# Robinson–Schensted correspondence (RSK)

$\lambda = (\lambda_1, \dots, \lambda_\ell)$ a partition.

$f_\lambda := \#$ SYT's of shape $\lambda$

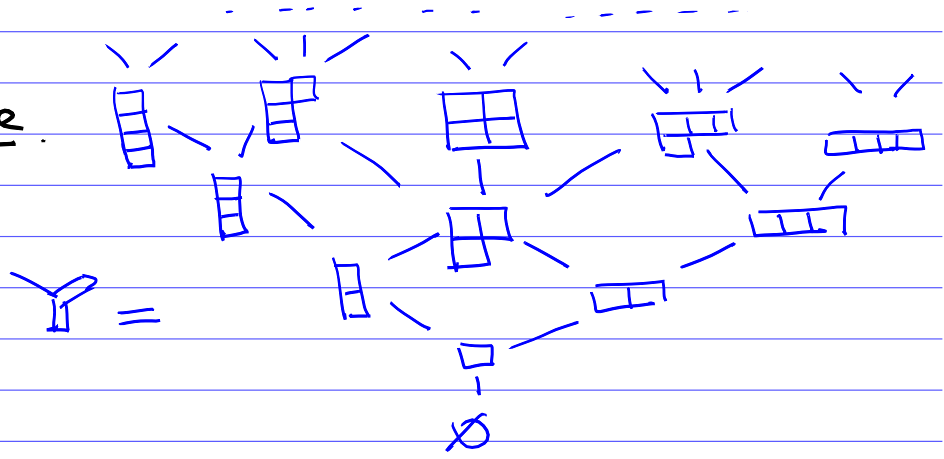
$= \#$ saturated chains in $\mathbb{Y}$ from $\varnothing$ to $\lambda$.

Young's lattice

**Theorem**

$$\sum_{\lambda \vdash n} (f_\lambda)^2 = n!$$

**Example.**




$n=4$: $f_{(1,1,1,1)} = 1$, $f_{(2,1,1)} = 3$, $f_{(2,2)} = 2$

$f_{(3,1)} = 3$, $f_{(4)} = 1$

$$1^2 + 3^2 + 2^2 + 3^2 + 1 = 24$$

**Remark.** Representation theoretic meaning of this identity.

For any finite group $G$, there are finitely many (equivalence classes) of irreducible representations $V_1, V_2, \dots, V_N$ of $G$ over $\mathbb{C}$.

$$\boxed{|G| = \sum_{i=1}^{N} (\dim V_N)^2}$$

the "sum of squares" formula

For $G = S_n$ (the symmetric group), irreducible representations $V_\lambda$ of $S_n$ can be labelled by partition $\lambda \vdash n$,

$$f_\lambda = \dim V_\lambda .$$

So $\sum_{\lambda \vdash n} (f_\lambda)^2$

$$= \# \left\{ (P, Q) \;\middle|\; \begin{array}{l} P, Q \text{ are SYT's} \\ \text{of the same} \\ \text{shape } \lambda \vdash n \end{array} \right\}$$

$$= n!$$

How to construct a bijection

$$S_n \longleftrightarrow \left\{ (P, Q) \;\middle|\; \begin{array}{l} \text{SYT's} \\ \text{of the} \\ \text{same shape} \\ \lambda \vdash n \end{array} \right\} \;?$$

- Such a bijection can be constructed using Schensted insertion algorithm. [Schensted 1961]

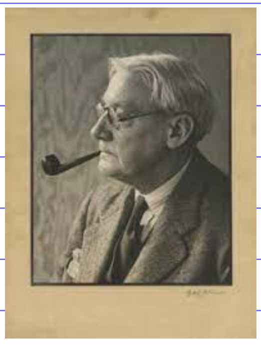
- It is called the Robinson-Schensted correspondence.

  Robinson discovered the same bijection (but in different form) earlier [Robinson 1938].

- It is a special case of the more general Robinson-Schensted-Knuth correspondence (RSK). [Knuth 1970]

  Knuth generalized Schented's construction to semi-standard Young tableaux.

  We will often denote this bijection RSK.



Gilbert de Beauregard Robinson
1906-1992

Craige Schensted

(changed name to Ea Ea)

1927 – 2021

Donald Ervin Knuth
(born 1938)
(Chinese name: 高德纳
Gāo dé nà)

# Schensted's Algorithm

Ea Ea drumming

**Input:** $w = w_1 \dots w_n \in S_n$

**Output:** $(P, Q)$ two SYT's of the same shape $\lambda \vdash n$.

$P$ is called the "insertion tableau"

$Q$ is called the "recording tableau"

**Example** $w = 3, 5, 2, 4, 7, 1, 6$

We'll construct $P$ & $Q$ by adding boxes one at a time.

Initially, set $P = \emptyset$, $Q = \emptyset$.

$P: \ \emptyset \xleftarrow{w_1 = 3} \quad \begin{array}{|c|}\hline 3 \\ \hline\end{array} \xleftarrow{w_2 = 5} \quad \begin{array}{|c|c|}\hline 3 & 5 \\ \hline\end{array} \xleftarrow{w_3 = 2}$

$\begin{array}{|c|c|}\hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array} \xleftarrow{4} \quad \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 & 5 \\ \hline\end{array} \xleftarrow{7} \quad \begin{array}{|c|c|c|}\hline 2 & 4 & 7 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} \xleftarrow{1}$

$\begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \xleftarrow{6} \quad \begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 & 7 \\ \hline 3 \\ \cline{1-1}\end{array} \leftarrow$ this is the resulting $P$-tableau

$Q$ records the growth of $P$-tableau

(If the $k^{th}$ insertion step results adding a box $(i,j)$, Then $Q$ contains $k$ in box $(i,j)$).

$$Q = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 7 \\ \hline 6 \\ \cline{1-1}\end{array}$$

## Insertion step

$\tilde{P} \leftarrow a = w_i$ ($\tilde{P}$ is some intermediate $P$-tableau)

1. If $a >$ all entries in $1^{st}$ row of $\tilde{P}$, then all a new box in the end of the first row and put $a$ in it.
2. Otherwise, find the smallest entry $a_1$ in the first row s.t. $a_1 > a$. Replace $a_1$ by $a$.
3. Repeat the same procedure trying to insert $a_1$ in the $2^{nd}$ row:
   If $a_1 >$ all entries in $2^{nd}$ row, then add a new box in the end of $2^{nd}$ row and place $a_1$ in it.
   Otherwise, find the smallest entry $a_2$ in $2^{nd}$ row, s.t. $a_2 > a_1$. Replace $a_2$ by $a_1$
4. Proceed by trying to insert $a_2$ in $3^{rd}$ row, etc.

Here is another example

$$\tilde{P} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 7 \\ \hline 3 & 8 \\ \cline{1-2} \end{array} \longleftarrow 4$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 7 \\ \hline 3 & 8 \\ \cline{1-2} \end{array} \quad \longleftarrow 5$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 \\ \cline{1-2} \end{array}$$

$$\longleftarrow 8$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 \\ \cline{1-2} 8 \\ \cline{1-1} \end{array}$$

The result of Schensted insertion

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<u>Theorem</u> The above construction $w \longmapsto (P, Q)$ gives a bijection between $S_n$ and $\{(P,Q) \mid \text{SYT's of same shape} \vdash n\}$

<u>Proof</u>. Construct the inverse map $(P, Q) \longmapsto w$. Basically, we need to invert the construction step by step.

<u>Exercise</u>. Carefully check that all steps of the Schensted construction are invertible.

## Example

$$(P, Q) = \left( \begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 2 & 5 & 7 \\ \hline 3 \\ \cline{1-1} \end{array} \ , \ \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & 7 \\ \hline 6 \\ \cline{1-1} \end{array} \right)$$

box in this position was inserted last

$$P := \begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 2 & 5 & 7 \\ \hline 3 \\ \cline{1-1} \end{array}$$

the largest entry in 1st row > 7

We get $w_n = 6$ and the intermediate $P$-tableau (just before the final one) was

$$\begin{array}{|c|c|c|} \hline 1 & 4 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$$

then look at the Q-tableau again. Find the position of the box of Q containing $n-1$

"Uninsert" this entry from P-tableau

the max entry in 2nd row <3

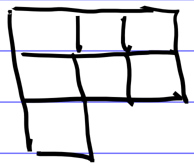
$$\begin{array}{|c|c|c|} \hline 1 & 4 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \rightsquigarrow \begin{array}{|c|c|c|} \hline 2 & 4 & 7 \\ \hline 3 & 5 \\ \cline{1-2} \end{array} \qquad w_{n-1} = 1$$

the previous intermediate P-tableau

etc.

**Definition.** Then shape $\lambda$ of the P & Q tableaux is called the Schensted shape of $w$.

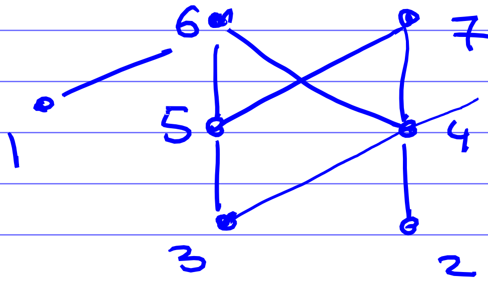
**Example** The Schensted shape of $w = 3\,5\,2\,4\,7\,1\,6$ is $\lambda = (3,3,1) =$



**Theorem.** Let $\lambda$ be the Schensted shape of $w \in S_n$.

- Then $\lambda_1$ (the size of 1st row of $\lambda$) equals the maximal size of an increasing subsequence in $w$.
- $\lambda'_1$ (the size of 1st column of $\lambda$) equals the maximal size of a decreasing subsequence in $w$.
- Moreover, $\lambda$ is the partition associated with the poset of $w$ by Greene's theorem

**Example.** $w = 3\ 5\ 2\ 4\ 7\ 1\ 6$




chains: $\ell_1 = 3, \quad \ell_2 = 6, \quad \ell_3 = 7$

$\lambda = (3, 3, 1) = $ ▭▭▭ / ▭▭▭ / ▭

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antichains: $m_1 = 3, \quad m_2 = 5, \quad m_3 = 7$

$\mu = (3, 2, 2) = \lambda'$

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**Exercise.** Prove the 2 two claims of the theorem (about max size of an increasing / decreasing subsequence of $w$)

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**Theorem.** If $w \overset{RSK}{\longmapsto} (P, Q)$, then

$w^{-1} \longmapsto (Q, P)$

the inverse permutation of $w$

**Special case.** $w \in S_n$ permutation

TFAE:

- $w$ has no decreasing subsequence of size 3

obvious $\Updownarrow$

- $w$ is 321-avoiding

prev. thm $\Updownarrow$

- the Schensted shape $\lambda$ of $w$ is a Young diagram with at most 2 rows (i.e. $\lambda_1' \leq 2$)

We obtain:

$$\left\{\begin{matrix}\text{321-avoiding}\\ \text{permutations}\\ \text{in } S_n\end{matrix}\right\} \overset{RSK}{\longleftrightarrow} \left\{\begin{matrix}(P,Q)\ \text{SYT's}\\ \text{of the same}\\ \text{shape} \vdash n\\ \text{with at most}\\ \text{2 rows}\end{matrix}\right\}$$

One can combine these 2 SYT's $P, Q$ into a single SYT $T$ of the rectangular shape $2 \times n$.

$$(P, Q) \rightsquigarrow T = 2\overset{n}{\boxed{P \quad \tilde{Q}}}$$

$\tilde{Q}$: rotate $Q$ by $180°$, replace each entry $i$ by $2n+1-i$

**Example**

$w = 3\ 1\ 2\ 6\ 4\ 5$

$\updownarrow$ RSK

$P$:

| | | | |
|---|---|---|---|
| 1 | 2 | 4 | 5 |
| 3 | 6 | | |

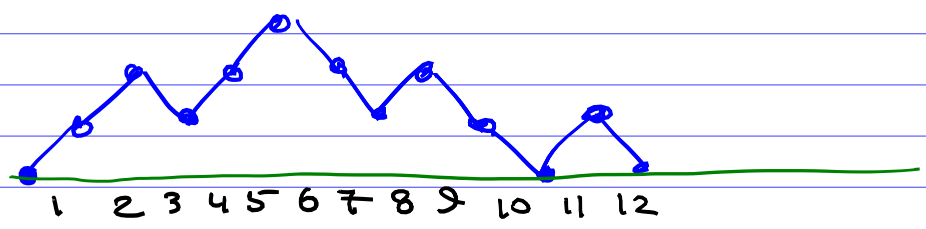
$Q$:

| | | | |
|---|---|---|---|
| 1 | 3 | 4 | 6 |
| 2 | 5 | | |

$T =$

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 4 | 5 | 8 | 11 |
| 3 | 6 | 7 | 9 | 10 | 12 |

Recall that SYT's of shape $2 \times n$ are in bijection with Dyck paths:




If $i \in 1^{st}$ row of $T$, then $i^{th}$ step is "up"

If $j \in 2^{nd}$ row of $T$, the $j^{th}$ step is "down"

So RSK is this case gives a bijection:

$$\left\{\begin{matrix}\text{321-avoiding}\\ \text{permutations}\\ \text{in } S_n\end{matrix}\right\} \overset{RSK}{\longleftrightarrow} \left\{\begin{matrix}\text{Dyck}\\ \text{paths with}\\ 2n \text{ steps}\end{matrix}\right\}$$

All this is nice, but in order to show that RSK works we need to prove some things...

Is there a simpler way to prove the identity

$$\sum_{\lambda \vdash n} (f_\lambda)^2 = n! \qquad ?$$

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## Up & Down operators on Young's lattice $\mathbb{Y}$

you can take your favorite field instead of $\mathbb{R}$

Let $\mathbb{R}[\mathbb{Y}]$ be the linear space of formal linear combinations of Young diagrams

**Example:**

$$3\,\ydiagram{2} - 27\,\ydiagram{2,1} + \pi\,\ydiagram{2,2,1}$$

an element of $\mathbb{R}[\mathbb{Y}]$

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Clearly, $\mathbb{R}[\mathbb{Y}]$ has a linear basis given by Young diagrams.

Define two linear operators $U$ and $D$ acting on $\mathbb{R}[\mathbb{Y}]$ by

$$U : \lambda \longmapsto \sum_{\mu : \mu \gtrdot \lambda} \mu$$

obtained from $\lambda$ by adding a box

$$D : \lambda \longmapsto \sum_{\mu : \mu \lessdot \lambda} \mu$$

obtained from $\lambda$ by removing a box

<u>Examples</u>

- $U(\ydiagram{1}) = \ydiagram{2} + \ydiagram{1,1}$
- $D(\ydiagram{1}) = \varnothing$

This is not $0$. This is a basis element of $\mathbb{R}[\mathbb{Y}]$ given by the empty Young diagram $\varnothing$

$\varnothing \neq 0$

- $UU(\ydiagram{1}) =$

$= U(\ydiagram{2} + \ydiagram{1,1})$

$= \ydiagram{3} + \ydiagram{2,1}$

$+ \ydiagram{2,1} + \ydiagram{1,1,1} = \ydiagram{3} + 2\,\ydiagram{2,1} + \ydiagram{1,1,1}.$

- $D D(\ydiagram{1}) = D(\varnothing) = 0.$

This is the "true" $0$.

There is no way to remove a box from the empty Young diagram $\varnothing$.

Notice that

$$U^n(\varnothing) = \sum_{\lambda \vdash n}' f_\lambda \cdot \lambda$$

The coefficient of $\lambda$ equals
# saturated chains from $\varnothing$ to $\lambda$
in $\mathbb{Y}$ $= f_\lambda$

Also notice that

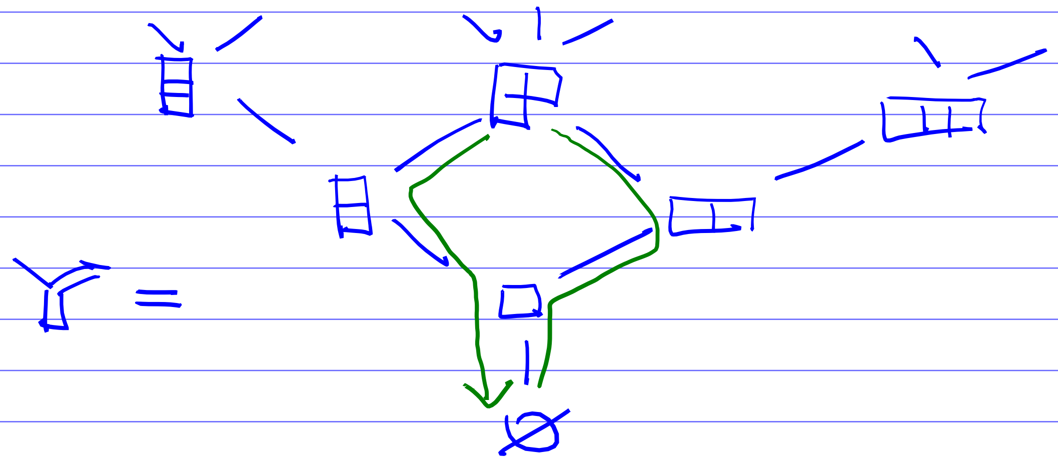
$$D^n(\lambda) = f_\lambda \cdot \varnothing$$

for any $\lambda \vdash n$

So $D^n U^n(\varnothing) =$ ← by the definitio

$$= \sum_{\lambda \vdash n}' (f_\lambda)^2 \varnothing.$$

Example. $n = 3$

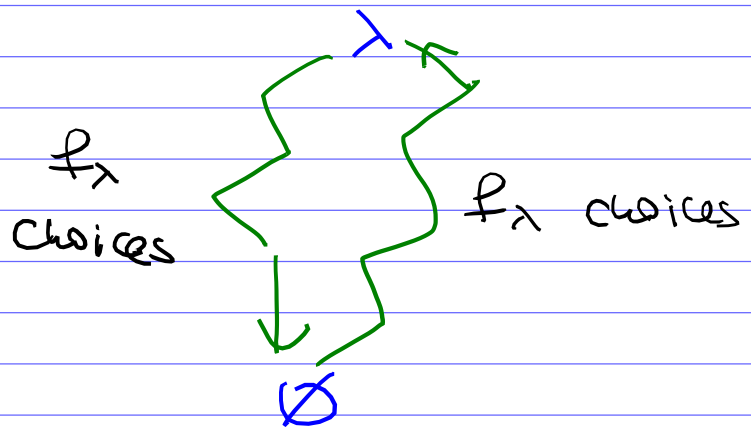


$D^3 U^3(\varnothing)$ counts paths in this graph that start at $\varnothing$ make 3 "up" steps and then 3 "down" steps.

All such paths should end at $\varnothing$ and # such path equals:

$$1\cdot 1 + 2\cdot 2 + 1\cdot 1 = 6.$$

In general, $D^n U^n(\emptyset)$ counts paths of the form




where $\lambda$ is any partition $\lambda \vdash n$.

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To prove $\sum_{\lambda \vdash n} (f_\lambda)^2 = n!$

we now need to show

<u>Claim</u> $D^n U^n(\emptyset) = n!\,\emptyset$.

<u>Crucial Lemma</u>. The operators $U$ & $D$ satisfy the relation:

- $D \cdot U - U \cdot D = I$ the identity operator
- $D(\emptyset) = 0$.

## Example

$$\lambda = \ydiagram{3,2}$$

$$U(\lambda) = \begin{ytableau} \; & \; & \; & \bullet \\ \; & \; \end{ytableau} + \begin{ytableau} \; & \; & \; \\ \; & \; & \bullet \end{ytableau} + \begin{ytableau} \; & \; & \; \\ \; & \; \\ \bullet \end{ytableau}$$

$$D(U(\lambda)) = \left( \ydiagram{3,2} + \ydiagram{4,1} \right)$$
$$+ \ydiagram{3,2}$$
$$+ \left( \ydiagram{3,2} + \ydiagram{3,1,1} + \ydiagram{2,2,1} \right).$$

$$D(\lambda) = \ydiagram{3,1} + \ydiagram{2,2}$$

$$U(D(\lambda)) = \left( \begin{ytableau} \; & \; & \; & \bullet \\ \; \end{ytableau} + \begin{ytableau} \; & \; & \; \\ \; & \bullet \end{ytableau} + \begin{ytableau} \; & \; & \; \\ \; \\ \bullet \end{ytableau} \right)$$
$$+ \left( \begin{ytableau} \; & \; & \bullet \\ \; & \; \end{ytableau} + \begin{ytableau} \; & \; \\ \; & \; \\ \bullet \end{ytableau} \right).$$

Now

lot of things cancel

$$D(U(\lambda)) - U(D(\lambda)) = \cdots\cdots =$$

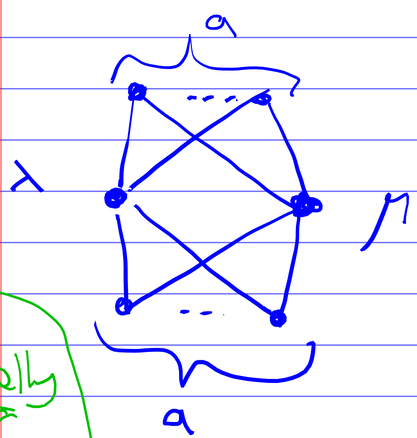
$$= \lambda.$$

Combinatorially, the relation

$$D\cdot U(\lambda) - UD(\lambda) = \lambda$$

is equivalent to the following properties of Young's lattice:

<u>Lemma</u>. (A) For $\lambda \neq \mu$
$|\lambda| = |\mu|$

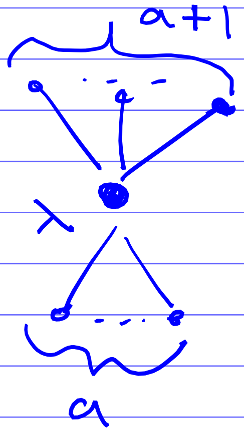



$$\#\{\nu : \nu \gtrdot \lambda \ \&\ \nu \gtrdot \mu\}$$

$$= \#\{\gamma : \gamma \lessdot \lambda \ \&\ \gamma \lessdot \mu\}$$

Actually for YL $a = 0$ or $1$

(B) For $\lambda = \mu \vdash n$

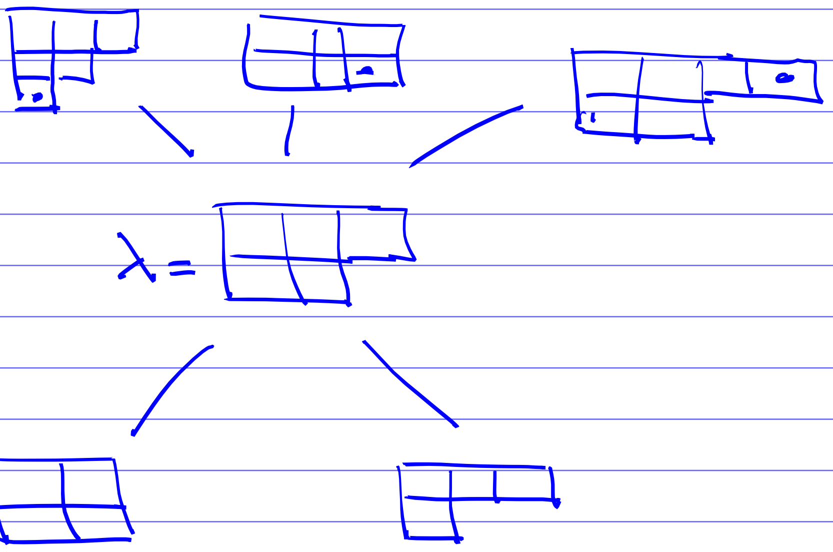



$$\#\{\nu : \nu \gtrdot \lambda\}$$

$$= \#\{\gamma : \gamma \lessdot \lambda\}$$

$$+ 1$$

## Example



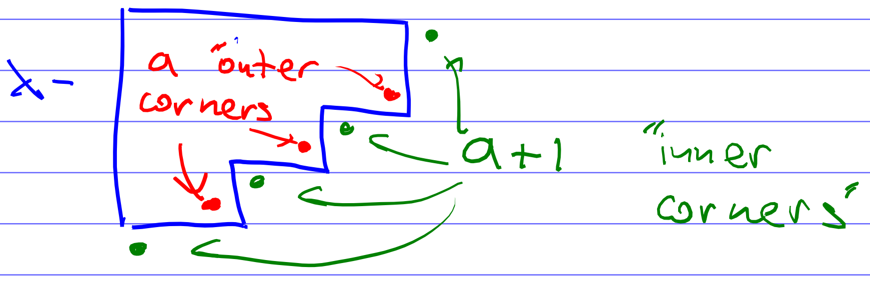

## Proof of Lemma

(A) $\lambda \neq \mu$: We want to add a box $x$ to $\lambda$ and then remove a <u>different</u> box $y \neq x$ to get $\mu$.

The result will be exactly the same if we first remove box $y$ and then add box $x$.

(B) $\lambda = \mu$

We want to count # ways to add a box $x$ to $\lambda$ and then remove the <u>same</u> box. There are exactly 1 more way to do this than # way to remove a box from $\lambda$ & then add the same box



Claim The relations

- $DU - DU = I$
- $D(\emptyset) = 0$

imply $D^n U^n (\emptyset) = n! \, \emptyset$

Examples: $n = 1$

$$DU(\emptyset) = (I + UD)(\emptyset)$$
$$= \emptyset + 0 = 1 \cdot \emptyset.$$

$n = 2$

$$D\underline{DU}U(\emptyset) =$$
$$= D(UD + I)U(\emptyset)$$
$$= DU(\emptyset) + DUDU(\emptyset)$$
$$= \emptyset + DU(\emptyset) = \emptyset + \emptyset$$
$$= 2! \, \emptyset.$$

etc.