q - analogs

A "q-analog" of some "classical" combinatorial number A is a polynomial $A(q) = a_0 + a_1 q + \dots + a_d q^d$ (typically, with nonnegative integer coefficients a_i) such that $A(1) = A$ (the "classical limit").

q-numbers.

$$[n]_q := 1 + q + q^2 + \dots + q^{n-1}$$
$$= \frac{1-q^n}{1-q}$$

q-factorials:

$$[n]_q! := [1]_q [2]_q \cdots [n]_q!$$

This is a bit confusing notation: $[n]_q! \neq ([n]_q)!$

Example:

$$[3]_q! := (1+q)(1+q+q^2)$$
$$= 1 + 2q + 2q^2 + q^3$$

Have we already seen $[n]_q!$ in this class?

Theorem. $$\sum_{w \in S_n} q^{\mathrm{inv}(w)} = [n]_q!$$

inversions in perm. w

the sum over all permutations of size n

Also $$\sum_{w \in S_n} q^{\mathrm{maj}(w)} = [n]_q!$$

If A counts # some combinatorial objects, then its q-analog $A(q)$ counts these object according to some statistics

q-binomial coefficients

a.k.a. the Gaussian coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q!\,[n-k]_q!}$$

Carl Friedrich Gauss
1777 – 1855

Example $$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{[4]_q [3]_q [2]_q}{[2]_q\ [2]_q}$$

$$= \frac{(1+q+q^2+q^3)(1+q+q^2)}{(1+q)}$$

$$= (1+q^2)\cdot(1+q+q^2)$$

$$= 1 + q + 2q^2 + q^3 + q^4$$

This turns out to be a polynomial with positive integer coefficient. But in general, this is not obvious. Why should the numerator be divisible by the denominator?

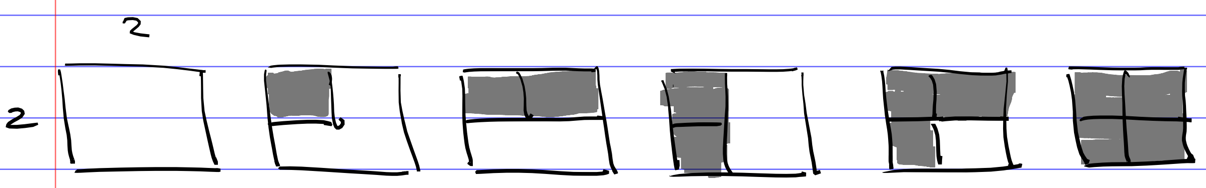
Theorem.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\lambda \subseteq k \times (n-k)} q^{|\lambda|}.$$

The sum over Young diagr. λ that fit inside the $k \times (n-k)$ rectangle

The coefficients of this expression are exactly the rank numbers r_k of the Young's lattice $L(k, n-k)$.

Example $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q =$



$$1 + q + q^2 + q^2 + q^3 + q^4$$

Proof #1. It is not hard to prove it by induction on n. One can easily show that both L.H.S. & R.H.S. satisfy the q-Pascal's recurrence relation:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$$

Exercise Check this.

Remark This is an easy but not very conceptual proof. We'll give another more interesting proof.

q-Pascal's triangle

Voici le q-analogue de mon triangle

Blaise Pascal
1623-1662

$$\begin{bmatrix}0\\0\end{bmatrix}_q$$

$$\begin{bmatrix}1\\0\end{bmatrix}_q \qquad \begin{bmatrix}1\\1\end{bmatrix}_q$$

$$\begin{bmatrix}2\\0\end{bmatrix}_q \qquad \begin{bmatrix}2\\1\end{bmatrix}_q \qquad \begin{bmatrix}2\\2\end{bmatrix}_q$$

$$\cdots$$

$$\|$$

$$1$$

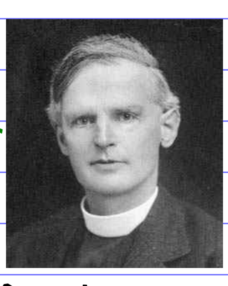
$$1 \qquad 1$$

$$1 \qquad 1+q \qquad 1$$

$$1 \qquad 1+q+q^2 \qquad 1+q+q^2 \qquad 1$$

$$1 \qquad 1+q+q^2+q^3 \qquad 1+q+2q^2+q^3+q^4 \qquad 1+q+q^2+q^3 \qquad 1$$

$$\cdots$$



My diagrams are not that young.

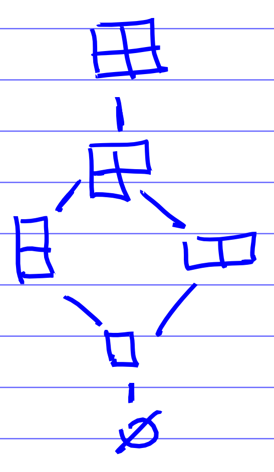
Alfred Young
1873 – 1940

Young's lattices

Recall, $L(m,n) := \mathbb{Y}_{m\times n}$ is the lattice of all Young diagrams λ that fit inside the $m\times n$ rectangle ordered by containment.

Example:

$L(2,2) =$



Clearly, Young's lattice $L(m,n)$ is a ranked poset with the rank function $\rho(\lambda) = |\lambda|$.

Let $r_k = r_k\big(L(m,n)\big)$, $k = 0, 1, \dots, m\cdot n$, denote the rank numbers of Young's lattice $L(m,n)$.

Explicitly,

$$r_k\big(L(m,n)\big) := \left\{ \begin{array}{l} \text{Young diagrams } \lambda \text{ s.t.} \\ \lambda \subseteq m\times n \\ |\lambda| = k \end{array} \right\}$$

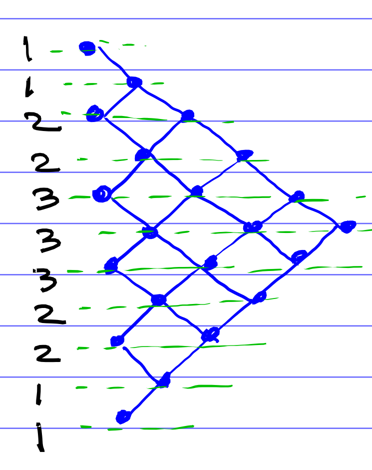
The generating function

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_q = r_0 + r_1 q + \dots + r_{mn}\, q^{m\cdot n}$$

is exactly the Gaussian q-binomial coefficient.

Example:

$L(5,2) =$



The rank numbers $r_k\big(L(5,2)\big)$ are: 1, 1, 2, 2, 3, 3, 3, 2, 2, 1, 1.

The Gaussian q-binomial coeff. is

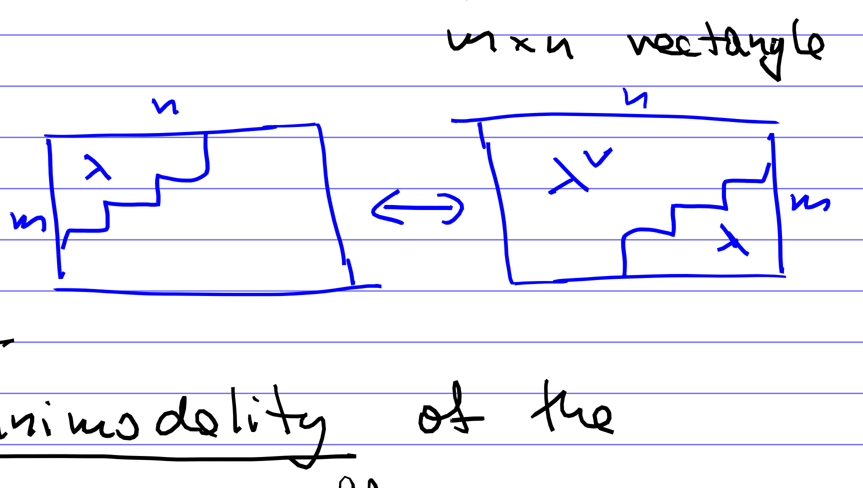
$$\begin{bmatrix} 7 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + 2q^3 + 3q^4 + 3q^5 + 3q^6 + 2q^7 + 2q^8 + q^9 + q^{10}.$$

Theorem. Young's lattice $L(m,n)$ is

- rank symmetric: $r_k = r_{m\cdot n - k}$
- rank unimodal: $r_0 \le r_1 \le \dots \le r_{\lfloor \frac{m\cdot n}{2} \rfloor} \ge \dots \ge r_{mn}$
- Sperner.

Remarks

- the rank symmetry is trivial: $\lambda \overset{1:1}{\longleftrightarrow} \lambda^\vee :=$ the complement of λ in the $m\times n$ rectangle



- the unimodality of the Gaussian coefficients is a notoriously hard result. The first proof was given by Sylvester in 1878. The first constructive proof was given more than 100 years later by K. O'Hara '1990

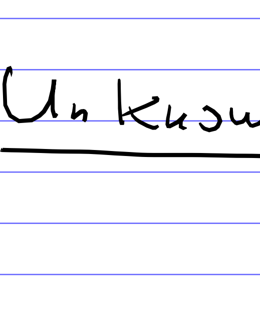
We will discuss Sylvesters proof

- The Sperner property of $L(m,n)$ was proved by R. Stanley '1980.

I probably looked like this when I proved the Sperner property of $L(m,n)$.

Does $L(m,n)$ have a symmetric chain decomposition?

Unknown, for general m & n.



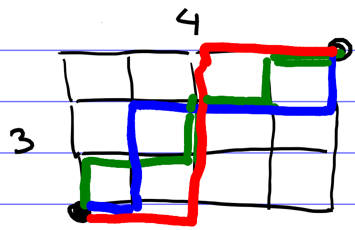
I conjecture that $L(m,n)$ has a symmetric chain decomposition for any positive m and n.

The Sperner property of $L(m,n)$ can be reformulated like this:

Let $P_1, P_2, \ldots, P_N$ be any collection of lattice paths from the lower left to the upper right corners in the $m \times n$ rectangle s.t. any two paths P_i & P_j intersect

Here we mean that the paths intersect in the strict sense, not just "touch" each other

For example,



Then $N \le r_{\lfloor \frac{m \cdot n}{2} \rfloor} := \#\left\{ \begin{array}{l} \text{Young diagr.} \\ \lambda \subseteq m \times n \\ \text{with} \\ |\lambda| = \lfloor \frac{m \cdot n}{2} \rfloor \end{array} \right\}$

A more conceptual proof that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\lambda \subseteq k \times (n-k)} q^{|\lambda|}.$$

<u>Proof #2</u>. Both sides are rational expressions in q. It is enough to check that they are equal to each other for infinitely many values of q.

We'll prove this when $q = p^r$ (a power of a prime number p).

There are infinitely many prime numbers!

Euclid 300 BC

Let $\mathbb{F}_q$ be the finite field with q- elements.

If not that fatal duel, I would have discovered a lot of other genius things

Évariste Galois
1811 — 1832

Finite fields are also known as Galois fields.

$GL_n(\mathbb{F}_q)$ – all invertible $n \times n$ matrices with elements in $\mathbb{F}_q$.

What is $\# GL_n(\mathbb{F}_q)$?

<u>Example</u> $q = 2$, $n = 2$

Invertible

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\# GL_2(\mathbb{F}_2) = 6.$$

Let's construct an invertible $n \times n$ matrix by picking its rows one by one.

$$\begin{bmatrix} \text{— row 1 —} \\ \text{— row 2 —} \\ \vdots \\ \text{— row } n \text{ —} \end{bmatrix}$$

Row 1 can be any n-vector over $\mathbb{F}_q$, except $(0,0,\ldots,0)$.
So we have $q^n - 1$ choices.

Row 2 can be any n-vector over $\mathbb{F}_q$, except Row 1 rescaled by an element of $\mathbb{F}_q$.
We have $q^n - q$ choices.

Row 3 can be any n-vector, except a linear combination of Row 1 & Row 2
We have $q^n - q$ choices

etc.

Theorem There are exactly
$(q^n-1)(q^n-q)(q^n-q^2)\ldots(q^n-q^{k-1})$
$k \times n$ matrices over $\mathbb{F}_q$ that have maximal possible rank r.

In particular,

$$\# GL_n(\mathbb{F}_q) =$$
$$= (q^n-1)(q^n-q)(q^n-q^2)\ldots(q^n-q^{n-1})$$
$$= q^{\binom{n}{2}} \cdot (q-1)^n \; [n]_q!$$

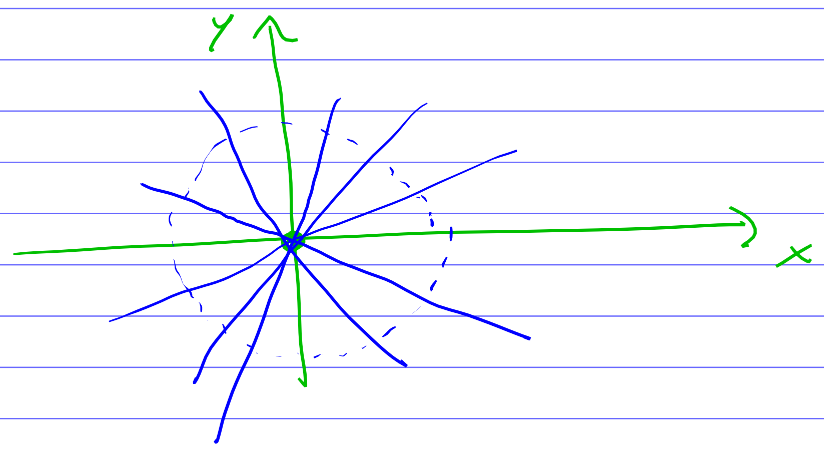
The Grassmannian $Gr(k,n;\mathbb{F})$ is the "space" of k-dim linear subspaces in $\mathbb{F}^n$.

Hermann Günter Graßmann
1809 – 1877

I invented a parametrization of $Gr(2,4;\mathbb{R})$

Julius Plücker
1801–1868

Example. $k=1$, $n=2$. The elements of $Gr(1,2)$ (a.k.a. the projective line) are lines that pass through the origin



$Gr(1,2,\mathbb{R}) \simeq$ a circle where any two opposite points are identified

$\simeq$ just a circle.

More concretely, $Gr(k,n;\mathbb{F}_q)$ is the space of $k\times n$ matrices over $\mathbb{F}_q$ of rank k modulo row operations

$$\underset{k}{\overset{k}{\boxed{\text{any invert. } k\times k \text{ matr.}}}} \Big\backslash \ \overset{n}{\boxed{\text{any } k\times n \text{ matrix of rank } k}}$$

So $\# Gr(k,n;\mathbb{F}_q) =$

$$= \frac{(q^n-1)(q^n-q)(q^n-q^2)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)(q^k-q^2)\cdots(q^k-q^{k-1})}$$

Theorem $\# Gr(k,n;\mathbb{F}_q)$

$$= \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

Let's count $\# Gr(k,n;\mathbb{F}_q)$ in a different way...

Gaussian elimination.

Elimination is named after me. But Chinese knew about it in 179 CE

We know from Linear Algebra that any $k\times n$ matrix of rank k can by transformed by row operations into a canonical form, called reduced row-echelon form.

Example: $k=5$, $n=10$.

pivots

$$k\begin{bmatrix} \boxed{1} & * & 0 & * & * & 0 & 0 & * & 0 & * \\ 0 & 0 & \boxed{1} & * & * & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & * \end{bmatrix}$$
n

"$*$" - any element of $\mathbb{F}_q$

$\#$ pivots = the rank of matrix.

So there should be exactly k = # rows pivots

$\# Gr(k,n;\mathbb{F}_q) =$

$=$ # matrices in the reduced row-echelon form.

This echelon form looks better if we remove the pivot columns:

$$k\begin{bmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix}$$
$n-k$

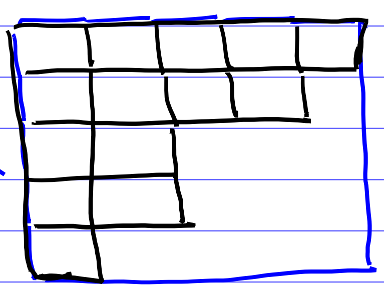
we are not loosing any information here. We can uniquely recover the positions of pivot columns from the shape formed by "$*$"s.

such matrices

$= q^{\# *'s}$

Notice that the pattern formed by "$*$"s is (the mirror image of) a Young diagram $\lambda \subseteq k\times(n-k)$

$$k\begin{bmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \rightsquigarrow k$$
$n-k$



$n-k$

$\lambda = (5,4,2,2,1)$

$\# *'s = |\lambda|$

We obtain

$\# Gr(k,n;\mathbb{F}_q)$

$= \# \left\{ \begin{array}{l} k\times n \text{ matrices of rank } k \\ \text{in reduced row-echelon} \\ \text{form} \end{array} \right\}$

$$= \sum_{\lambda \subseteq k\times(n-k)} q^{|\lambda|}.$$

Comparing this with the previous expression

$\# Gr(k,n;\mathbb{F}_q) = \left[{n \atop k}\right]_q$

we deduce the result. $\square$

We've got

$$[n]_q! = \sum_{w \in S_n} q^{inv(w)}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\lambda \subseteq k \times (n-k)} q^{|\lambda|}$$

How about the multinomial coefficients?

Let $n_1 + \dots + n_r = n$ (a composition of n)

The multinomial coeffs. are defined as

$$\binom{n}{n_1\, n_2 \dots n_r} := \frac{n!}{n_1!\, n_2! \dots n_r!}$$

__Proposition__ $\binom{n}{n_1\, n_2 \dots n_r} =$

$=$ # __permutations of the multiset__

$\{1^{n_1}, 2^{n_2}, \dots, r^{n_r}\}$

$:= \{\underbrace{1, \dots, 1}_{n_1}, \underbrace{2, \dots, 2}_{n_2}, \dots, \underbrace{r, \dots, r}_{n_r}\}$

i.e. words with exactly n_1 1's, n_2 2's, etc.

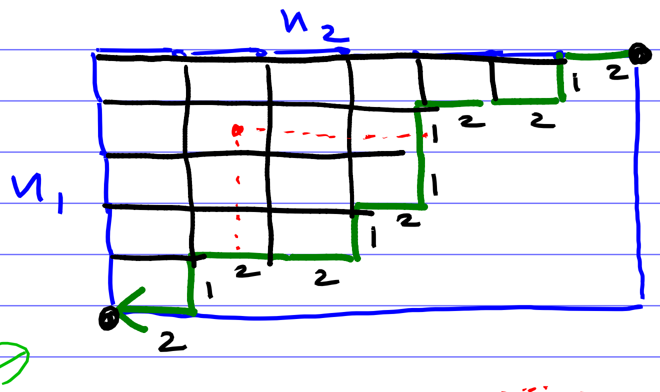
For a permutation of multiset (or word) $w = w_1, \dots, w_n$ define the <u>number of inversions</u>

$$\mathrm{inv}(w) := \#\{1 \le i < j \le n \mid w_i > w_j\}$$

Clearly, if $w \in S_n$ is a usual permutation of $1, 2, \dots, n$, then $\mathrm{inv}(w)$ is the inversion number of w that we discussed before.

<u>Example.</u> $r = 2$

Let us identify a Young diagram λ that fits inside the $n_1 \times n_2$ rectangle we the following permutation w of the multiset $\{1^{n_1}, 2^{n_2}\}$:



$n_1 = 5$, $n_2 = 7$

$\lambda = (6, 4, 4, 3, 1)$

$$w = 2, 1, 2, 2, 1, 1, 2, 1, 2, 2, 1, 2$$

walk along the border of shape λ from the upper right corner to the lower left corner

Then

$$\mathrm{inv}(w) = |\lambda|$$

inversions of w are in bijection with boxes of λ

q-multinomial coefficients

$$\begin{bmatrix} n \\ n_1\, n_2 \dots n_r \end{bmatrix}_q := \frac{[n]_q!}{[n_1]_q! \dots [n_r]_q!}$$

Theorem. $\begin{bmatrix} n \\ n_1\, n_2 \dots n_r \end{bmatrix}_q$ is a polynomial in q with non-negative integer coefficients.

$$\begin{bmatrix} n \\ n_1 \dots n_r \end{bmatrix}_q = \sum_{\substack{w \text{ is} \\ \text{a perm. of} \\ \text{the multiset} \\ \{1^{n_1}, 2^{n_2}, \dots, r^{n_r}\}}} q^{\mathrm{inv}(w)}$$

The 2 proofs that for the q-binomial coefficients (based on recurrence relation and on finite field) can be easily extended to all q-multinomial coeffs...

It is also easy to deduce the polynomiality of the q-multinomial coeffs. from the polynomiality of q-binomial coeffs.

Indeed,

$$\begin{bmatrix} n \\ n_1\, n_2 \dots n_r \end{bmatrix}_q =$$

$$= \begin{bmatrix} n \\ n_1 \end{bmatrix}_q \begin{bmatrix} n-n_1 \\ n_2 \end{bmatrix} \begin{bmatrix} n-n_1-n_2 \\ n_3 \end{bmatrix}_q \cdots \begin{bmatrix} n_{r-1}+n_r \\ n_{r-1} \end{bmatrix}_q$$

$\square$

Example $\quad u_1, u_2, u_3 = 1, 1, 2$

Perms. of the multiset $\{1,2,3,3\}$

w	$\mathrm{inv}(w)$
1 2 3 3	0
1 3 2 3	1
1 3 3 2	2
2 1 3 3	1
2 3 1 3	2
2 3 3 1	3
3 1 2 3	2
3 1 3 2	3
3 2 1 3	3
3 2 3 1	4
3 3 1 2	4
3 3 2 1	5

$$\begin{bmatrix} 4 \\ 1 \; 1 \; 2 \end{bmatrix}_q = 1 + 2q + 3q^2 + 3q^3 + 2q^4 + q^5$$

$$\begin{bmatrix} 4 \\ 1 \; 1 \; 2 \end{bmatrix}_q = [4]_q [3]_q$$

Notice that the coeffs. 1, 2, 3, 3, 2, 1 are symmetric & unimodal.

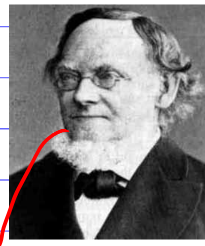
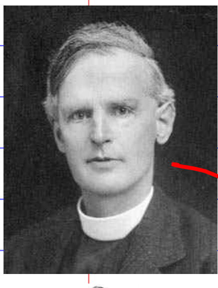
The symmetry is easy, the unimodality is hard.

Next week we'll discuss how to prove the unimodality.

Question. Is there a generalization of the major index $\mathrm{maj}(w)$ to permutations of multisets, which is equidistributed with the inversion number $\mathrm{inv}(w)$?

How about my index?

Percy Alexander MacMahon
1854 – 1929


Don't forget to upload your Problem Set!