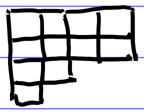
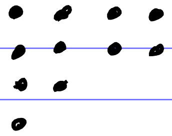
Theory of Partitions

$\lambda = (\lambda_1, \ldots, \lambda_\ell)$ integer partition

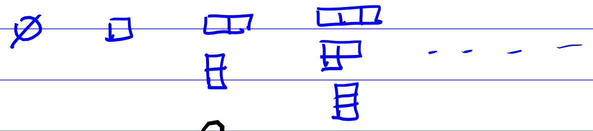
$\updownarrow$

Young diagram aka Ferrers diagram

$\lambda = (4, 4, 2, 1) =$  $=$ 

$p(n) :=$ the number of partitions of n

n	0	1	2	3	4	5	6	7	8 ...
$p(n)$	1	1	2	3	5	7	11	15	22 ...



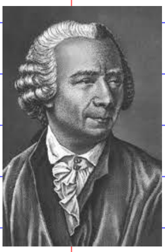
Asymptotic formula:

[Hardy-Ramanujan '1918] [Uspensky 1920]

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}} \quad \text{as } n \to \infty$$

G.H. Hardy 1877 – 1947 Srinivasa Ramanujan 1887 – 1920

Generating function: [Euler 1740]



Leonhard Euler
1707 – 1783

$$\boxed{\sum_{n\geq 0} p(n)\, x^n = \prod_{k\geq 1} \frac{1}{1-x^k}}$$

We alredy proved this using the following notation for partitions

$$\lambda = (1^{m_1}\, 2^{m_2}\, 3^{m_3} \ldots)$$

where $m_k :=$ # parts in λ which are equal to k

$$\lambda \longleftrightarrow (m_1, m_2, m_3, \ldots)$$

any sequence of non-negative integers m_k such that only finitely many m_k are $\neq 0$.

If $\lambda \vdash n$, then

$$n = 1\cdot m_1 + 2\cdot m_2 + 3\cdot m_3 + \ldots$$

$$\sum_{n\geq 0} p(n)\, x^n = \sum_{(m_1, m_2, \ldots)} x^{1\cdot m_1 + 2m_2 + 3m_3 + \ldots}$$

$$= \Big(\sum_{m_1\geq 0} x^{m_1}\Big)\cdot\Big(\sum_{m_2\geq 0} x^{2m_2}\Big)\Big(\sum_{m_3\geq 0} x^{3m_3}\Big)\cdots$$

$$= \frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^3}\cdot\;\ldots$$

$\square$

Partitions with restricted parts

• Odd parts

Let $p_{odd}(n) :=$ # partitions $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$ such that all λ_i are odd.

n	0	1	2	3	4	5	$\ldots$
$p_{odd}(n)$	1	1	1	2	2	3	$\ldots$
	$\emptyset$	(1)	(11)	(111)	(1111)	(11111)	
				(3)	(31)	(311)	
						(5)	

• Distinct parts

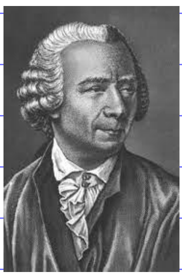
Let $p_{dist}(n) :=$ # partitions $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$ such that all λ_i's are distict from each other,

i.e. $\lambda_1 \underset{\neq}{>} \lambda_2 \underset{\neq}{>} \lambda_3 \underset{\neq}{>} \ldots$

such partitions are also called strict partitions

n	0	1	2	3	4	5	$\ldots$
$p_{dist}(n)$	1	1	1	2	2	3	$\ldots$
	$\emptyset$	(1)	(2)	(3)	(4)	(5)	
				(21)	(31)	(41)	
						(32)	

Theorem [Euler 1748]



$$P_{odd}(n) = P_{dist}(n)$$

for any n.

Proof (Using generating functions)

$$\lambda = (1^{m_1} \, 2^{m_2} \, 3^{m_3} \ldots)$$

all parts of λ are odd $\iff$ $m_{2k} = 0 \quad \forall k.$

all parts of λ are distict $\iff$ $m_k \in \{0, 1\} \quad \forall k$

$$\sum_{n \geq 0} P_{odd}(n) \, x^n = \sum_{m_1, m_3, m_5, \ldots} x^{1 \cdot m_1 + 3m_3 + 5m_5 + \ldots}$$

$$= \left(\sum_{m_1 \geq 0} x^{m_1} \right) \left(\sum_{m_3 \geq 0} x^{3m_3} \right) \left(\sum_{m_5 \geq 0} x^{5m_5} \right) \cdots$$

$$= \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdot \ldots$$

On the other hand,

$$\sum_{n \geq 0} P_{dist}(n) \, x^n = \sum_{\substack{(m_1, m_2, \ldots) \\ m_k \in \{0,1\} \, \forall k}} x^{1 \cdot m_2 + 2m_2 + \ldots}$$

$$= \left(\sum_{m_1 \in \{0,1\}} x^{m_1} \right) \left(\sum_{m_2 \in \{0,1\}} x^{2m_2} \right) \cdots$$

$$= (1+x)(1+x^2)(1+x^3)(1+x^4) \cdots$$

How tho see that these two indinite products are equal to each other ?

$$\frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdot \dots$$

only odd powers of

even power of

$$= \frac{(1-x^2)(1-x^4)\cdot(1-x^6)(1-x^8)\dots}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)\dots}$$

all powers of q

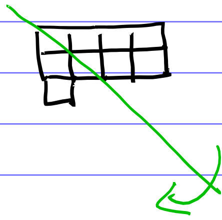
$$= \prod_{k\geq 1} \frac{(1-x^{2k})}{(1-x^k)}$$

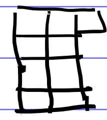
$$= \prod_{k\geq 1} \frac{\cancel{(1-x^k)}(1+x^k)}{\cancel{(1-x^k)}}$$

$$= \prod_{k\geq 1} (1+x^k),$$ as needed. $\square$

<u>Exercise</u>. Find a bijective proof that $p_{odd}(n) = p_{dist}(n)$.

Conjugate partitions

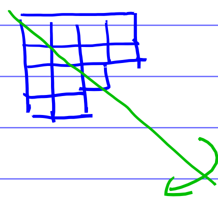
Example $\lambda = (4,4,1) =$ 

$\lambda' = (3,2,2,2) =$ 

λ' is the conjugate partition of λ

Self-conjugate partitions

λ is called <u>self-conjugate</u> if $\lambda' = \lambda$

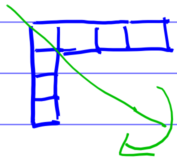
Example $\lambda = (4,4,3,2) =$ 

<u>Theorem</u>. The number of self-conjugate partitions of n equals the number of partitions of n with distinct odd parts.

Young diagram is symmetric w.r.t. reflections around the main diagonal.

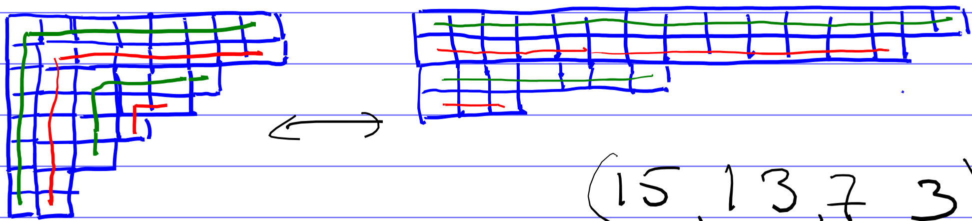
$\lambda = (\lambda_1 \underset{\neq}{>} \lambda_2 \underset{\neq}{>} \dots)$ all λ_k are odd

Bijective proof

Let us subdivide the Young diagram of self-conjugate partition into hooks  (as shown below)

and then "unfold" each hook.

Example



(8, 8, 6, 5, 4, 3, 2, 2)
a self-conjugate partition

(15, 13, 7, 3)
a partition with distinct and odd parts

$\square$

Generating function

$$\sum_{n \geq 0} \#\left\{ \begin{matrix} \text{self-conj.} \\ \text{partitions of} \\ n \end{matrix} \right\} x^n$$

$$= \prod_{k \geq 1} \left(1 + x^{2k-1}\right)$$

gen. funct for partitions with distinct & odd parts

Euler's Pentagonal Number Theorem

Let's expand the infinite product:

$$\prod_{k\geq 1}(1-x^k) = (1-x)(1-x^2)(1-x^3)\cdots$$

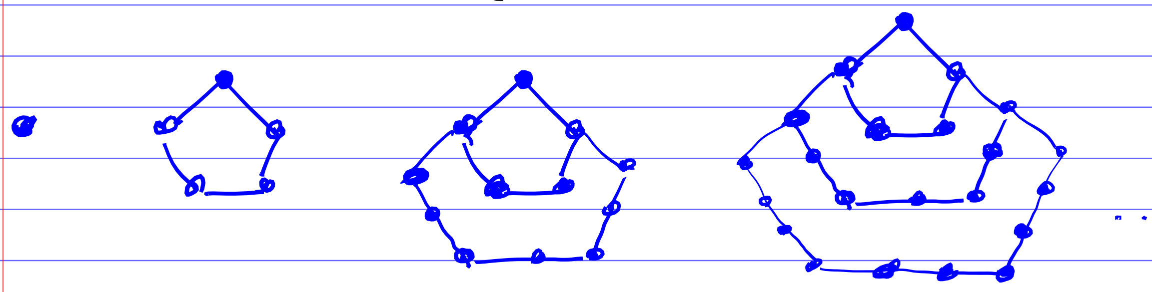
$$= 1 - x - x^2 + x^5 + x^7 - x^{12} - $$
$$- x^{15} + x^{22} + x^{26} - \cdots$$

A stricking feature of this expansion is that there are going to be a lot of concellations:

- all coeffs. $\in \{0, 1, -1\}$
- most coeffs. are actually 0

There is a sparse set of non-zero terms.

The numbers $1, 5, 12, 22, 35, \ldots$ (every second exponent in the expansion) are called the pentagonal numbers.

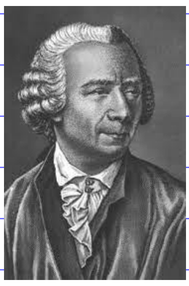


1   5   12   22

General formula: The k^{th} pentagonal number is

$$\frac{k(3k-1)}{2}$$

Theorem (Euler's Pentagonal Thm)



Euler conjectured this in 1741 and proved 1750

$$\boxed{\prod_{n=1}^{\infty} (1-x^n) = \sum_{k \in \mathbb{Z}} (-1)^k \, x^{k(3k-1)/2}}$$

Notice that the summation is over all positive and negative k's

Q: where the exponents

$2, 7, 15, 26, \ldots$

in the expansion are coming from? (They are not pentagonal numbers.)

A. They correspond to negative values of k.

So we can think of them as k^{th} pentagonal numbers for $k < 0$.

Remark If you don't like summations over all integer k's, you can write the R.H.S. as

$$\sum_{k \in \mathbb{Z}} (-1)^k \, x^{k(3k-1)/2} =$$

$$= \underline{1} + \sum_{k=1}^{\infty} (-1)^k \left(x^{k(3k-1)/2} + x^{k(3k+1)/2} \right).$$

Notice that

$$\left(\sum_{n\geq 0} p(n)\, x^n\right)\cdot\left(\prod_{n\geq 1}(1-x^n)\right) = 1.$$

$$\left(\sum_{n\geq 0} p(n)\, x^n\right)\left(1-x-x^2+x^5+x^7-x^{12}-\dots\right) = 1$$

<u>Corollary</u> (of pentagonal thm)
(also due to Euler)

The partition numbers satisfy the recurrence:

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \dots$$

<u>Note</u>. This is an effective way to calculate $p(n)$.

Proof of pentagonal theorem

(Proof by involution priciple.)

All terms in the expansion of $\prod\limits_{k\geq 0}(1-x^k)$ correspond to partitions with distinct parts.

Example

$$(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)\cdots$$

$$\downarrow$$

$\lambda = (4,3,1) =$

• • • •
• • •
•

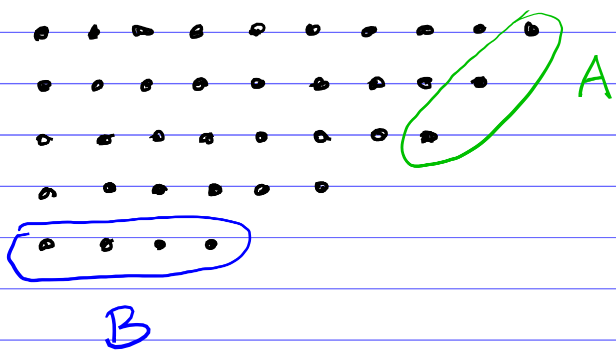
a partition with distinct parts

So $\prod\limits_{k\geq 1}(1-x^k) =$

$$= \sum_{n\geq 0}\Bigg(\sum_{\substack{\lambda \vdash n \\ \text{partition with} \\ \text{distinct parts}}} (-1)^{\#\text{ parts in }\lambda}\Bigg)x^n.$$

We need to show that this expr. is zero for most values of n (except pentagonal numbers).

Let's us construct a sign-reversing involution τ on partitions with distinct parts.

$\lambda =$ 

Ferrers diagram of λ

Consider 2 subsets of dots in the Ferrers diagram of λ (or boxes of the Young diagr. of λ)

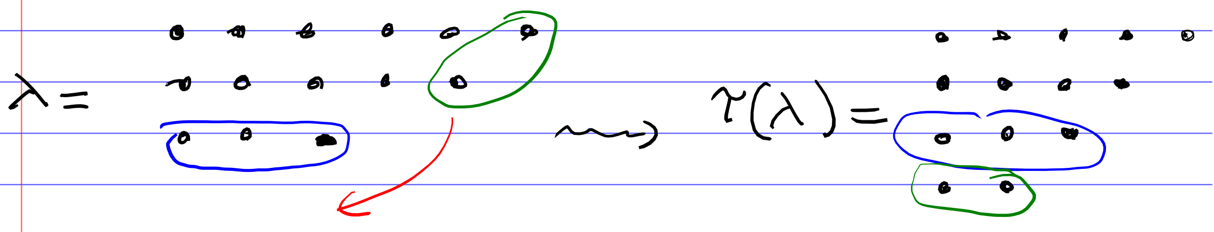
A : all dots on the diagonal starting from the top rightmost dot

B : all dots in the last row.

Define $\tau(\lambda)$ as follows:

I. If $|A| < |B|$, then $\tau(\lambda)$ is obtained from λ by removing all dots in A and adding a new row of size $|A|$.

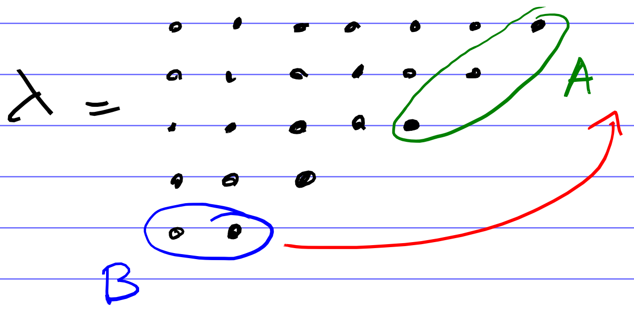
Example



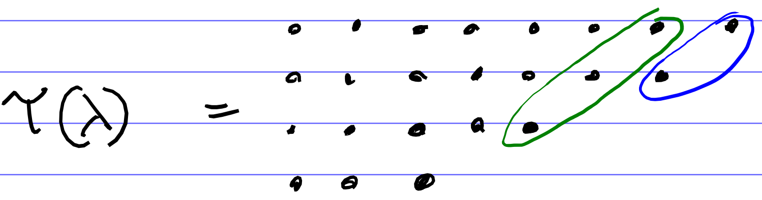
$(6,5,3)$ $(5,4,3,2)$

II. If $|A| \geq |B|$, then $\tau(\lambda)$ is obtained from λ by removing all dots in B (the last row) and adding a new diagonal with B dots to the right of A.

Example.



$(7,6,5,3,2)$



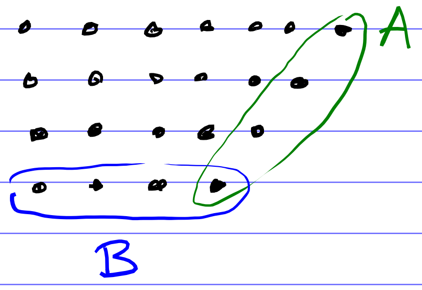
$(8,7,5,3)$

III. This operation τ will not produce a valid Ferrers diagram only if

- If collections of dots A & B overlap (at one box)
- $|A| = |B|$ or $|A| = |B| - 1$.

In this case τ is not defined.

<u>Example</u>

$\lambda =$ 

$\tau(\lambda)$ is not defined.

$(7, 6, 5, 4)$

<u>Notice:</u>

- τ preserves # dots in Ferrers diagram
- τ changes # rows by ± 1.
- τ is an <u>involution</u>:

if $\tau(\lambda) = \mu$ then $\tau(\mu) = \lambda$.

So τ will cancel all terms

in $$\sum_{\substack{\lambda \vdash n \\ \lambda \text{ has distict} \\ \text{parts}}} (-1)^{\#\text{parts in } \lambda},$$

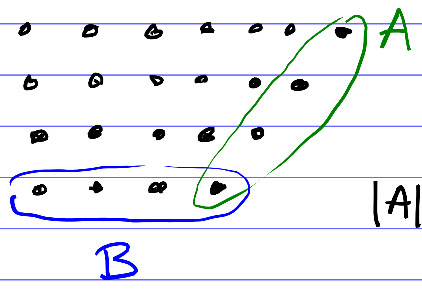
except the contribution of diagrams for which τ is not defined.

This "surviving" diagrams correspond to the terms:

$$\sum_{k \geq 1} (-1)^k x^{k(3k-1)/2} \quad \text{and} \quad \sum_{k\geq 1} (-1)^k x^{k(3k+1)/2}$$

in the expansion.

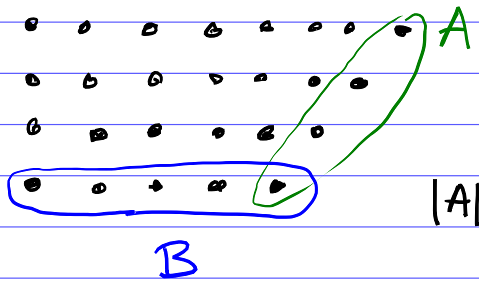
Case III with $|A| = |B|$ corresponds to the "true" pentagonal numbers



$|A| = |B|$

This Ferrers diagram like like a pentagon

Case III with $|A| = |B| - 1$ corresponds to the "negative" pentagonal numbers



$|A| = |B| - 1.$

This proves Euler's pentagonal Theorem. $\square$