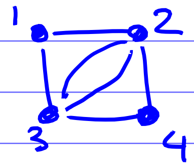
18.212 Lecture 20 04/05/2021

Kirchhoff's Matrix Tree Theorem (MTT)

G - a graph on n vertices

Example $G =$



$L = (\ell_{ij})$ - the Laplacian matrix of G

$n \times n$ matrix

$$\ell_{ij} = \begin{cases} -\#\text{ edges between } i \text{ \& } j & \text{if } i \neq j \\ \deg_G(i) & \text{if } i = j \end{cases}$$

Reduced Laplacian matrix:

$\tilde{L} = L$ with n^{th} row & n^{th} column removed $\quad ((n-1)\times(n-1)$ matrix

Instead of n, we can pick any other $i = 1,..,n$

Example

$$L = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 4 & -2 & -1 \\ -1 & -2 & 4 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}, \quad \tilde{L} = \begin{bmatrix} 2 & -1 & -1 & \cancel{0} \\ -1 & 4 & -2 & \cancel{-1} \\ -1 & -2 & 4 & \cancel{-1} \\ \cancel{0} & \cancel{-1} & \cancel{-1} & \cancel{2} \end{bmatrix}$$

$4 \times 4 \qquad\qquad 3 \times 3$

Matrix Tree Theorem.

$\#$ Spanning trees of $G = \det \tilde{L}$.

Incidence Matrix of G

should not be confused with the adjacency matrix

$n \times m$ matrix

n = # vertices of G

m = # edges of G

The (i,j) entry of the incidence matrix is 1 if i^{th} vertex is incident to the j^{th} edge. Otherwise, it is 0.

We'll need a modification of the incidence matrix:

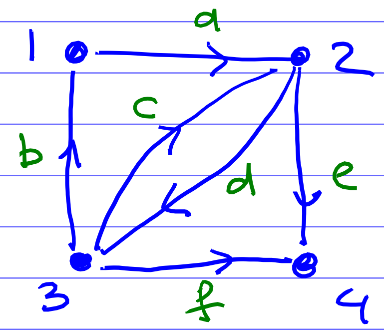
Oriented Incidence Matrix

Orient all edges of G.

$B = (b_{ij})$ $\quad n \times m$ matrix

$$b_{ij} = \begin{cases} 1 & \text{if the } i^{th} \text{ vertex is the source of the } j^{th} \text{ edge} \\ -1 & \text{if the } i^{th} \text{ vertex is the target of the } j^{th} \text{ edge} \\ 0 & \text{otherwise} \end{cases}$$

Example

$G =$ 

$$B = \begin{array}{c} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \\ \begin{matrix} a & \ \ b & \ \ c & \ \ d & \ \ e & \ \ f \end{matrix} \end{array}$$

Lemma $L = B \cdot B^T$

Proof $(B \cdot B^T)_{ij}$ is the dot-product of the i^{th} and j^{th} rows of B.

$i = j$: # ± 1's in the i^{th} row of B $= \deg_G(i)$.

So $(B B^T)_{ii} = \deg_G(i)$.

$i \neq j$: For any edge between vertices i & j, the corresp. entries of the i^{th} & j^{th} rows are 1 and -1, or -1 and 1. For any other edge, at least one of these entries is 0.

So $(B B^T)_{ij} = -$ # edges between vertices i & j.

$\square$

Reduced Oriented Incidence Matrix

$\tilde{B} = B$ with the n^{th} row removed.

$(n-1) \times m$ matrix.

Example

$$\tilde{B} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & -1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{-1} & \cancel{1} \end{bmatrix}$$

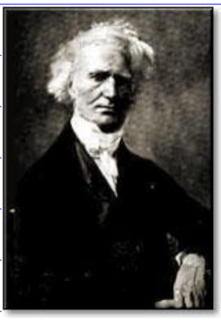
3×6 matrix

Lemma $\tilde{L} = \tilde{B} \cdot \tilde{B}^T$

Proof. The same proof. $\square$

Cauchy – Binet Formula

Baron Augustin-Louis Cauchy (1789 - 1857)



Jacques Philippe Marie Binet (1786 - 1856)

Let $k \le m$. C – $k \times m$ matrix,

D – $m \times k$ matrix

$$\det(C \cdot D) = \sum_{\substack{S \subseteq \{1,2,\ldots,m\} \\ |S| = k}} \det(C_S) \cdot \det(D_S)$$

where C_S and D_S are square $k \times k$ submatrices of C and D, respectively, located in column set S.

Example

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 5 \\ 1 & 1 \\ -2 & 3 \end{bmatrix}$$

$$C \cdot D = \begin{bmatrix} -1 & 16 \\ 7 & 6 \end{bmatrix}$$

$$\det(C \cdot D) = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} \cdot \begin{vmatrix} 3 & 5 \\ 1 & 1 \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} \cdot \begin{vmatrix} 3 & 5 \\ -2 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix}$$

$$= (-5) \cdot (-2) + (-7) \cdot 19 + 1 \cdot 5$$

$$= 10 - 133 + 5 = -118.$$

$$\begin{vmatrix} -1 & 16 \\ 7 & 6 \end{vmatrix} = -118.$$

Proof of Cauchy–Binet

Consider the product of square $(k+m)\times(k+m)$ matrices:

$$\begin{array}{c} \\ k \\ m \\ \\ \end{array}\begin{array}{c} \begin{array}{cc} k & m \end{array} \\ \left[\begin{array}{c|c} I_k & C \\ \hline O & I_m \end{array}\right] \end{array} \cdot \begin{array}{c} \begin{array}{cc} m & k \end{array} \\ \left[\begin{array}{c|c} C & O \\ \hline -I_m & D \end{array}\right] \end{array}\begin{array}{c} k \\ m \end{array}$$

$$= \begin{array}{c} k \\ \\ \end{array}\left[\begin{array}{c|c} O & C\cdot D \\ \hline -I_m & D \end{array}\right]\begin{array}{c} k \\ \\ \end{array} .$$
$$\qquad\qquad m \qquad\qquad k$$

Here I_k is $k\times k$ identity matrix

O is the zero matrix

$$\det \begin{bmatrix} I_k & C \\ O & I_m \end{bmatrix} = 1$$

$$\det \left[\begin{array}{c|c} O & C\cdot D \\ \hline -I_m & D \end{array}\right] = \pm \det(C\cdot D)$$

(more precisely, $= (-1)^{m\cdot(k+1)} \det(C\cdot D)$).

$$\det \left[\begin{array}{c|c} C & O \\ \hline -I_m & D \end{array}\right] = \pm \text{ R.H.S of}$$

Cauchy-Binet formula.

(Check that the signs agree!)

<u>Example</u> (C, D from prev. example)

$$\det \begin{bmatrix} C & O \\ -I_m & D \end{bmatrix} =$$

$$= \det \left[\begin{array}{ccc|cc} 1 & 2 & 3 & 0 & 0 \\ 2 & -1 & -1 & 0 & 0 \\ \hline -1 & 0 & 0 & 3 & 5 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & -1 & -2 & 3 \end{array}\right]$$

expand this using the definition of det as an alternating sum over permutations...

Back to Matrix Tree Theorem

Proof of MTT

reduced oriented incidence matrix

$$\det(\tilde{L}) = \det(\tilde{B} \cdot \tilde{B}^T)$$

Cauchy Binet

$$= \sum_{\substack{S \subseteq \{1,\dots,m\} \\ |S| = n-1}}' (\det \tilde{B}_S)^2 .$$

sum over all $(n-1)$-element subsets of edges

n = # vertices
m = # edges

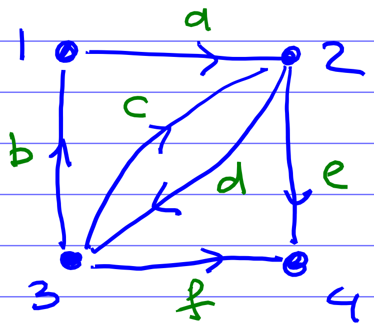
Lemma

$\det(\tilde{B}_S) = \pm 1$ if

S is the set of edges of a spanning tree of G

Otherwise, $\det(\tilde{B}_S) = 0$.

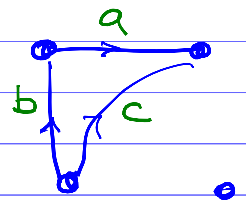
Example

$G =$ 

$$\tilde{B} = \begin{array}{c} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & -1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{-1} & \cancel{1} \end{bmatrix} \\ \begin{array}{cccccc} a & b & c & d & e & f \end{array} \end{array}$$

$$\det\left(\tilde{B}_{\{a,b,c\}}\right) = \begin{array}{c} \begin{vmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{vmatrix} \\ \begin{array}{ccc} a & b & c \end{array} \end{array} = 0$$

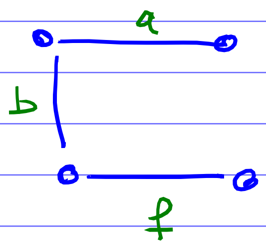
why?



not a spanning tree

$$\det\left(\tilde{B}_{\{a,b,f\}}\right) = \begin{array}{c} \begin{vmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix} \\ \begin{array}{ccc} a & b & c \end{array} \end{array} =$$

$$= 1.$$



a spanning tree

Clearly the last lemma implies $\det \tilde{L} = \det(\tilde{B} \cdot \tilde{B}^T)$ $= \#$ spanning trees of G.

It remains to prove the lemma.

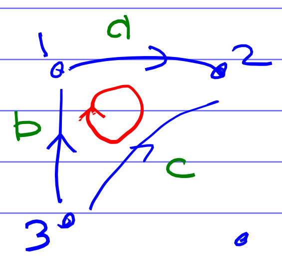
<u>Proof of Lemma</u>

1. If S is <u>not</u> the set of edges of a spanning tree.

S forms a subgraph H of G with $n-1$ edges, which is not a tree
$\Rightarrow$ The subgraph contains a cycle C.

$\Rightarrow$ the columns of $\tilde{B}$ corresponding to edges of C are linearly dependent
$\Rightarrow \det(\tilde{B}_S) = 0$

<u>Example</u> $S = \{a, b, c\}$

H: 

$$\tilde{B} = \begin{array}{c} \left[\begin{array}{cccc} 1 & -1 & 0 & \cdots \\ -1 & 0 & -1 & \cdots \\ 0 & 1 & 1 & \cdots \end{array}\right] \\ \begin{array}{cccc} a & b & c & \end{array} \end{array}$$

$$+\text{Column}_a + \text{Column}_b - \text{Column}_c = \vec{0}$$

"+" if the orient. of edge agrees with the orient. of cycles

"−" if the orient. of edge is opposite to the orient. of cycle

2. If S is the set of edges of a spanning tree T of G.

Induction on n.

Base $n=1$. (Easy to check)

Inductive step $n \geq 2$.

Find a leaf ℓ of T, s.t. $\ell \neq n$. (Any tree on ≥ 2 vertices has ≥ 2 leaves.)

The row of $\tilde{B}_S$ corresp. to ℓ contains exactly one non-zero entry, which is equal to ± 1.

Expand $\det(\tilde{B}_S)$ by ℓ^{th} row. We get

$$\det(\tilde{B}_S) = \pm \det(\tilde{\tilde{B}})$$

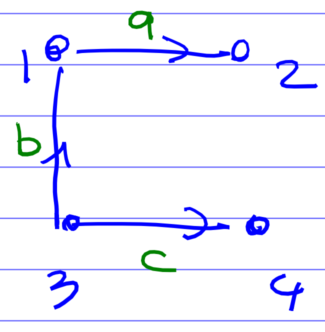
similar (but smaller) matrix corresponding to the tree obtained from T by removing the leaf ℓ.

$$\underbrace{= \dots =}_{\text{keep removing leaves one by one.}} \pm 1.$$

Example $S = \{a, b, f\}$

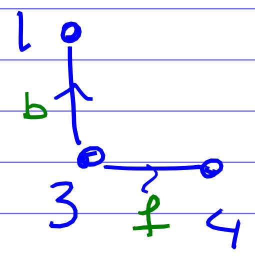
$$\tilde{B} = \begin{array}{c} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \\ \begin{array}{ccc} a & b & f \end{array} \end{array}$$

$$\tilde{B}_{\{a,b,f\}} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{row } 2$$

$T =$ 

leaf $\ell = 2$

$$\ell=2 \quad \begin{vmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \pm \begin{array}{c} \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} \\ \begin{array}{cc} b & f \end{array} \end{array} = \pm |1|$$



$= \pm 1.$

$\square$

This finishes the proof of the MTT.

Example. Let's deduce Cayley's formula from the MTT.

$G = K_n$.

$$L = \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & & \\ \vdots & & \ddots & \\ -1 & -1 & \cdots & n-1 \end{bmatrix}$$ $n \times n$ matrix

$$\tilde{L} = \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & & \\ \vdots & & \ddots & \\ -1 & -1 & \cdots & n-1 \end{bmatrix}$$ $(n-1) \times (n-1)$ matrix.

How to calculate $\det(\tilde{L})$?

Let's find all eigenvalues of $\tilde{L}$.

$$\tilde{L} - n I_{n-1} = \begin{bmatrix} -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 \\ \cdots & & & \cdots \\ -1 & -1 & \cdots & -1 \end{bmatrix}$$

$(n-1) \times (n-1)$ matrix of rank 1.

So its all eigenvalues, except one, equal to zero.

Also $\sum$ eigenvalues $=$ trace

Thus the eigenvalues of

$\tilde{L} - n\underline{I}_{n-1}$ are

$$\underbrace{0, 0, \ldots, 0,}_{n-2} \; -n+1$$

$\uparrow$ $tr(\tilde{L} - n\underline{I})$

$\Rightarrow$ the eigenvalues of $\tilde{L}$ are

$$\underbrace{n, n, \ldots, n,}_{n-2} \; 1$$

$\Rightarrow$ $\det \tilde{L} = \prod$ eigenvalues

$$= \underbrace{n \cdot n \cdot \ldots \cdot n}_{n-2} \cdot 1 = n^{n-2}.$$

Hypercubes

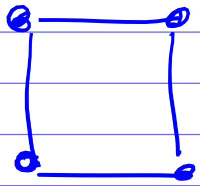
Let's now calculate # spanning trees of the d-hypercube graph H_d.

H_d = the 1-skeleton of the d-dimensional hypercube.

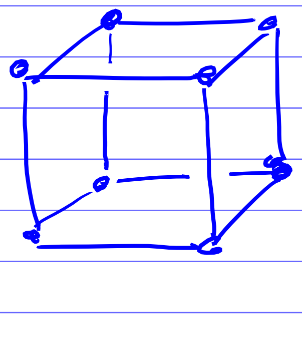
Examples

$H_1 =$ 

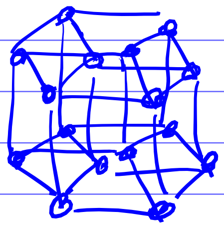
1 spanning tree

$H_2 =$ 

4 spanning trees

$H_3 =$ 

? spanning trees.

$H_4 =$ 

Product of graphs

$G = (V_1, E_1)$, $\quad H = (V_2, E_2)$

$G \times H := (V, E)$

where $\quad V = V_1 \times V_2$

$\quad := \{(u,v) \mid u \in V_1,\ v \in V_2\}$

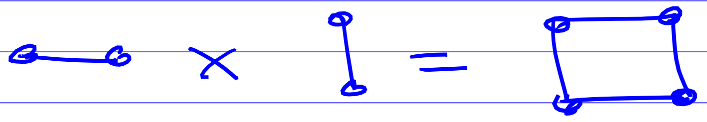
$E = \{((u,v),(u',v')) \mid$

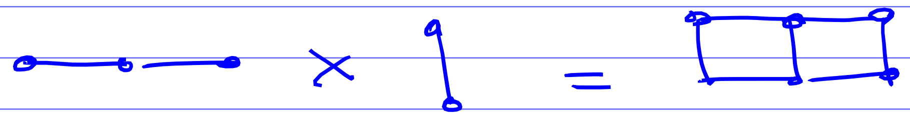
$\quad u, u' \in V_1;\ v, v' \in V_2$

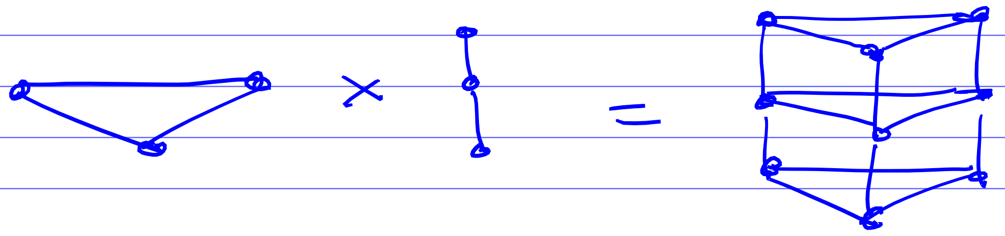
$\quad u = u'$ and $(v, v') \in E_2$, or

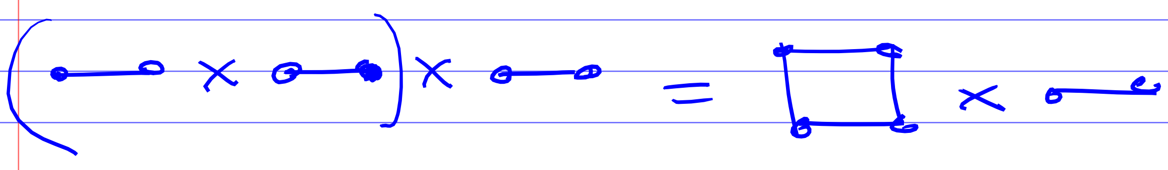
$\quad v = v'$ and $(u, u') \in E_1 \}$

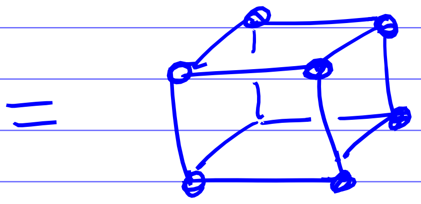
Exemples











In general, the hypercube graph is the product:

$$H_d = \underbrace{\bullet\!-\!\bullet \times \bullet\!-\!\bullet \times \cdots \times \bullet\!-\!\bullet}_{d}$$

Let $A(G)$ be the adjacency matrix of G.

Lemma. Suppose that $A(G)$ has eigenvalues $\alpha_1, \dots, \alpha_m$; and $A(H)$ has eigenvalues $\beta_1, \beta_2, \dots, \beta_n$.

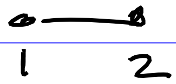
Then $A(G \times H)$ has $m \cdot n$ eigenvalues $\alpha_i + \beta_j$, $i \in [m]$, $j \in [n]$.

Proof idea. Use Linear Algebra. This is related to tensor products of matrices

$$A(G \times H) = A(G) \otimes I_n + I_m \otimes A(H).$$

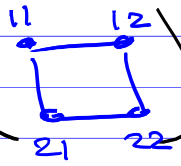
Exercise. Prove this lemma.

Example

$G = H =$ 

$$A(G) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

eigenvalues are $1, -1$

$A(G \times H) = A\Big($  $\Big)$

$$\begin{array}{c} 11 \\ 12 \\ 21 \\ 22 \\ \ \end{array}\begin{array}{c} \left[\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right] \\ \begin{array}{cccc} 11 & 12 & 21 & 22 \end{array} \end{array}$$

has eigenvalues

$$1+1, \quad 1-1, \quad -1+1, \quad -1-2$$

$$= 2, \ 0, \ 0, \ -2$$

In general $A(H_d)$ has eigenvalues

$$\underbrace{\pm 1 \pm 1 \pm 1 \cdots \pm 1}_{d \text{ terms}}$$

2^d choices for signs

We've got

Lemma. $A(H_d)$ has eigenvalues $-d+2k$ with multiplicity $\binom{d}{k}$ for $k=0,\dots,d$

Example $A(H_3)$ has eigenvalues

$$\underbrace{-3,}_{\binom{3}{0}=1}\ \underbrace{-1,\ -1,\ -1,}_{\binom{3}{1}=3}\ \underbrace{1,\ 1,\ 1,}_{\binom{3}{2}=3}\ \underbrace{3}_{\binom{3}{3}=1}$$

How does it help us?

We need the (reduced) Laplacian matrix, not the adjacency matrix?

Observation H_d is a d-regular graph.
(all vertices have the same degree d).

So $L(H_d) = dI - A(H_d)$

Its eigenvalues are
$d -$ the eigenvalues of $A(H_d)$.

<u>Corollary</u> $L(H_d)$ has the eigenvalues:

$2k$ with multiplicity $\binom{d}{k}$
for $k = 0, 1, \ldots, d$.

<u>Theorem</u> (follows from MTT).
Suppose $L(G)$ has eigenvalues $\lambda_1 = 0, \lambda_2, \ldots, \lambda_n$

Then # spanning trees of G

$$= \det(\tilde{L}) = \boxed{\frac{\lambda_2 \cdot \lambda_3 \cdot \ldots \cdot \lambda_n}{n}}$$

(where n is the number of vertices of G).

<u>Exercise</u>. Let L be any symmetric $n \times n$ matrix with all row and column sums equal to zero. Let $\tilde{L}$ be the $(n-1) \times (n-1)$ matrix obtained from L by removing the last row & the last column. Let $\lambda_1 = 0, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of L.

Then $\det(\tilde{L}) = \dfrac{\lambda_2 \cdot \lambda_3 \cdot \ldots \cdot \lambda_n}{n}$.

Now we deduce.

<u>Theorem</u>. The hypercube graph H_d has

$$\frac{1}{2^d}\prod_{k=1}^{d}(2k)^{\binom{d}{k}}$$

$$= \boxed{2^{2^d-d-1}\prod_{k=1} k^{\binom{d}{k}}}$$

spanning trees.