18.212 Lecture 21 04/07/2021

Matrix Tree Theorem (cont'd)

G connected graph on n vertices labelled $1, 2, \ldots, n$.

Adjacency matrix $A = (a_{ij})_{i,j \in [n]}$

a_{ij} = # edges between vert. i & j

$(a_{ii} = 0 \quad \forall i)$

Laplacian matrix

$$L = \mathrm{diag}(d_1, \ldots, d_n) - A$$

$$d_i = \deg_G(i).$$

reduced Laplacian matrix

$\tilde{L}^{(i)}$ = L with i^{th} row & i^{th} col. removed ($(n-1) \times (n-1)$ matrix)

MTT: # spanning trees of G $= \det(\tilde{L}^{(i)})$.

for any $i = 1, \ldots, n$.

Q: Why is the R.H.S. independent of i?

Let C be any symmetric $n \times n$ matrix with zero row & column sums.

$\tilde{C}^{(i)} = C$ with i^{th} row & i^{th} col. removed.

($\det(\tilde{C}^{(i)})$ are called <u>principal</u> <u>cofactors</u> of C).

Let $\mu_1 = 0, \mu_2, \dots, \mu_n$ be the eigenvalues of C.
(One eigenvalues should be 0, because $\det(C) = 0$.)

<u>Lemma</u> For any $i = 1, \dots, n$

$$\det(\tilde{C}^{(i)}) = \frac{\mu_2 \mu_2 \cdots \mu_n}{n}$$

all principal cofactors of C are equal to each other

<u>Exercise</u>. Prove this lemma.

<u>Alternative formulation of MTT.</u>

Let $\lambda_1 = 0, \lambda_2, \dots, \lambda_n$ be the eigenvalues of the Laplacian matrix L. Then

$$\boxed{\#\text{ spanning trees of } G = \frac{\lambda_2 \lambda_3 \cdots \lambda_n}{n}}$$

Def. The **spectrum** of graph G is the collection $\alpha_1, \alpha_2, \ldots, \alpha_n$ of its **adjacency** matrix A.

"**Spectral Graph Theory**" is the area of math that studies property of a graph in terms of its spectrum.

But for MTT we need the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the Laplacian matrix; not the eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_n$ of the adjacency matrix.

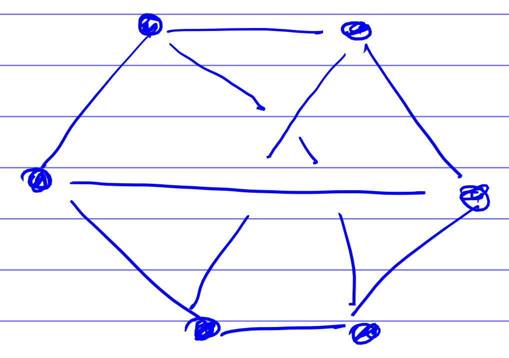
Q: Are $\lambda_1, \ldots, \lambda_n$ related to $\underbrace{\alpha_1, \ldots, \alpha_n}_{\text{the spectrum of } G}$.

A: In general, no.

But for regular graphs they are related.

Def. A graph G is called *d-regular* if the degree of any vertex in G equals d.

Example



a 3-regular graph

(Actually, this graph is $K_{3,3}$)

For a d-regular graph, the Laplacian matrix is
$$L = dI - A.$$

Clearly, we have

Lemma For a d-regular graph G, the eigenvalues λ_i of its Laplacian are related to the eigenvalues α_i of its adjacency matrix (the spectrum of G) as
$$\lambda_i = d - \alpha_i \quad \forall i$$

Proof

Recall that the eigenvalues of a matrix A are the roots of its characteristic equation $\det(A - xI) = 0$.

The characteristic equation of L is

$$\det(L - xI) =$$
$$\det(dI - A - xI)$$
$$= \pm \det(A - (d-x)I) = 0.$$

So its roots are $d -$ the roots of the char. equation of A. $\square$

Remark. It is usually easier to calculate the eigenvalues α_i of the adjacency matrix A, than the eigenvalues λ_i of the Laplacian matrix.

For d-regular graphs we can relate them, and express # spanning trees in terms of α_i's.

Theorem Let G be a d-regular graph, and $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$ be its spectrum.

Then $\alpha_1 = d$ (the largest eigenvalue equals d)

spanning trees of G is

$$\frac{1}{n}(d - \alpha_2)(d - \alpha_3) \dots (d - \alpha_n).$$

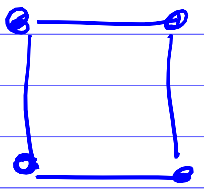
Hypercubes

Let's now calculate # spanning trees of the d-hypercube graph H_d.

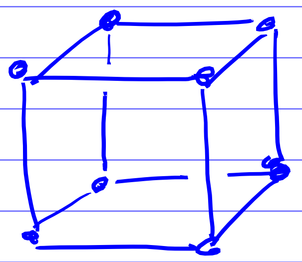
H_d = the 1-skeleton of the d-dimensional hypercube.

Examples $H_1 =$ 

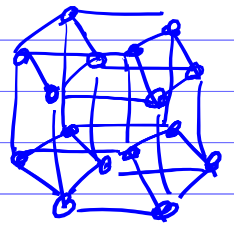
1 spanning tree

$H_2 =$ 

4 spanning trees

$H_3 =$ 

? spanning trees.

$H_4 =$ 

Observation H_d is a d-regular graph.

So here "d" stands both for "<u>d</u>imension" and "<u>d</u>egree".

Product of graphs

$$G = (V_1, E_1), \quad H = (V_2, E_2)$$

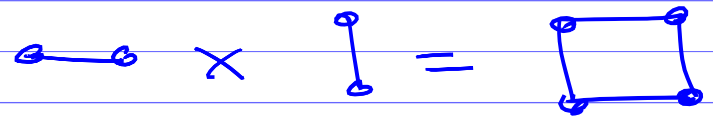
$$G \times H := (V, E)$$

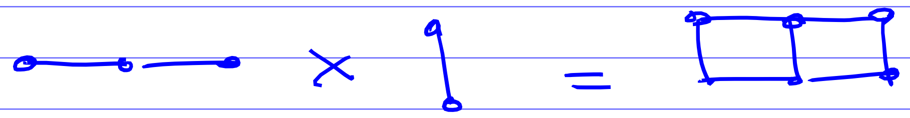
where $V = V_1 \times V_2$

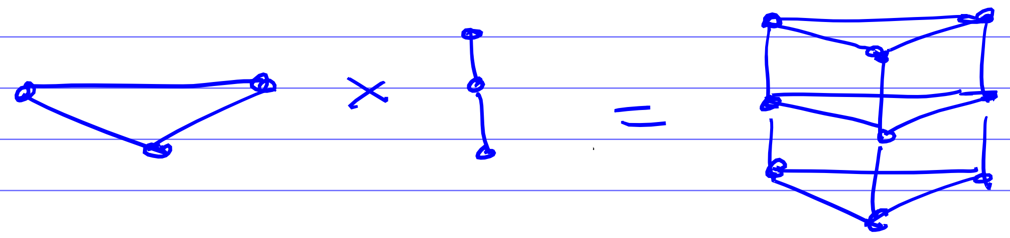
$$:= \{(u, v) \mid u \in V_1,\ v \in V_2\}$$

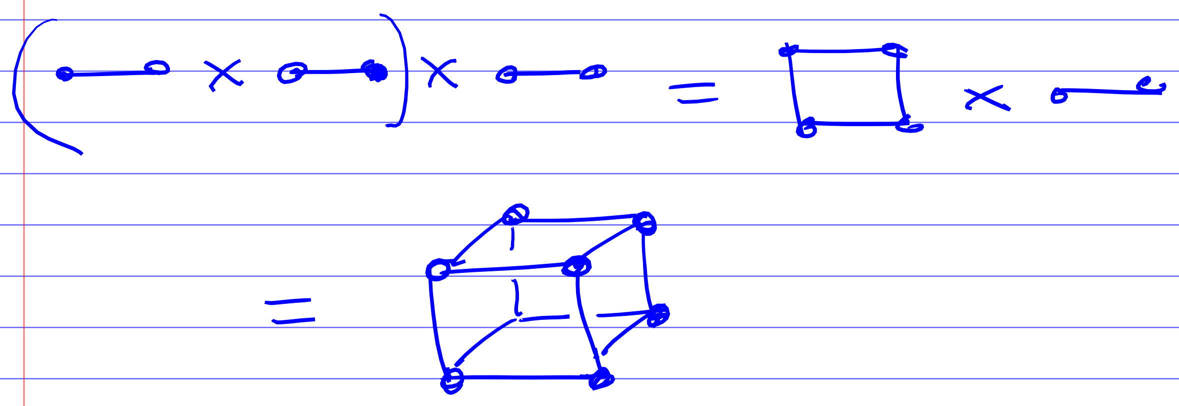
$$E = \{((u,v),(u',v')) \mid u, u' \in V_1;\ v, v' \in V_2,\ u = u' \text{ and } (v, v') \in E_2, \text{ or } v = v' \text{ and } (u, u') \in E_1\}$$

Exemples









In general, the hypercube graph is the product:

$$H_d = \underbrace{\bullet\!-\!\bullet \times \bullet\!-\!\bullet \times \cdots \times \bullet\!-\!\bullet}_{d}$$

Let $A(G)$ be the adjacency matrix of G.

Lemma. Suppose that

$A(G)$ has eigenvalues $\alpha_1, \dots, \alpha_m$; and

$A(H)$ has eigenvalues $\beta_1, \beta_2, \dots, \beta_n$

Then $A(G \times H)$ has $m \cdot n$ eigenvalues $\alpha_i + \beta_j$, $i \in [m]$, $j \in [n]$.

Idea of proof. (Use Linear Algebra) We can explicitly express the eigenvectors of $A(G \times H)$ in terms of the eigenvectors of $A(G)$ and $A(H)$, and see that the eigenvalues of $A(G \times H)$ as expressed as above. $\square$

Remark

This is related to tensor products of matrices

$$A(G \times H) = A(G) \otimes I_n + I_m \otimes A(H).$$

Exercise. Prove this lemma.

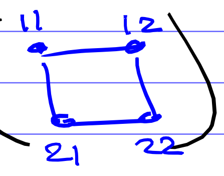
Example

$G = H = \underset{1}{\bullet}\!\!-\!\!\underset{2}{\bullet}$

$$A(G) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

and the eigenvalues are $1, -1$

$A(G \times H) = A\Big($  $\Big)$

$$= \begin{array}{c} \\ 11 \\ 12 \\ 21 \\ 22 \end{array} \begin{array}{c} \begin{array}{cccc} 11 & 12 & 21 & 22 \end{array} \\ \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \\ \begin{array}{cccc} 11 & 12 & 21 & 22 \end{array} \end{array}$$

eigenvectors of this matrix

are $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

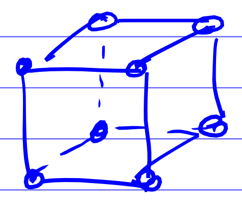
The eigenvalues are

$1+1, \quad 1-1, \quad -1+1, \quad -1-1$

$= 2, \; 0, \; 0, \; -2$

Example

$H_3 = (\bullet\!-\!\bullet \times \bullet\!-\!\bullet) \times \bullet\!-\!\bullet$

$=$ 

The adjacency matrix $A(H_3)$

has eigenvalues $\pm 1 \pm 1 \pm 1$

(8 choices for the signs) :

$1+1+1, \; 1+1-1, \; 1-1+1, \; -1+1+1,$

$1-1-1, \; -1+1-1, \; -1-1+1, \; -1-1-1$

$$= \underbrace{3}_{\binom{3}{0}}, \; \underbrace{1, 1, 1}_{\binom{3}{1}}, \; \underbrace{-1, -1, -1}_{\binom{3}{2}}, \; \underbrace{-3}_{\binom{3}{3}}.$$

In general, $A(H_d)$ has

eigenvalues

$$\underbrace{\pm 1 \pm 1 \pm 1 \ldots \pm 1}_{\substack{d \text{ terms} \\ 2^d \text{ choices for} \\ \text{signs}}}$$

We've got

Lemma. $A(H_d)$ has eigenvalues $-d+2k$ with multiplicity $\binom{d}{k}$ for $k=0,\dots,d$

Since H_d is a d-regular graph,

Corollary The eigenvalues of the Laplacian matrix L of H_d are

$2k$ with multiplicity $\binom{d}{k}$ for $k=0,1,2,\dots,d$.

Now we deduce.

Theorem. [R. Stanley, Enumerative Combinatorics, Vol. 2, p. 62]

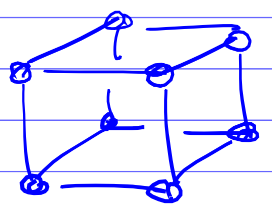
The number of spanning trees of the hypercube graph H_d is

$$\frac{1}{2^d} \prod_{k=1}^{d} (\underbrace{2k}_{\text{non-zero eigenvalues of } L})^{\binom{d}{k}}$$

the multiplicities of eigenvalues

vertices of H_d

$$= \boxed{2^{2^d-d-1} \prod_{k=1}^{d} k^{\binom{d}{k}}}$$

Example $H_3 =$ 

has $2^{2^3-3-1} \, 1^3 \cdot 2^3 \cdot 3^1$

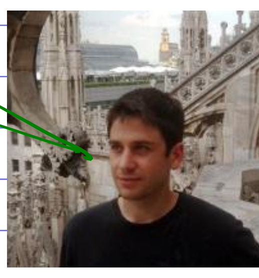
$= 3 \cdot 2^7 = \boxed{384}$

spanning trees.

Remark. Stanley asked for a bijective proof of this formula. This was solved by Olivier Bernardi, 2012.

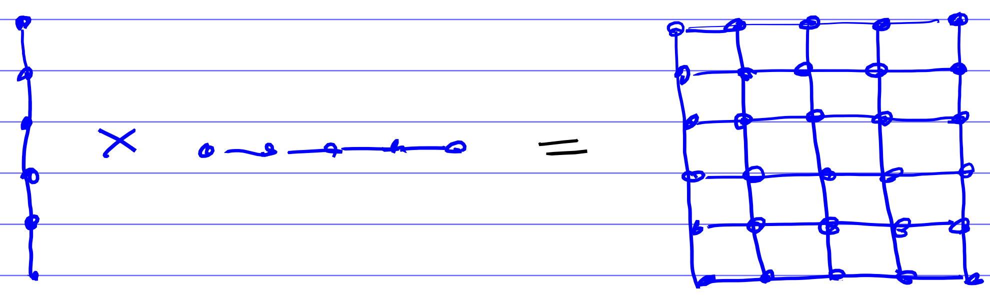
How about a bijective proof?

Ok!



Grid graphs

The product of m-chain and n-chain is the m×n grid graph.



6×5 grid

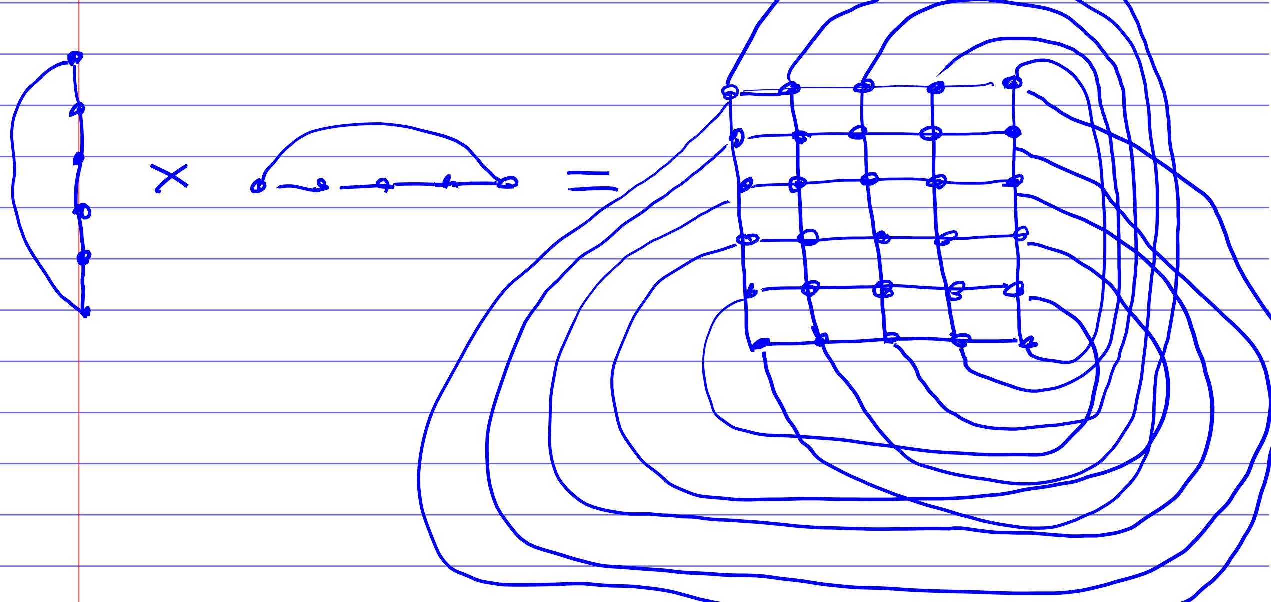
Can we use the previous approach and get a closed formula for the number of spanning trees of the m×n grid graph?

One problem is that the grid graph is not a regular graph.

We can calculate the spectrum (eigenvalues of the adjacency matrix). But we cannot express the eigenval. of the Laplacian matrix in terms of them.

However, if the take the product of m-cycles & n-cycle, we do get a regular graph.

Example



6×5 "toric grid graph"

Exercise. Find a closed formula for the number of spanning trees of the m×n toric grid graph.

Reciprocity for Spanning trees

Let $G = (V, E)$ be a simple graph; and

$\overline{G} = (V, \overline{E})$ be its complementary graph:

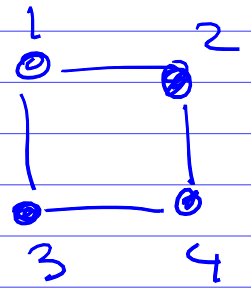
For $u \neq v \in G$,

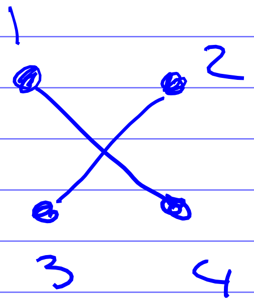
if $(u, v) \in E$ the

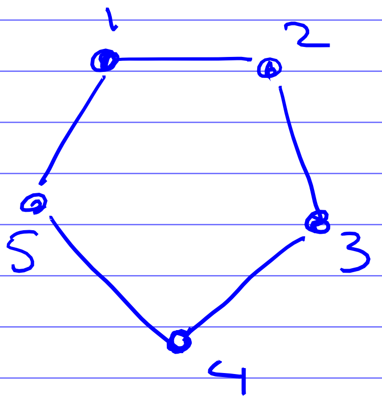
$(u, v) \notin \overline{E}$

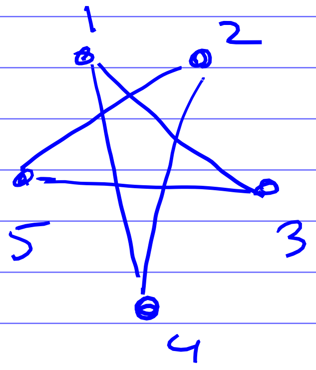
and vise versa.

Examples.

(1) $G =$ 

$\overline{G} =$ 

(2) $G =$ 

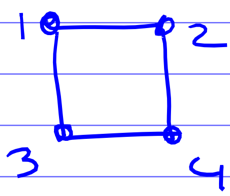
$\overline{G} =$ 

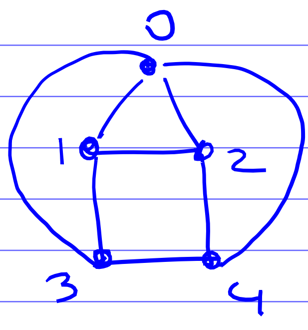
Q: Are spanning trees of G & $\overline{G}$ somehow related to each other?

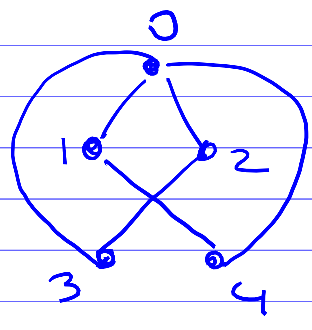
Assume that G is a simple graph on n vertices $1, 2, \dots, n$.

Let G^+ be the graph on $n+1$ vertices $0, 1, 2, \dots, n$ obtained from G by adding the vertex 0 connected with all other vertices $1, 2, \dots, n$ by edges.

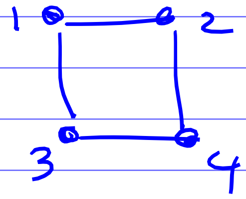
Similarly, we have the graph $\overline{G}^+$

<u>Example.</u> $G =$ 

$G^+ =$ 

$\overline{G}^+ =$ 

Let $F_G(x) = \sum'_{\substack{T \text{ spanning} \\ \text{tree of } G^+}} x^{\deg_T(0)-1}$

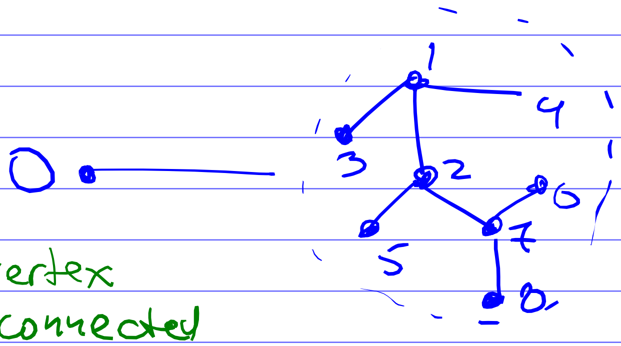
Example $G =$ 

$$F_G(x) = x^3 + 4\cdot 2x^2 + (4\cdot 3 + 2\cdot 2\cdot 2)\, x + 4\cdot 4$$

$$F_{\bar{G}}(x) = x^3 + 4x^2 + 4x + 0.$$

Clearly, $F_G(0) =$

$= n \cdot \#\{\text{spanning trees of } G\}$



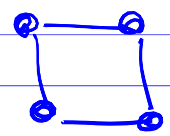
The vertex 0 is connected to one of the vertices $1, 2, \ldots, n$.

a spanning tree of G

Reciprocity Formula

[S. D. Bedrosian, 1964]

$$\boxed{F_G(x) = (-1)^{n-1} F_{\bar{G}}(-x-n)},$$

Example. $G =$ 

$$F_G(x) = x^3 + 8x^2 + 20x + 16$$

$$= (-1)^3 F_{\bar{G}}(-x-4)$$

$$= -\Big((-x-4)^3 + 4(-x-4)^2$$

$$+ 4(-x-4)\Big).$$

Exercise Prove the reciprocity formula. (For example, you can deduce it from the MTT, or give a direct combinatorial proof.)

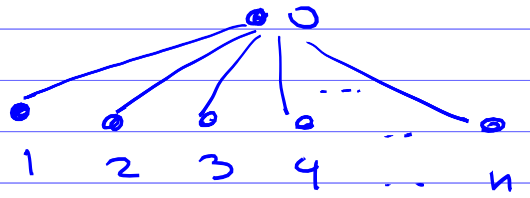
Suppose we know nothing about spanning trees, except the reciprocity formula.

We can use it to deduce many other formulas.

Some examples

1. Let $O_n = ([n], \emptyset)$ be the empty graph on n vertices.

$$O_n^+ =$$



$$F_{O_n}(x) = x^{n-1}$$

$$\overline{O_n} = K_n$$

$$F_{K_n}(x) = (-1)^{n-1}(-x-n)^{n-1} = (x+n)^{n-1}$$

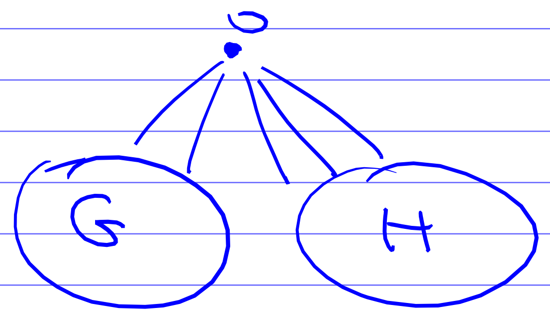
spanning trees in K_n

$$= \frac{1}{n} F_{K_n}(0) = \frac{1}{n} \cdot n^{n-1} = \boxed{n^{n-2}}$$

So we've got Cayley formula.

2. If $G \cup H$ is the disjoint union of two graphs, then

$$F_{G \cup H}(x) = x F_G(x) \cdot F_H(x)$$



So

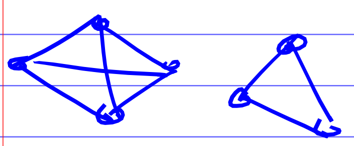
$$F_{K_m \cup K_n}(x) =$$

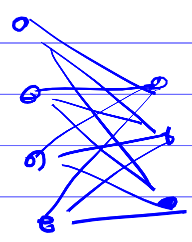
$$= x F_{K_m}(x) \, F_{K_n}(x)$$

by the previous example

$$= x \, (x+m)^{m-1} \cdot (x+n)^{n-1}$$

Now $\overline{K_m \cup K_n} = K_{m,n}$ the complete bipartite graph



K_m K_n $\xleftrightarrow[\text{graph}]{\text{complementary}}$ $K_{m,n} =$ 

So we obtain

$$F_{K_{m,n}} = (-1)^{m+n-1} (-x-m-n) \cdot$$

$$(-x-m-n+m)^{m-1} \cdot (-x-m-n+n)^{n-1}$$

$$= (x+m+n) \cdot (x+n)^{m-1} \cdot (x+m)^{n-1}$$

In particular, # spanning trees of $K_{m,n}$ is

$$\frac{1}{m+n} F_{K_{m,n}}(0) =$$

$$= \frac{1}{m+n} \cdot (m+n) \, n^{m-1} \cdot m^{n-1}$$

$$= \boxed{n^{m-1} \cdot m^{n-1}}$$

Etc.