18.212 Lecture 7 03/03/2021

Permutations $w \in S_n$

- w is a rearrangement $w_1, \dots, w_n$ of $1, 2, \dots, n$.
- w is a bijection $w : [n] \to [n]$
  $w : i \mapsto w_i$, for $i = 1, \dots, n$.

Several notations for permutations:

- **1-line notation**: $w = w_1, \dots, w_n$

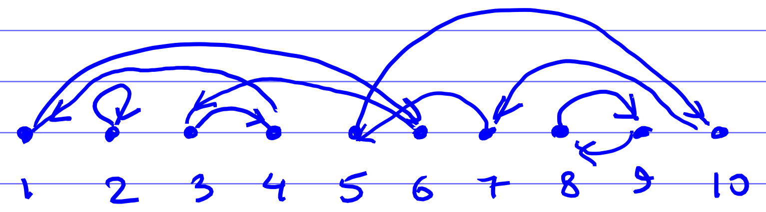
Example: $n = 10$

$w = 6, 2, 4, 1, 10, 3, 5, 9, 8, 7$

- **2-line notation**: $w = \begin{pmatrix} 1 & 2 & \dots & n \\ w_1 & w_2 & \dots & w_n \end{pmatrix}$

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 2 & 4 & 1 & 10 & 3 & 5 & 9 & 8 & 7 \end{pmatrix}$$

- **graphical notation**:



- **cycle notation**:

cycles of w

$w = (1, 6, 3, 4)\,(2)\,(5, 10, 7)\,(8, 9)$

a fixed point $w(i) = i$

This permutation has 4 cycles, including one fixed point.

The cycle $(1, 6, 3, 4)$ means that w cyclically shifts the entries $1, 6, 3, 4$: $w : 1 \mapsto 6$
$6 \mapsto 3$
$3 \mapsto 4$
$4 \mapsto 6$

So $(1, 6, 3, 4) = (6, 3, 4, 1) =$
$= (3, 4, 1, 6) = (4, 1, 6, 3)$.

this is the same cycle

Statistics on permutations

Definition: A *statistics* on permut. is a function $\sigma : S_n \to \mathbb{Z}_{\geq 0}$

Its *generating function* is

$$f_\sigma(x) := \sum_{w \in S_n}' x^{\sigma(w)}$$

Two statistics σ and μ on S_n are *equidistributed* if

$$f_\sigma(x) = f_\mu(x).$$

It turns out that many interesting statistics on S_n fall into 3 classes of equidistributed statistics, called

- *Mahonian* statistics

  named after Percy Alexander MacMahon

- *Eulerian* statistics
- no common name

  Should we call them *"Stirlingian"* statistics?

Definition: For a permutation $w \in S_n$

- an inversion is a pair (i,j) s.t. $1 \leq i < j \leq n$ and $w_i > w_j$
- a desent is an index $i \in \{1, \dots, n-1\}$ s.t. $w_{i+1} > w_i$

Let $\mathrm{inv}(w) := \#$ inversions in w

$\mathrm{des}(w) := \#$ descent in w.

Also let $\mathrm{cyc}(w) := \#$ cycles in w, (including fixed points)

Example:

16 inversions

$w = 6\ 2\ 4\ 1\ 10\ 3\ 5\ 9\ 8\ 7$

5 descents

$\mathrm{inv}(w) = 16$,

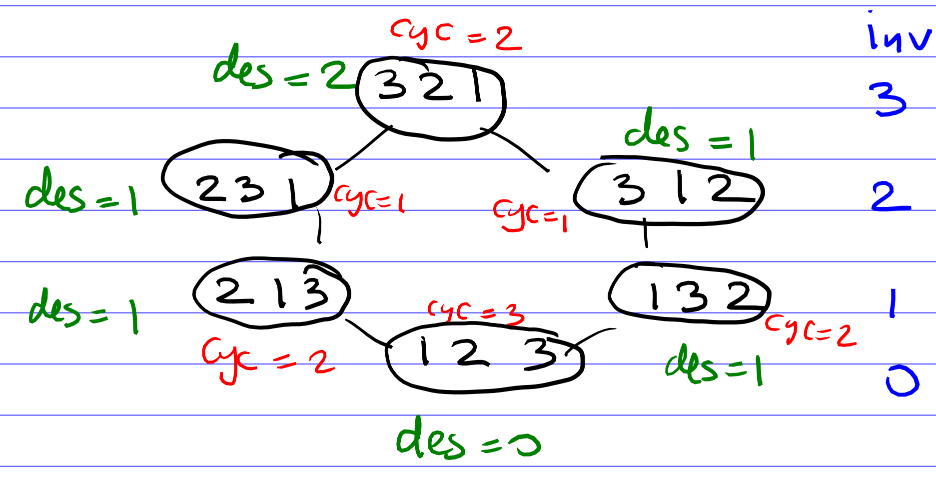
$\mathrm{des}(w) = 5$

$\mathrm{cyc}(w) = 4$

- A Mahonian statistic is any statistic equidistributed with inv.
- A Eulerian statistic is any statistic equidistributed with des.
- A "Stirlingian" statistic is any statistic equidistributed with cyc.

Example: $n = 3$

S_3



$$f_{inv}(x) = 1 + 2x + 2x^2 + x^3$$
$$= (1+x)(1+x+x^2).$$

$$f_{des}(x) = 1 + 4x + x^2$$

$$f_{cyc}(x) = 2x + 3x^2 + x^3$$
$$= x(x+1)(x+2)$$

Theorem. $\displaystyle\sum_{w \in S_n} x^{inv(w)}$

$$= (1+x)(1+x+x^2)\cdots(1+x+x^2+\dots+x^{n-1})$$

Proof. Easy to prove by induction on n. If we take a permutation $w \in S_{n-1}$ written in 1-line notation, there n possible positions to insert a new entry "n":

$$w = \underset{0}{\sqcup}\, w_1 \underset{1}{\sqcup}\, w_2 \underset{2}{\sqcup} \cdots\, w_{n-1} \underset{n-1}{\sqcup}$$

Such insertions of "n" preserve all inversions of w, and add $0, 1, 2, \dots,$ or $n-1$ new inversions depending on position of "n".

So $$\sum_{u \in S_n} x^{inv(u)} =$$

$$= \Big(\sum_{w \in S_{n-1}} x^{inv(w)}\Big)(1+x+\dots+x^{n-1})$$

$$\overset{ind.}{=} (1+x)(1+x+x^2)\cdots(1+x+\dots+x^{n-1})$$

$\square$

Def. The Eulerian numbers are

$A_{n,k} :=$ # permutation $w \in S_n$ with $des(w) = k$

The Eulerian polynomials:

$$A_n(x) := \sum_{k=0}^{n} A_{nk}\, x^k = \sum_{w \in S_n} x^{des(w)}.$$

Example above: $A_3(x) = 1 + 4x + x^2$.

More examples of permutation statistics:

- **Major index**

$$\mathrm{maj}(w) = \sum_{\substack{i \text{ is a descent} \\ \text{of } w}}' i$$

Named after major Persi MacMahon.

Example: $w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 7 > & 3 > & 1 & 6 & 8 > & 4 \end{pmatrix}$

$\mathrm{des}(w) = 3$

$\mathrm{maj}(w) = 3 + 4 + 7 = 14.$

Exercise Prove that maj is equidistributed with inv.
(So it is a Mahonian statistics)

- **Records**

Def. A record of a permutation $w \in S_n$ is an entry w_i which is greater than all preceding entries $w_1, w_2, \dots, w_{i-1}$.

$\mathrm{rec}(w) := \#$ records in w.

Example: $w = 2, 5, 7, 3, 1, 6, 8, 4$

records are $2, 5, 7, 8$

$\mathrm{rec}(w) = 4.$

Theorem The statistics cyc and rec are equidistributed.

Proof. Let's construct a bijection $f: S_n \to S_n$ such that if $f: w \mapsto \tilde{w}$ the $\mathrm{cyc}(w) = \mathrm{rec}(\tilde{w})$.

We'll use both the cyclic & the 1-line notations for permutations.

There are many different ways to write a permutation in cyclic notation, e.g.,

$$w = (2,5,3)(1,6)(4)$$
$$= (6,1)(4)(5,3,2)$$
$$= (4)(3,2,5)(1,6) = \dots$$

Let's make a choice of cyclic notation for w

$$w = (a_1 \dots)(a_2 \dots)(a_3 \dots) \dots$$

unique by requiring

- the entries $a_1, a_2, \dots$ are the maximal entries in the cycles
- the cycles are arranged so that $a_1 < a_2 < \dots$

The map $w \mapsto \tilde{w}$:
write the permutation w in cyclic notation, as above. Then erase all parentheses and regard it as 1-line notation for $\tilde{w}$.

Example:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 7 & 3 & 1 & 6 & 8 & 4 \end{pmatrix}$$
$$= (1,2,5)(3,7,8,4)(6)$$
$$= (\underline{5},1,2)(\underline{6})(\underline{8},4,3,7)$$
$$\downarrow$$
$$\tilde{w} = \underline{5},1,2,\underline{6},\underline{8},4,3,7$$

Notice that the first elements $a_1, a_2, \dots$ in the cycles of w are exactly the records of $\tilde{w}$.

The inverse map $\tilde{w} \mapsto w$:
Write $\tilde{w}$ in 1-line notation, insert ")(" in front of each record of $\tilde{w}$, and consider it as cycle notation for w.

Example: $\tilde{w} = \underline{5}\ 1\ 2\ \underline{6}\ \underline{8}\ 4\ 3\ 7$

$$w = (\underline{5}\ 1\ 2)(\underline{6})(\underline{8}\ 4\ 3\ 7)$$

So the map $w \mapsto \tilde{w}$ is a bijection $S_n \to S_n$. $\square$

Exceedances

Def. For $w \in S_n$, an index $i \in \{1, \dots, n\}$ is an <u>exceedance</u> if $w_i > i$.

Let $exc(w) := \#$ exceedances in w.

Example.

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \wedge & \wedge & \wedge & & & & \wedge & \\ 2 & 5 & 7 & 3 & 1 & 6 & 8 & 4 \end{pmatrix}$$

$exc(w) = 4.$

Theorem. The statistics exc and des are equidistributed.

Proof. It is a bit easier to work with <u>anti-exceedances</u> of w

(Def: i is an <u>anti-exceedance</u> of w if $w_i < i$.)

Notice that $\#$ exceedances of w $= \#$ anti-exceedances of w^{-1}.

So exc is equidistributed with $\#$ anti-exceedances.

Now the map $w \mapsto \tilde{w}$ (from the proof of previous Thm) converts $\#$ anti-exceedances in w to $\#$ descents of $\tilde{w}$

(Check this!) $\square$

Another related statistic:

wexc(w) := # weak exceedances in w

$:= \#\{ i \in [n] \mid w_i \geq i \}$

$= exc(w) + \#$ fixed points in w.

Example:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \wedge\!| & \wedge\!| & \wedge\!| & & & \wedge\!| & \wedge\!| & \\ 2 & 5 & 7 & 3 & 1 & 6 & 8 & 4 \end{pmatrix}$$

$exc(w) = 4$

$wexc(w) = 5$

Exercise. For $n \geq 2$, show that the statistics on S_n

$exc(w)$ and $wexc(w) - 1$

are equidistributed.

Example $n = 3$

w	exc(w)	wexc(w)	wexc(w)−1
123	0	3	2
213	1	2	1
132	1	2	1
312	1	1	0
231	2	2	1
321	1	2	1

gen. functions: $1 + 4x + x^2 \quad = \quad 1 + 4x + x^2$

Stirling Numbers

Definition. For $0 \le k \le n$, the Stirling number of the first kind is

$$s(n,k) := (-1)^{n-k}\, c(n,k),$$

where $c(n,k)$ is the signless Stirling number of the first kind, defined as

$$c(n,k) := \#\text{ permutations } w \in S_n \text{ with } \operatorname{cyc}(w) = k.$$

Also recall, the Stirling numbers of the second kind:

$$S(n,k) := \#\text{ set partitions of } [n] \text{ with } k \text{ blocks.}$$

Recall the rising & falling factorials:

$$x^{(n)} := x(x+1)(x+2)\cdot\ldots\cdot(x+n-1)$$

$$x_{(n)} := x(x-1)(x-2)\cdot\ldots\cdot(x-n+1).$$

Theorem

$$\sum_{k=0}^{n} S(n,k)\, x^{k} = x_{(n)}$$

Equivalently, $\sum_{k=0}^{n} c(n,k) x^{k} = x^{(n)}$.

Compare this with the formula for the Stirling numbers of 2^{nd} kind from the previous lecture:

$$\sum_{k=0}^{n} S(n,k)\, x_{(k)} = x^{n}.$$

Consider the $(d+1)$-dimensional linear space V of all polynomials $f(x) = a_0 + a_1 x + \dots + a_d x^d$ of degree $\leq d$.

This space has the following 2 linear bases:

$$1, x, x^2, \dots, x^d$$

and $\quad x_{(0)} = 1, \quad x_{(1)} = x, \quad x_{(2)} = x(x-1), \quad \dots, x_{(d)}$.

The above formulas mean that the Stirling numbers (of both kinds) are the entries of change of basis matrices between these two bases.

<u>Corollary</u> The following 2 $(d+1)\times(d+1)$ matrices are inverse to each other:

$$\left(s(n,k)\right)_{0 \leq n,k \leq d} \quad \text{and}$$

$$\left(S(n,k)\right)_{0 \leq n,k \leq d}.$$

<u>Example</u>. $d=3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix}^{-1}$$

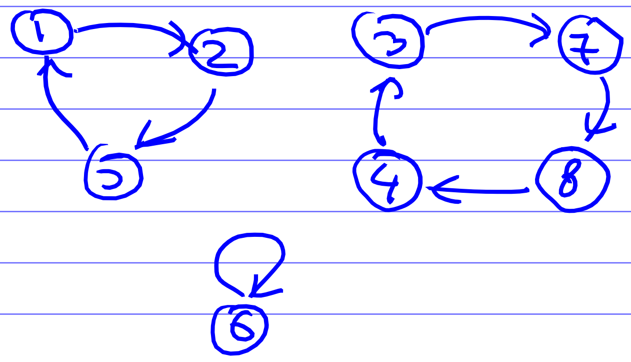
Stirling #'s of 1st kind

Stirling #'s of 2nd kind

Let's prove the above Theorem (that $\sum_{k=0}^{n} s(n,k)\, x^k = x_{(n)}$).

<u>Proof #1.</u> (Using the exponential formula from the previous lecture)

Permutations $w \in S_n$ (expressed in graphical notation) can be viewed as certain directed graphs on n labelled vertices:

$w =$ 

Their connected components correspond to permutations $w \in S_n$ with exactly 1 cycle ($cyc(w) = 1$).

The number of such permutations is

$$\#\{w \in S_n \mid cyc(w) = 1\} = (n-1)!$$

Indeed, in cyclic notation

$$w = (1, \underbrace{a, b, \ldots, z}_{\text{some rearrangement of } 2,3,\ldots,n})$$

Let $c(x) := \sum_{n\geq 1} \#\left\{ \begin{array}{c} w\in S_n \\ cyc(w)=1 \end{array}\right\} \frac{x^n}{n!}$

$$= \sum_{n\geq 1} (n-1)!\, \frac{x^n}{n!}$$

$$= \sum_{n\geq 1} \frac{x^n}{n} = -\log(1-x).$$

$$d(x,y) := \sum_{n\geq 0} \frac{x^n}{n!} \left(\sum_{w\in S_n} y^{cyc(w)} \right)$$

According to the exponential formula,

$$d(x,y) = \exp\left(y \cdot c(x)\right)$$

$$= \exp(-y \cdot \log(1-x))$$

$$= \left(\exp(\log(1-x))\right)^{-y}$$

$$= (1-x)^{-y}$$

$$= \sum_{n\geq 0} \binom{-y}{n} (-x)^n$$

binomial formula

$$= \sum_{n\geq 0} \underbrace{(-1)^n\, n! \binom{-y}{n}} \cdot \frac{x^n}{n!}$$

So $\sum_{k=0}^{n} c(n,k)\, x^k = (-1)^n n! \binom{-y}{n}$

$$= y(y+1)\cdots(y+n-1)$$

$$= y^{(n)}$$

Recall $\binom{z}{n} := \frac{z(z-1)\cdots(z-n+1)}{n!}$

and $\sum_{k=0}^{n} s(n,k)\, x^k = y(y-1)\cdots(y-n+1)$

$$= y_{(n)}. \qquad \square$$

There are shorter (and nicer) proofs of this formula.

Exercise. Find a proof of $\sum_{k=0}^{n} s(n,k)\, x^k = x_{(n)}$ based on a bijection or on a recurrence relation.

3 Pascal-like triangles:

1. Signless Stirling triangle of 1st kind

$$\begin{array}{ccccc} & & C(0,0) & & \\ & C(1,0) & & C(1,1) & \\ C(2,0) & & C(2,1) & & C(2,2) \end{array} =$$

. . . . . .

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 0 & & 1 & & & & \\ & & & 0 & & 1 & & 1 & & & \\ & & 0 & & 2 & & 3 & & 1 & & \\ & 0 & & 6 & & 11 & & 6 & & 1 & \\ 0 & & 24 & & 50 & & 35 & & 10 & & 1 \end{array}$$

. . . .

$$\boxed{C(n,k) = C(n-1,k-1) + (n-1)\,C(n-1,k)}$$

2. Stirling triangle of 2nd kind

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 1 & & & & \\
 & & & 0 & & 1 & & 1 & & & \\
 & & 0 & & 1 & & 3 & & 1 & & \\
 & 0 & & 1 & & 7 & & 6 & & 1 & \\
0 & & 1 & & 15 & & 25 & & 10 & & 1
\end{array}
$$

. . . .

$$\boxed{S(n,k) = S(n-1,k-1) + k\,S(n-1,k)}$$

3. Eulerian triangle

Eulerian numbers

$$A(n,k) := \#\{w \in S_n \mid \mathrm{des}(w) = k\}$$

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 4 & & 1 & & & \\
 & & 1 & & 11 & & 11 & & 1 & & \\
 & 1 & & 26 & & 66 & & 26 & & 1 & \\
1 & & 57 & & 302 & & 302 & & 57 & & 1
\end{array}
$$

. . . .

$$\boxed{A(n,k) = (n-k)A(n-1,k-1) + (k+1)A(n-1,k)}$$

Exercise. Prove these recurrence relations for the Stirling numbers of both kinds and for the Eulerian numbers.