$ on the plane (with steps of two types $(1,1)$ and $(1,-1)$) that start at $(0,0)$, end at $(a,b)$, and always stay in the uppper half-plane $\{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$. For example, for $(k,l) = (2n,0)$, these paths are the usual Dyck paths with $2n$ steps, so $b_{2n,0}$ is the Catalan number $C_n$.

Find and prove an explicit formula for $b_{k,l}$ using two methods:

(a) the reflection method,
(b) the hook length formula.

**Problem 9.** Prove that the number of non-crossing set partitons of $[n]$ with $k$ blocks equals the number $N(n,k)$ of Dyck paths with $2n$ steps and $k$ peaks (the Narayna number).

**Problem 10.** Show that the two statistics on permutations $w \in S_n$

- the inversion number $\mathrm{inv}(w) := \#\{1 \leq i < j \leq n \mid w_i > w_j\}$,
- the major index $\mathrm{maj}(w) := \sum_{i\,:\,w_i > w_{i+1}} i$

are equidistributed.

**Problem 11.** Show that the following 3 statistics on permutations $w \in S_n$ are equidistributed with each other:

- the number of descents $\mathrm{des}(w) := \#\{i \in [n-1] \mid w_i > w_{i+1}\}$,
- the number of exceedances $\mathrm{exc}(w) := \#\{i \in [n] \mid w_i > i\}$,
- the number of weak exceedances minus one $\mathrm{wexc}(w) - 1 := \#\{i \in [n] \mid w_i \geq i\} - 1$.

**Problem 12.** For $1 \leq k \leq n/2$, find a bijection $f$ between $k$-element subsets of $\{1,\ldots,n\}$ and $(n-k)$-element subsets of $\{1,\ldots,n\}$ such that $f(I) \supseteq I$, for any $k$-element subset $I$.