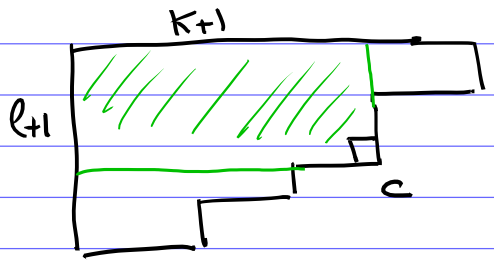
Let's finish the "hook walk" proof of the hook length formula:

$$\boxed{f_\lambda = \frac{n!}{H(\lambda)}}$$

f_λ: # SYT's of shape $\lambda \vdash n$

$H(\lambda) = \prod_{a \in \lambda} h(a)$

Fix $\lambda \vdash n$ and a corner box c, and consider the "green rectangle":



It's boxes a have weights $wt(a) := \frac{1}{h(a)-1}$ of the following form:

$k+1$

$\ell+1$

$\frac{1}{x_1+y_1}$	$\frac{1}{x_2+y_1}$	$\cdots$	$\frac{1}{x_k+y_1}$	$\frac{1}{y_1}$
$\frac{1}{x_1+y_2}$	$\frac{1}{x_2+y_2}$	$\cdots$	$\frac{1}{x_k+y_2}$	$\frac{1}{y_2}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\frac{1}{x_1+y_\ell}$	$\frac{1}{x_2+y_\ell}$	$\cdots$	$\frac{1}{x_k+y_\ell}$	$\frac{1}{y_\ell}$
$\frac{1}{x_1}$	$\frac{1}{x_2}$	$\cdots$	$\frac{1}{x_k}$	1

corner box c

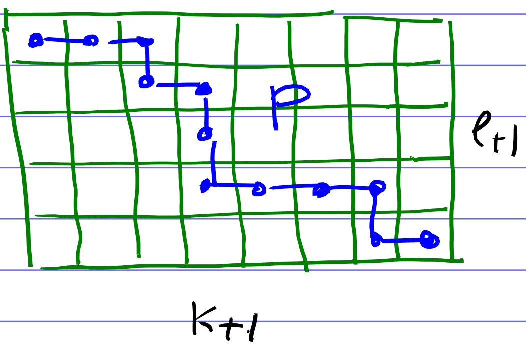
For a path P, in the green rectangle

$$wt(P) := \prod_{a \in P} wt(a).$$

Lemma 1.

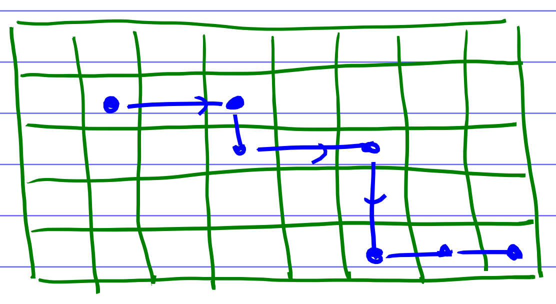
$$\sum_{P} wt(P) = \frac{1}{x_1 x_2 \cdots x_k \, y_1 y_2 \cdots y_\ell}$$

P is a lattice path from the upper left to the bottom right corner of the rectangle



Proof: Easy induction on n.

Let's now return to hook walks (in the green rectangle) what can start at any box and where we are allowed to skip over rows & columns. (But a hook walk should end at the lower right corner (the corner c) of the rectangle.)



a hook walk

Lemma 2.
$$\sum_{\substack{P \text{ is a} \\ \text{hook walk in} \\ \text{the } (k+1)\times(\ell+1) \\ \text{rectangle}}} wt(P) =$$

$$= \left(1+\frac{1}{x_1}\right)\left(1+\frac{1}{x_2}\right)\cdots\left(1+\frac{1}{x_k}\right)$$
$$\cdot \left(1+\frac{1}{y_1}\right)\left(1+\frac{1}{y_2}\right)\cdots\left(1+\frac{1}{y_\ell}\right).$$

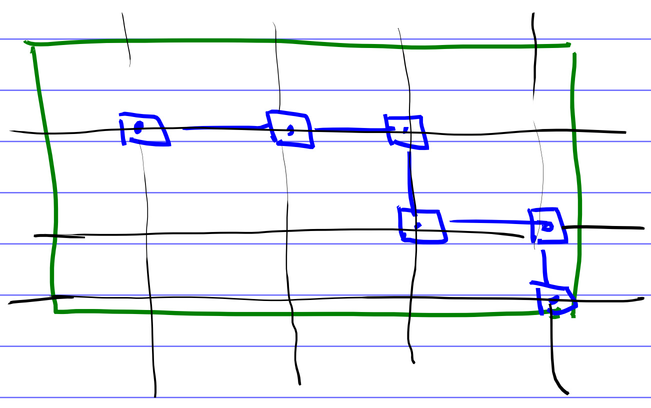
Example
$k=\ell=1$

$\frac{1}{x_1+y_1}$	$\frac{1}{y_1}$
$\frac{1}{x_1}$	1

$$\frac{1}{x_1+y_1}\cdot\frac{1}{y_1} + \frac{1}{x_1+y_1}\cdot\frac{1}{x_1} + \frac{1}{x_1} + \frac{1}{y_1} + 1$$

$$= \left(1+\frac{1}{x_1}\right)\left(1+\frac{1}{y_1}\right).$$

Proof. Follows from Lemma 1. Indeed, any hook walk is a lattice path for some sub-rectangle formed by some subsets of rows & columns of the green rectangle. Picking a term in the expansion of the R.H.S. corresponds to picking subsets of rows & columns.



$\square$

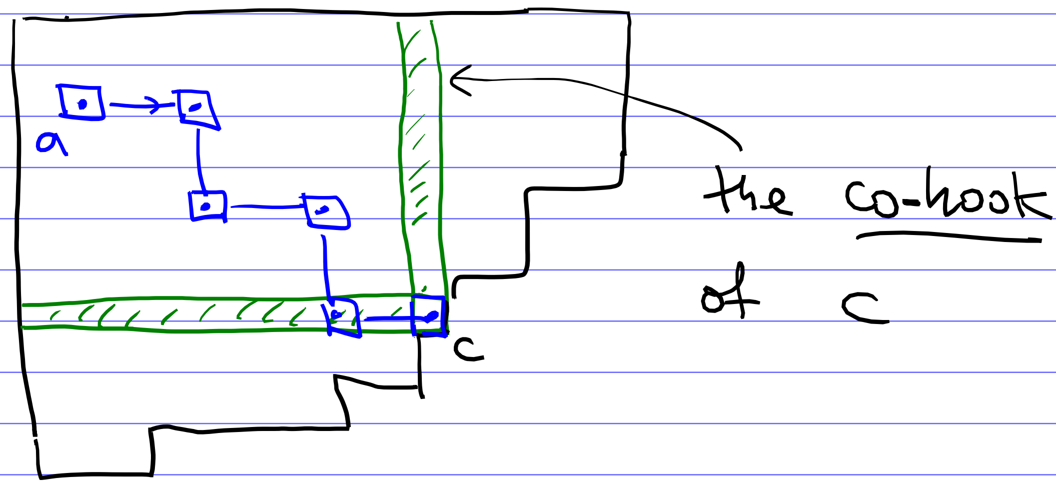
We apply Lemma 1 to the sub-rectangle formed by rows & columns containing the boxes of a hook walk.

back to the probabilities in random hook walks...

$$P(a,c) := \sum_{\substack{p \text{ is a hook} \\ \text{walk from} \\ a \text{ to } c}} wt(p)$$

$$P(c) := \sum_{a} \frac{1}{n} P(a,c)$$

the probability that a random hook walk arrives to the corner box c.



$$P(c) := \frac{1}{n} \sum_{a} P(a,c)$$

$$\overset{\text{Lemma 2}}{=} \frac{1}{n}\left(1+\frac{1}{x_1}\right)\cdots\left(1+\frac{1}{x_k}\right)\cdot$$

$$\cdot\left(1+\frac{1}{y_1}\right)\cdots\left(1+\frac{1}{y_\ell}\right)$$

$$= \frac{1}{n} \prod_{\substack{b \in \text{cohook}(c) \\ b \neq c}} \left(1+\frac{1}{h(b)-1}\right)$$

$$= \frac{1}{n} \cdot \prod_{\substack{b \in \text{co-hook}(c) \\ b \neq c}} \frac{h(b)}{h(b)-1}$$

$$= \frac{1}{n} \frac{H(\lambda)}{H(\lambda - c)}$$

when we remove a corner box c, only the hook lengths for boxes b in the co-hook (c) change (decrease by 1).

Conclusion:

We proved

$$P(c) = \frac{1}{n} \frac{H(\lambda)}{H(\lambda - c)}$$

$$\sum_{\substack{c \text{ corner} \\ \text{of } \lambda}} P(c) = 1$$

This implies the needed recurrence relation

$$\frac{n!}{H(\lambda)} = \sum_{\substack{c \text{ corner} \\ \text{of } \lambda}} \frac{(n-1)!}{H(\lambda - c)}$$

for the R.H.S. of the hook length formula.

This proves the hook length formula by induction on n.

$\boxtimes$

Let's talk more about partitions...

Partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$ are <u>integer partitions</u> (or partitions of integers). Later in this course we'll talk about <u>partition theory</u> — the area of math that studies such partitions.

But there are also different kinds of partitions (which we will also discuss in this course):

- <u>plane partitions</u> and
- <u>set partitions</u>.

<u>Set partitions</u>

Notation: $[n] := \{1, 2, \dots, n\}$.

<u>Definition</u>. A <u>set partition</u> of $[n]$ is a way to subdivide the set $[n]$ into a disjoint union of non-empty blocks.

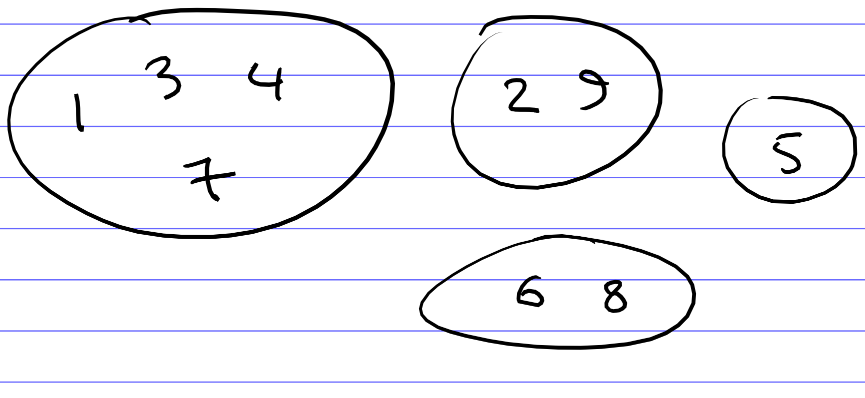
<u>Example</u>. $n = 9$.

blocks

$$\pi = (1, 3, 4, 7 \mid 2, 9 \mid 5 \mid 6, 8)$$

is the set partition

$$[9] = \{1, 3, 4, 7\} \cup \{2, 9\} \cup \{5\} \cup \{6, 8\}.$$



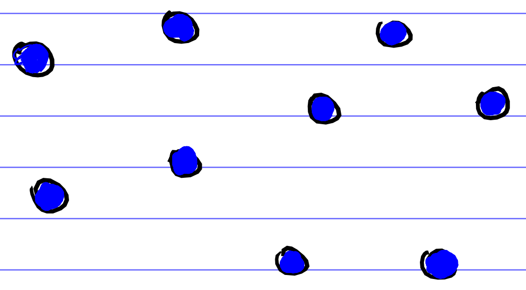
Note that the order of the blocks and the ordering of elts. in the blocks do not matter, e.g.,

$$\pi = (9, 2 \mid 6, 8 \mid 5 \mid 4, 3, 1, 7)$$

is considered the same set partition as the one above.

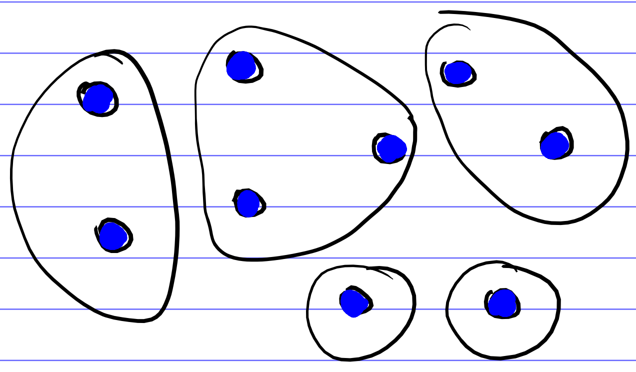
Set partitions vs integer partitions

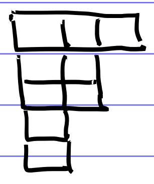
(labelled vs unlabelled objects)



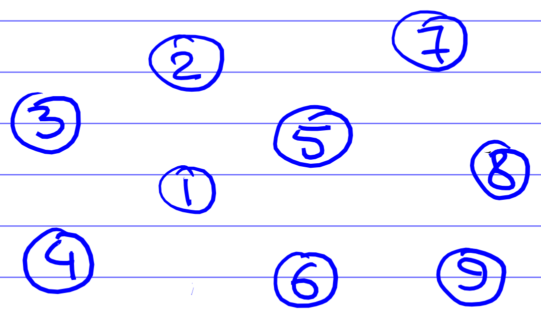
n indistinguishable (or unlabelled) balls.

An integer partition corresponds to a way to subdivide the balls into a disjoint union of non empty blocks:



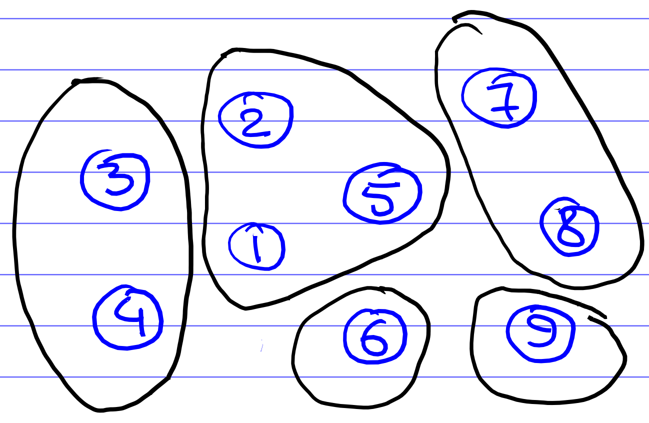
$\lambda = (3, 2, 2, 1, 1) =$ 

Let's us now consider n labelled balls:



all n balls are different from each other

A set partition corresponds to a way to subdivide n labelled balls into a disjoint union of non-empty blocks.



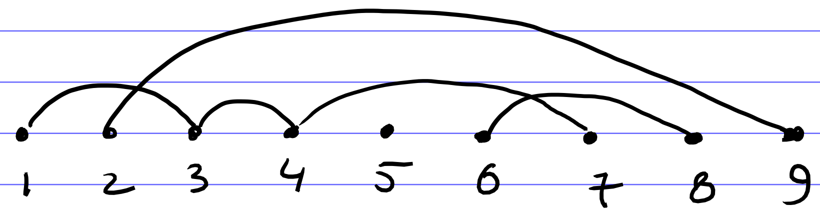
$\pi = (3, 4 \mid 1, 2, 5 \mid 7, 8 \mid 6 \mid 9)$

So set partitions can be viewed as a labelled version of integer partitions, and integer partitions can be viewed as an unlabelled version of set partitions.

Later we'll talk more about labelled vs unlabelled combinatorial objects.

Here is another graphical way to represent set partitions by arc diagrams.

<u>Example</u>: $\pi = (1\,3\,4\,7 \mid 2\,9 \mid 5 \mid 6\,8)$

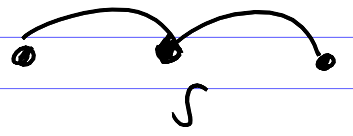


- arrange the vertices $1, 2, \ldots, n$ from left to right
- connect adjacent entries in the blocks by arcs.

<u>Remark</u>. Earier we mentioned arc diagrams corresponding to <u>perfect matchings</u> (each vertex belongs to exactly 1 arc).

Here the condition is:

each vertex v belongs to at most 1 arc entering v from the left & at most 1 arc entering v from the right:

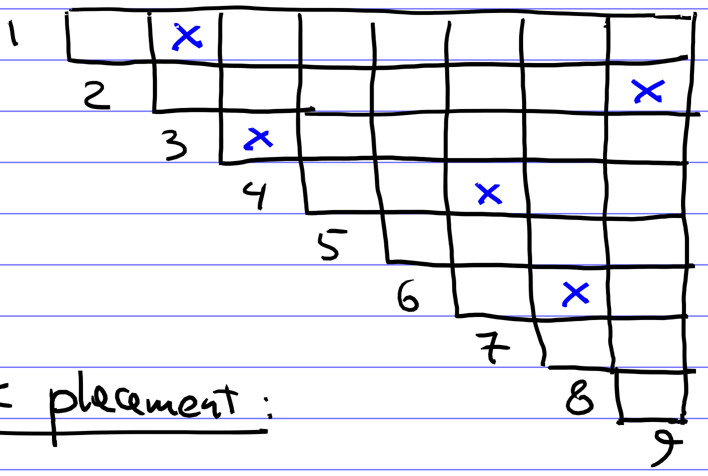


Rook Placements

These diagrams can also be drawn as non-attacking <u>rook placements</u> on a triangular board.

<u>Example</u>: $n = 9$

$\pi = (1, 3, 4, 7 \mid 2, 9 \mid 5 \mid 6, 8)$



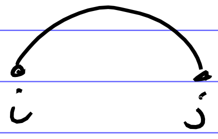
<u>Def.</u>

A <u>rook placement</u>: $\nexists$ two rooks (shown by X's) in the same column or the same row.

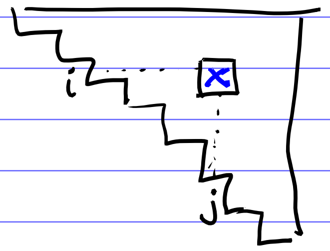
Correspondence: $\{\text{arc diagrams}\}$

$\Updownarrow$

$\{\text{rook placements}\}$

An arc 

corresponds to placing a rook in position (i, j):



Definitions:

- The Bell number $B(n)$ is the number of all set partitions on $[n]$
- The Stirling number of the second kind $S(n,k)$ is the number of set partitions of $[n]$ with k blocks.

$$B(n) = \sum_k S(n,k).$$

Example

n	set partitions	B(n)
1	(1)	1
2	(12) (1|2)	2
3	(123) (12|3) (13|2) (1|23) (1|2|3)	5
4	(15 set partitions)	15
5	(52 set partitions)	52

. . . .

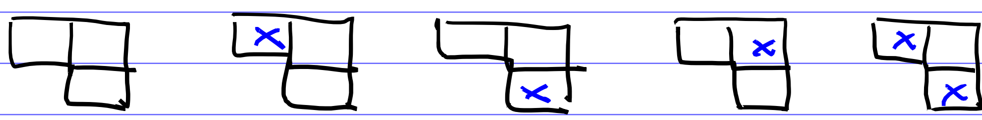
We proved

Theorem. $B(n)$ equals the number of rook placements (with any number of rooks, including 0 rook) on the triangular board with rows of sizes $n-1, n-2, \dots, 1$.

$S(n,k)$ equals the number of rook placements with exactly $n-k$ rooks on this triangular board.

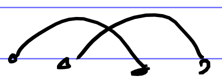
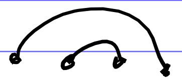
Example $n = 3$

$B(3) = 5$ rook placements:



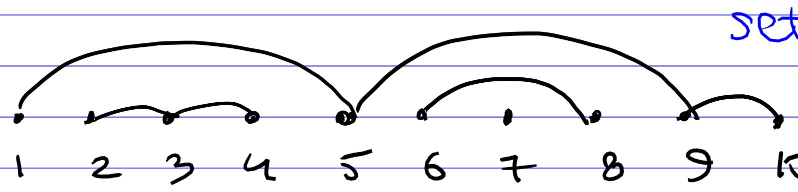
$S(3,3)=1$ $\qquad$ $S(3,2)=3$ $\qquad$ $S(3,1)=1$

Non-crossing & non-nesting set partitions.

Def A set partition of $[n]$ is called *non-crossing* (resp., *non-nesting*) if its arc diagram contains no pair of crossing arcs  (resp., no pair of nesting arcs )

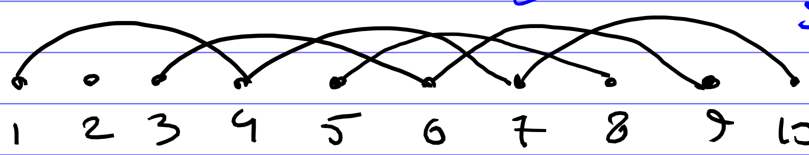
Example.

a non-crossing set partition



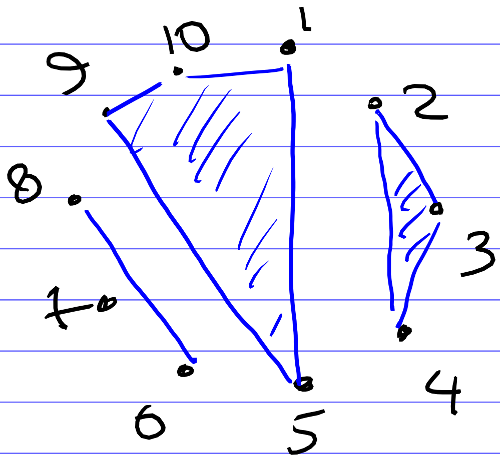
$$(1,5,9,10 \mid 2,3,4 \mid 6,8 \mid 7)$$

a non-nesting set partition



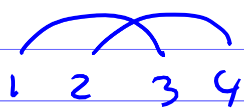
$$(1,4,7,10 \mid 2 \mid 3,6,9 \mid 5,8)$$

Non-crossing set partitions are also often represented by a collection of pairwise non-crossing polygons (and line segments) drawn inside a circle, e.g.:



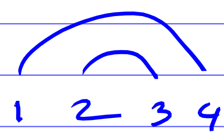
$$\pi = (1,5,9,10 \mid 2,3,4 \mid 6,8,7)$$

For $n \leq 3$, all set partitions are both non-crossing and non-nesting.

For $n = 4$, all set partitions, except $(13 \mid 24)$ 

)

are non-crossing.

Also all set partitions of $[4]$ except $(14 \mid 23)$ 

)

are non-nesting.

n	1	2	3	4	5
Bell number $B(n)$	1	2	5	15	52
# non-crossing set partitions	1	2	5	14	?
# non-nesting set partitions	1	2	5	14	?

__Theorem__ # non-crossing set partitions of $[n]$

$=$ # non-nesting set partitions of $[n]$

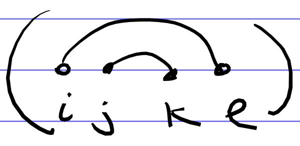
$=$ the Catalan number $C(n)$.

__Remark__. Earlier we discussed non-crossing & non-nesting perfect matchings.

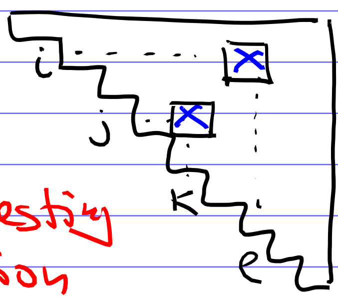
Here we have another instance of a "non-crossing vs non-nesting phenomenon".

Let's prove ½ of this theorem by constructing a bijection between non-nesting set partitions and Dyck paths.

{non-nesting set partitions of [n]} $\xleftrightarrow{\text{bij.}}$ {Dyck paths with 2n steps}

A pair of nested arcs  corresponds to a pair of rooks in the following relative position to each other

not allowed in a non-nesting set partition

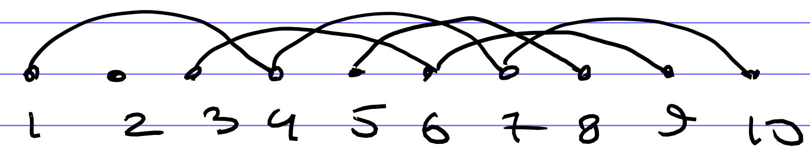


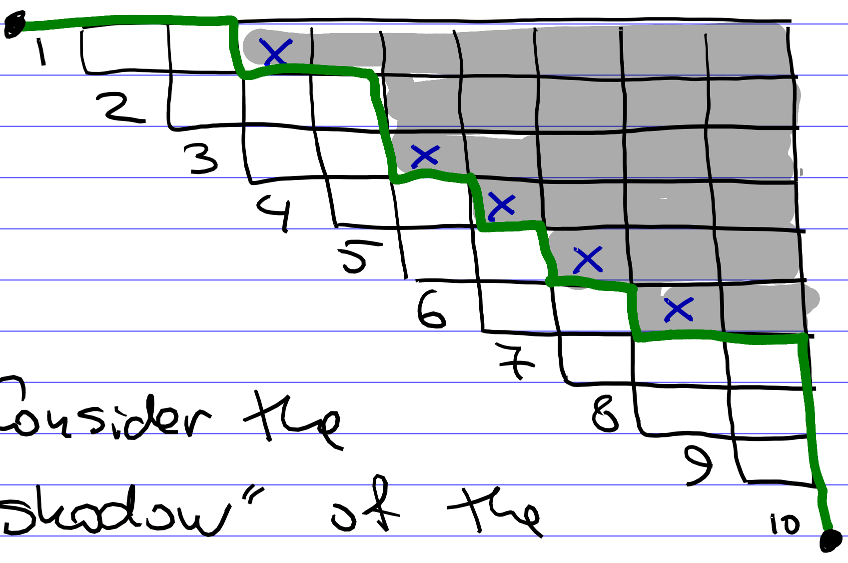
one rook is on SW of another rook.

So non-nesting set partitions correspond to rook placements (with any number of rooks) on the triangular board $(n-1, n-2, \ldots, 1)$ satisfying the following condition:

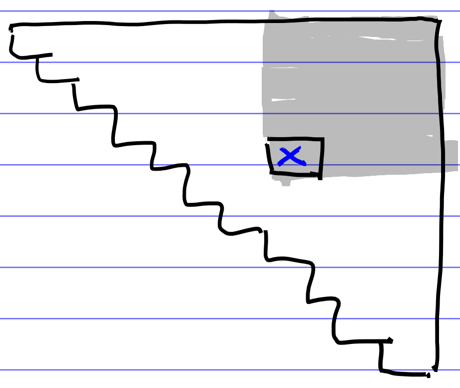
Any two rooks are located on the SE - NW of each other.

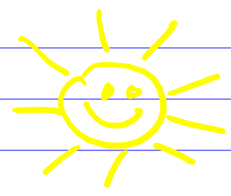
<u>Example</u> $\pi = (1,4,7,10 \,|\, 2 \,|\, 3,6,9 \,|\, 5,8)$





Consider the "shadow" of the rooks: Each rook casts the shadow of the form:



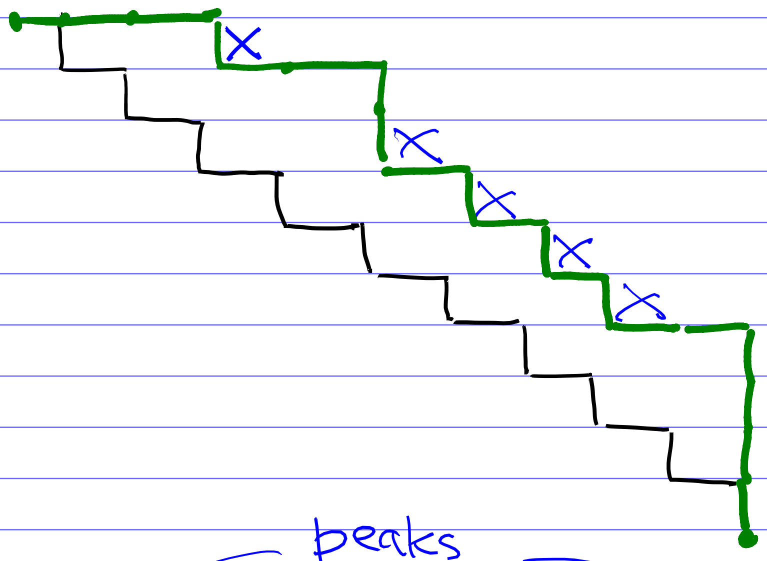


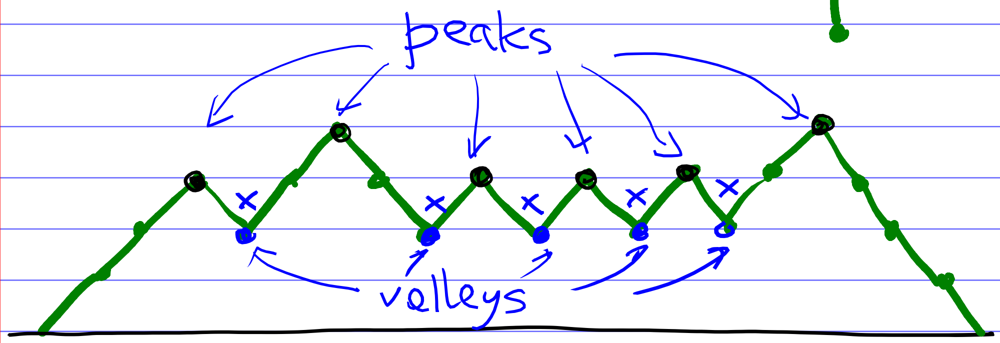
The corresponding Dyck path (shown in green) is the border between the shadowed & the illuminated regions.

<u>Exercise</u>. Prove the other part of the theorem:

$\#\{$non-crossing set partitions of $[n]\} = C_n$.

Looking more closely at the previous bijection, we see that the rooks correspond to the valleys in the Dyck path:





<u>Def</u>. A <u>valley</u> in a Dyck path is a "down" step immediately followed by an "up" step.

A peak in an "up" step immediately followed by a "down" step.

a peak

. U U U D U U . . .

a valley

Clearly, # peaks in a Dyck path = # valleys + 1.

The Narayana numbers

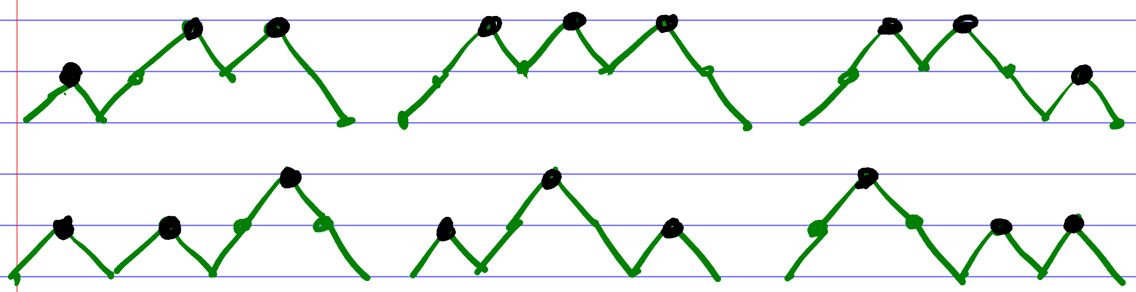
Definition The *Narayana number* $N(n,k)$ is the number of Dyck paths with $2n$ steps and exactly k peaks (equiv, $k-1$ valleys).

We proved

Theorem. The number of non-nesting set partitions of $[n]$ with k blocks equals the Narayana number $N(n, n-k+1)$.

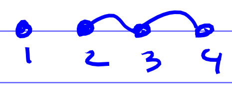
Example $N(4,3) = 6$

6 Dyck paths with 3 peaks:

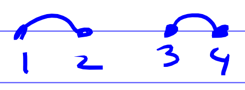


correspond to 6 non-nesting set partitions of $[4]$ with 2 blocks:

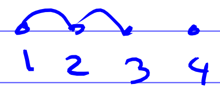
$(1|234)$



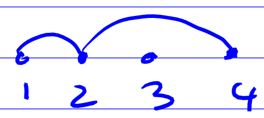
$(12|34)$



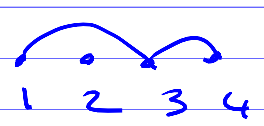
$(123|4)$



$(124|3)$



$(134|2)$



$(13|24)$