Ordinary generating function vs exponential generating functions:

$\sum a_n x^n$ — ordinary gen. funct.

$\sum a_n \frac{x^n}{n!}$ — exponential gen. funct.

Which one should you use?

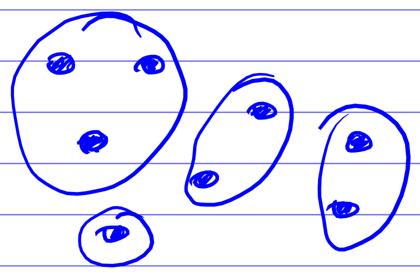
If a_n counts the number of some kind of combinatorial objects on n unlabelled nodes then use ordinary gen. funct.

If a_n counts some comb. objects on n labelled nodes then use exponential gen. funct.

Example The partition number

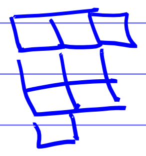
$p(n) :=$ the number of integer partitions of n

(= # Young diagrams with n boxes)

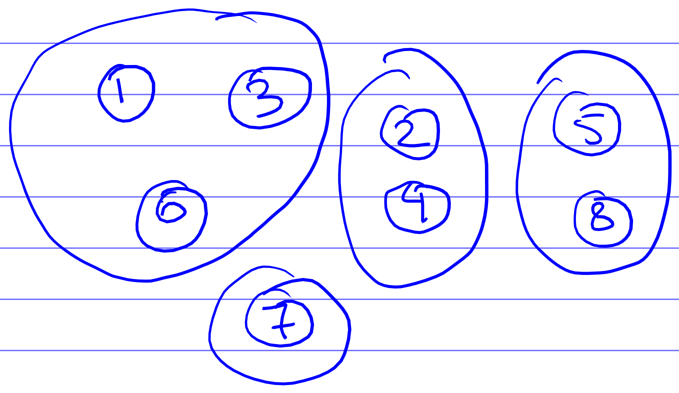


unlabelled objects $\rightsquigarrow$ use the ordinary g.f.

$\lambda = (3,2,2,1) =$



The Bell number $B(n) :=$ the number of set partitions of $[n] := \{1, 2, \dots, n\}$



$\pi = (1,3,6 \mid 2,4 \mid 5,8 \mid 7)$

labelled objects $\rightsquigarrow$ use the exponential g.f.

Lets calculate these gen. funct.

Integer partitions

Another notation for partitions:

$\lambda = (\lambda_1, \lambda_2, \dots) = 1^{m_1} 2^{m_2} 3^{m_3} \dots$

$m_i :=$ # times part i appears in λ

Example $\lambda = (5,4,4,2,1,1,1)$

$= 1^3 2^1 3^0 4^2 5^1$

$(m_1, m_2, m_3, \dots)$ can be any sequence of non-negative integers such that there are finitely many indices i with $m_i \neq 0$.

If $\lambda \vdash n$ then

$n = 1 \cdot m_1 + 2 \cdot m_2 + 3 \cdot m_3 + \dots$

Ordinary gen. funct.:

$$\sum_{n \geq 0} p(n) x^n = \sum_{\lambda \text{ partition}} x^{|\lambda|}$$

$$= \sum_{m_1, m_2, m_3, \dots \geq 0} x^{1 \cdot m_1 + 2 m_2 + 3 m_3 + \dots}$$

(only finitely many non-zero entries)

$$= \left(\sum_{m_1 \geq 0} x^{m_1}\right)\left(\sum_{m_2 \geq 0} x^{2m_2}\right)\left(\sum_{m_3 \geq 0} x^{3m_3}\right) \dots$$

$$= \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \dots$$

$$= \boxed{\prod_{k \geq 1} \frac{1}{1-x^k}}.$$

this is an infinite product. However for any fixed n only finitely many terms (for $k = 1, 2, \dots, n$) contribute to the coeff of x^n.

How about exp. g. f. for the Bell numbers $B(n)$?

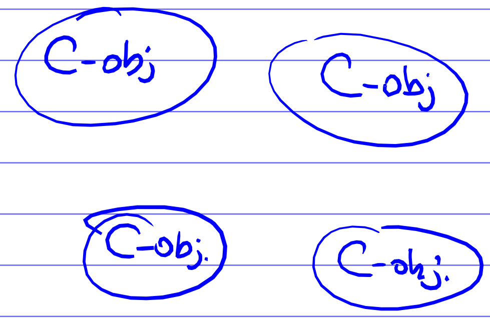
The Exponential Formula

c_n = # of some kind of connected objects on n labelled nodes.
(Let's call them C-objects.)

d_n = # of the same kind of objects on n labelled nodes that are not necessarily connected.
(Let's call them D-objects)

In other words, a D-object on n nodes consists of a set partition π of $[n]$ and a choice of a C-object on each block of π.

a D-object :

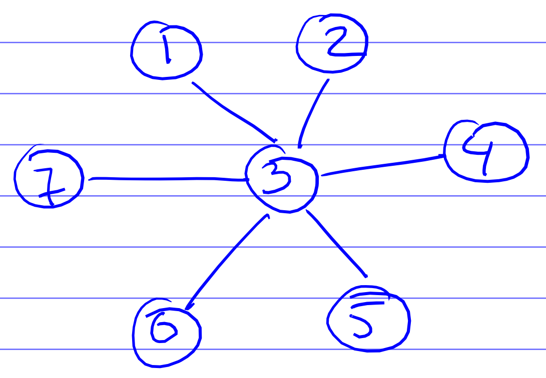


Let $$c(x) := \sum_{n \geq 1} c_n \frac{x^n}{n!}$$

$$d(x) := \sum_{n \geq 0} d_n \frac{x^n}{n!}$$

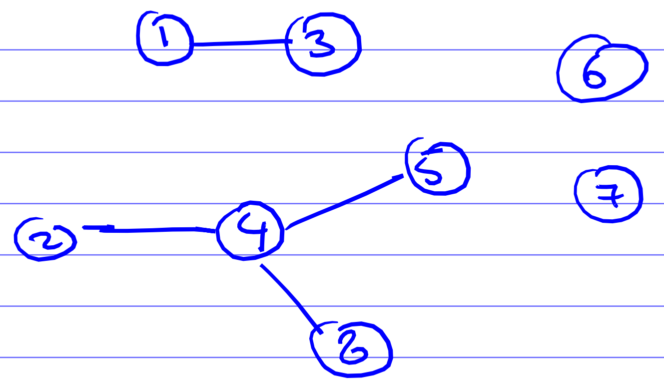
Then
$$\boxed{d(x) = e^{c(x)}}$$
the exponential formula

Example A **star graph** is a connected simple graph s.t. all vertices, except one vertex, have degrees 1.



C-objects : star graphs with labelled vertices

D-objects : simple graphs with labelled vertices such that each connected component is a star



$$c_n = \begin{cases} 1 & n = 1, 2 \\ n & n \geq 3 \end{cases}$$

$$C(x) = \frac{x}{1} + \frac{x^2}{2!} + \frac{3x^3}{3!} + \frac{4x^4}{4!} + \dots$$

$$= \sum_{n \geq 1} \frac{n \cdot x^n}{n!} - \frac{x^2}{2}$$

$$= x \sum_{n \geq 1} \frac{x^{n-1}}{(n-1)!} - \frac{x^2}{2} = x e^x - \frac{x^2}{2}.$$

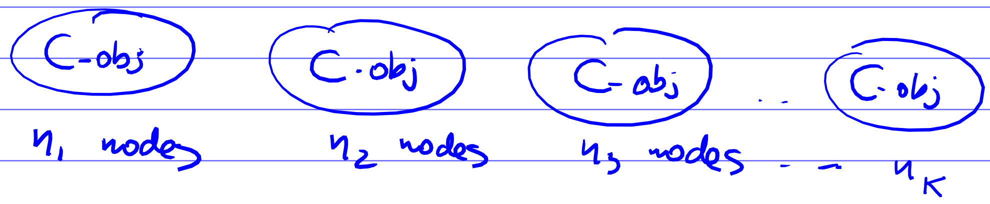
So we have

$$d(x) = e^{C(x)} = e^{x \cdot e^x - \frac{x^2}{2}},$$

and $d_n = n! \left(\text{the coeff of } x^n \text{ in } e^{xe^x - \frac{x^2}{2}} \right).$

Proof of exp. formula

A D-object on n (labelled) nodes:



$$n = n_1 + n_2 + n_3 + \dots + n_k$$

So

$$d_n = \sum_{k \geq 0} \sum_{\substack{n_1, \dots, n_k \geq 1 \\ n_1 + \dots + n_k = n}} \frac{1}{k!} \binom{n}{n_1, n_2, \dots, n_k} c_{n_1} c_{n_2} \cdots c_{n_k}$$

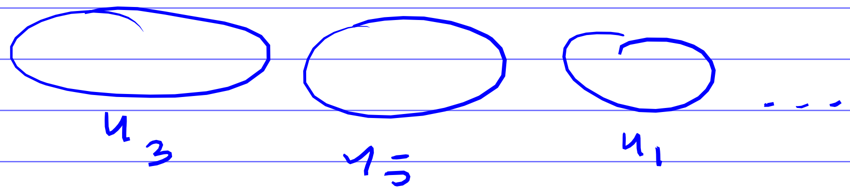
- the multinomial coefficient

$$\binom{n}{n_1, n_2, \dots, n_k} = \frac{n!}{n_1! \, n_2! \, n_3! \cdots n_k!}$$

= # ways to subdivide the set of nodes $[n]$ into k blocks with $n_1, n_2, n_3, \dots, n_k$ nodes.

- $\frac{1}{k!}$ because we don't's order the blocks.

(If we list the block in any different order, e.g.,



we get the same D-object.)

- $c_{n_1} c_{n_2} \cdots c_{n_k}$ = # way to pick C-structures on all k blocks.

Let's rewrite this formula:

$$\frac{d_n}{n!} = \sum_{k \geq 0} \sum_{\substack{n_1, \dots, n_k \geq 1 \\ n_1 + \dots + n_k = n}} \frac{1}{k!} \frac{c_{n_1}}{n_1!} \frac{c_{n_2}}{n_2!} \cdots \frac{c_{n_k}}{n_k!}$$

So $\quad d(x) := \sum_{n \geq 0} \frac{d_n x^n}{n!}$

$$= \sum_{k \geq 0} \sum_{n_1, \dots, n_k \geq 1} \frac{1}{k!} \frac{c_{n_1} x^{n_1}}{n_1!} \cdots \frac{c_{n_k} x^{n_k}}{n_k!}$$

$$= \sum_{k \geq 0} \frac{1}{k!} \left(\sum_{m \geq 1} \frac{c_m x^m}{m!} \right)^k$$

this summation starts with $m = 1$, because we don't allow empty connected components

$$= \exp\left(c(x) \right).$$

$\square$

Example : The Bell numbers $B(n)$

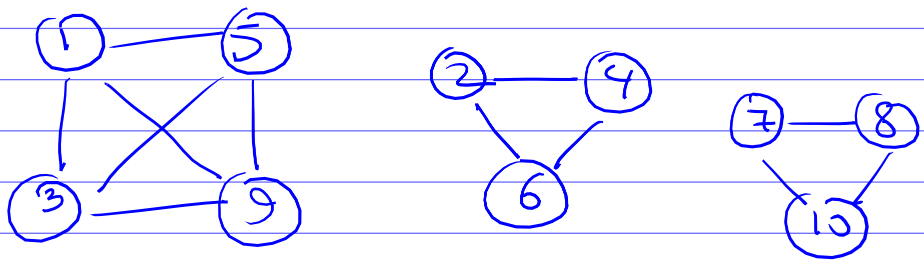
C- objects : just sets without any other structure

$c_n = 1.$

Then D-objects are set partitions

$d_n = B(n)$

If you prefer to talk about graphs: C-objects are complete graphs & D-objects are graphs with labelled vertices s.t. any connected component is a complete graph.



$\pi = (1,3,5,9 \mid 2,4,6 \mid 7,8,10)$

We obtain

$$C(x) = \sum_{n \geq 1} 1 \cdot \frac{x^n}{n!}$$

$$= \sum_{n \geq 0} \frac{x^n}{n!} - 1 = e^x - 1$$

So we obtain:

<u>Corollary.</u> (exp. g. f. for the Bell numbers)

$$\boxed{\sum_{n \geq 0} B(n) \frac{x^n}{n!} = e^{(e^x - 1)}}$$

How about generating functions for the <u>Stirling numbers</u> of the second kind?

$S(n, k) :=$ # set partitions of $[n]$ with exactly k blocks.

We can easily adapt the exponential formula to keep track of # of connected components, as follows:

Same setup as before

$c_n :=$ # C-objects on n labelled nodes

$d_{n,k} :=$ # D-objects on n labelled nodes with k connected components.

(In particular, $c_n = d_{n,1}$)

$$d(x,y) := \sum_{n \geq 0} \sum_{k \geq 0} d_{n,k} \frac{x^n}{n!} y^k$$

Theorem

$$\boxed{d(x,y) = e^{y \cdot c(x)}}$$

The proof is basically the same as before. (We just need to insert the extra factor y^k in all expressions.)

<u>Example</u>: The Stirling numbers of the 2^{nd} kind.

$$\sum_{n,k\geq 0} S(n,k)\,\frac{x^n}{n!}\,y^k$$

$$= \boxed{e^{y(e^x-1)}}$$

If we fix n, we obtain

$$\sum_{k=0}^{n} S(n,k)\,y^k =$$

$= n!$. the coeff. of x^n in $e^{y(e^x-1)}$.

Actually, there is another nicer looking generating function for the Stirling numbers (of 2^{nd} kind).

Let $y_{(k)} := y(y-1)(y-2)\cdot\ldots\cdot(y-k+1)$

(These expressions $y_{(k)}$ are called the <u>falling factorials</u> and

$y^{(k)} := y(y+1)(y+2)\cdot\ldots\cdot(y+k-1)$

are called the <u>rising factorials</u>.)

Note: $y_{(k)} = k!\binom{y}{k}$ the binomial coefficients

Theorem

$$\boxed{\sum_{k=0}^{n} S(n,k)\, y_{(k)} = y^n}$$

Example. $n = 2$

$S(2,0) = 0$, $S(2,1) = S(2,2) = 1$

$$y_{(1)} + y_{(2)} = y + y\cdot(y-1) =$$

$$= y(1 + y - 1) =$$

$$= y^2.$$

To give more examples, let's list the first few Stirling numbers $S(n,k)$. We'll arrange them in the

<u>Stirling triangle</u> of 2^{nd} kind (similar to Pascal's triangle)

$$\begin{array}{ccccc}
 & & S(0,0) & & \\
 & S(1,0) & & S(1,1) & \\
S(2,0) & & S(2,1) & & S(2,2) \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

$$\begin{array}{lccccccccccc}
n=0 & & & & & & 1 & & & & & \\
n=1 & & & & & 0 & & 1 & & & & \\
n=2 & & & & 0 & & 1 & & 1 & & & \\
n=3 & & & 0 & & 1 & & 3 & & 1 & & \\
n=4 & & 0 & & 1 & & 7 & & 6 & & 1 & \\
n=5 & 0 & & 1 & & 15 & & 25 & & 10 & & 1
\end{array}$$

. . . .

Recurrence relation

$$\boxed{S(n,k) = S(n-1,k-1) + k\,S(n-1,k)}$$

<u>Exercise.</u> Prove this recurrence relation.

back to the formula

$$\sum_{k=0}^{n} S(n,k)\, y_{(k)} = y^n .$$

Example $n=4$

$$0\cdot y_{(0)} + 1\cdot y_{(1)} + 7\, y_{(2)} +$$
$$+ 6\, y_{(3)} + 1\, y_{(4)} = y^4$$

Proof (of formula $\sum_k S(n,k)\, y_{(k)} = y^n$)

Both sides are polynomials in y of degree n. In order to check that 2 polynomials are equal to each other, it is enough to check that their values argee for for infinitely many values of y.

We'll show that

$$\sum_{k=0}^{n} S(n,k)\, y_{(k)} = y^n$$

for all positive integer values of y.

R.H.S. $y^n = \#$ all functions

$$f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, y\}$$

there are y options for each $f(i)$, $i = 1, \ldots, n$.

L.H.S. $\displaystyle\sum_{k=0}^{n} S(n,k)\, y_{(k)}$

$$= \sum_{k=0}^{n} S(n,k) \cdot k! \binom{y}{k}.$$

$S(n,k)\, k! = \#$ set partitions on $[n]$ with k blocks add a choice of ordering of the blocks.

$[n] = B_1 \cup B_2 \cup \cdots \cup B_k$ (a disjoint union)

now we care about which block goes first, second, etc.

$S(n,k)\, k! = \#$ Surjective functions

$$f : [n] \to \{i_1, \ldots, i_k\}$$

some k element set

Indeed, take f such that

$$f^{-1}(i_1) = B_1,\ f^{-1}(i_2) = B_2, \ldots, f^{-1}(i_k) = B_k.$$

Cleary, <u>any</u> function $f : [n] \to \{1, \dots, y\}$ is the same thing as a choice subset $S \subset \{1, \dots, y\}$ and a <u>surjective</u> function $[n] \to S$.

So $$y^n = \sum_{k=0}^{n} \binom{y}{k} \underbrace{S(n,k) \cdot k!}$$

all functions $[n] \to \{1, \dots, y\}$

k-element subsets in $\{1, 2, \dots, y\}$

surjective functions $[n] \to$ (k-element set)

$\square$

You might be wondering what the Stirling numbers of the first kind are. (If you don't already know this.)

Permutations & cycles

Definition: A permutation is a bijective map $w : [n] \to [n]$.

Several notations for permutations:

- 2-line notation:

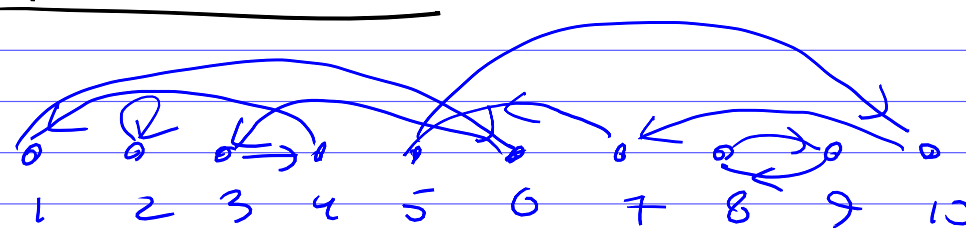
$$w = \begin{pmatrix} 1 & 2 & \cdots & n \\ w(1) & w(2) & & w(n) \end{pmatrix}$$

Example: $w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 2 & 4 & 1 & 10 & 3 & 5 & 9 & 8 & 7 \end{pmatrix}$

- 1-line notation:

$w = 6, 2, 4, 1, 10, 3, 5, 9, 8, 7$

- graphical notation:



- cycle notation

Example

cycles of w

$w = (1, 6, 3, 4)(2)(5, 10, 7)(8, 9)$

a fixed point $w(i) = i$

This permutation has 4 cycles, including the fixed point

Definition

The signless Stirling number of the first kind $c(n,k)$ is the number of permutations of size n with exactly k cycles (including fixed points)

The (signed) Stirling number of the first kind is $s(n,k) := (-1)^{n-k} c(n,k)$.

Let's do a bit more practice with the exponential formula.

Permutations are objects combinatorial objects on n labeled node (D-object)

"Connected" objects (C-objects) are permutations with exactly 1 cycle.

There are exactly $(n-1)!$ of permutations with exactly 1 cycle: $w = (1, \underbrace{a, b, \ldots, z}_{\substack{\text{some permutation} \\ \text{of } 2, 3, \ldots, n.}})$

$$C(x) := \sum_{n\geq 1} \#\left\{\begin{array}{l}\text{perm of size } n\\ \text{with exactly one}\\ \text{cycle}\end{array}\right\} \frac{x^n}{n!}$$

$$= \sum_{n\geq 1} (n-1)!\, \frac{x^n}{n!}$$

$$= \sum_{n\geq 1} \frac{x^n}{n} = -\log(1-x).$$

$$d(x,y) = \sum_{\substack{n\geq 0\\ k\geq 0}} c(n,k) \frac{x^n}{n!} y^k$$

$$\underset{\substack{\text{exp.}\\ \text{formula}}}{=} e^{y\, c(x)} = e^{y(-\log(1-x))}$$

$$= \left(e^{\log(1-x)}\right)^{-y} = (1-x)^{-y}$$

By substituting $x \to -x$, $y \to -y$ we obtain the gen. function for the (signed) Stirling numbers of 1^{st} kind.

Theorem

$$\sum_{n,k\geq 0} S(n,k)\,\frac{x^n}{n!}\,y^k$$

$$= (1+x)^y$$

$$= \sum_{n\geq 0} \binom{y}{n} x^n \qquad \text{(the binomial formula)}$$

$$= \sum_{n\geq 0} y_{(n)}\,\frac{x^n}{n!}$$

Equivalently,

$$\boxed{\sum_{k=0}^{n} s(n,k)\,y^k = y_{(n)}}$$

Compare this with the formula for the Stirling numbers of the second kind:

$$\boxed{\sum_{k=0}^{n} S(n,k)\,y_{(k)} = y^n}$$

Exercise Give a bijective proof of the formula

$$\sum_{k=0}^{n} s(n,k)\,y^k = y_{(n)}$$

Corollary: The Stirling numbers (of both kinds) are the coefficient of the change of basis matrices for the following 2 bases of the space $\mathbb{R}[y]$ of polynomials in y:

$$1, y, y^2, y^3, \dots \quad \text{(usual powers)}$$

$$y_{(0)}, y_{(1)}, y_{(2)}, y_{(3)} \quad \text{(falling factorials)}$$

Corollary The following 2 (infinite) matrices are inverses of each other

$$S = \left(S(n,k) \right)_{k,n \geq 0}$$

$$s = \left(s(n,k) \right)_{k,n \geq 0}$$

If you don't like infinite matrices, you can truncate them to $(N+1)\times(N+1)$ matrices

$$\left(S(n,k)\right)_{0\le n,k\le N} \quad \text{and}$$

$$\left(s(n,k)\right)_{0\le n,k\le N}.$$

Example. $N = 4$

$$\underbrace{\begin{bmatrix} 1 & & & & \\ 0 & 1 & & 0 & \\ 0 & 1 & 1 & & \\ 0 & 1 & 3 & 1 & \\ 0 & 1 & 7 & 6 & 1 \end{bmatrix}^{-1}}_{S(n,k)} = \underbrace{\begin{bmatrix} 1 & & & & \\ 0 & 1 & & 0 & \\ 0 & -1 & 1 & & \\ 0 & 2 & -3 & 1 & \\ 0 & -6 & 11 & -6 & 1 \end{bmatrix}}_{s(n,k)}$$

$$y(y-1) = -y + y^2$$
$$y(y-1)(y-2) = 2y - 3y^2 + y^3$$
$$y(y-1)(y-2)(y-3) = -6y + 11y^2 - 6y^3 + y^4$$

The (signless) Stirling triangle of the first kind:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 1 & & & & \\
 & & & 0 & & 1 & & 1 & & & \\
 & & 0 & & 2 & & 3 & & 1 & & \\
 & 0 & & 6 & & 11 & & 6 & & 1 & \\
0 & & 24 & & 50 & & 35 & & 10 & & 1
\end{array}$$

$\cdots\cdots$

Recurrence Relation:

$$\boxed{c(n,k) = c(n-1,k-1) + (n-1)\,c(n-1,k)}$$

Exercise Prove this recurrence relation.