

18.212 PROBLEM SET 3 (due Friday, May 10, 2019)

**Problem 1.** Give a bijective proof of the formula for the number of spanning trees in the complete bipartite graph  $K_{m,n}$ .

**Problem 2.** Find (and prove by any method) a closed formula for the number of spanning trees in the complete tripartite graph  $K_{m,n,k}$ .

( $K_{m,n,k}$  is the graph on  $m + n + k$  vertices subdivided into 3 blocks of sizes  $m, n, k$  such that two vertices are connected by an edge if and only if they belong to different blocks.)

**Problem 3.** Let  $G$  be a graph on  $n$  vertices, and let  $\bar{G}$  be the complementary graph on the same set of vertices such that two vertices  $u \neq v$  are connected by an edge in  $\bar{G}$  if and only if they are not connected by an edge in  $G$  (and vice versa).

Let  $G^+$  be the extended graph obtained from  $G$  by adding a new vertex 0 connected to all vertices of  $G$ , and let

$$F_G(x) := \sum_T x^{\deg_T(0)-1},$$

where the sum is over all spanning trees  $T$  of the extended graph  $G^+$ .

Prove the reciprocity formula for spanning trees

$$F_{\bar{G}}(x) = (-1)^{n-1} F_G(-x - n).$$

**Problem 4.** Prove the following two identities

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} y(y + kz)^{k-1} (x - kz)^{n-k},$$

and

$$(x + y)(x + y + nz)^{n-1} = \sum_{k=0}^n \binom{n}{k} x(x + kz)^{k-1} y(y + (n - k)z)^{n-k-1}.$$

Hint: Try to interpret these identities using the directed matrix tree theorem for the directed graph on  $n+2$  vertices with two special vertices  $A$  and  $B$  such that

- there is a directed edge  $A \rightarrow B$ ,
- there are directed edges from  $A$  and  $B$  to all other  $n$  vertices,
- all other  $n$  vertices are pairwise connected by edges in both directions.

**Problem 5.** The  $n$ -cube graph is the 1-skeleton (i.e., the graph formed by all vertices and edges) of the  $n$ -cube. Consider it as a bidirected graph with each edge directed both ways. Find the number of directed Eulerian cycles in this graph. (A directed Eulerian cycle should pass each edge of the  $n$ -cube once in each direction.)

**Problem 6.** Consider the electrical network given by the  $n$ -cube graph such that each edge has resistance 1 (Ohm). Find the resistance between a pair of opposite vertices in this graph.

**Problem 7.** Alice and Ben play the following “Dicewalk” game. There are  $n$  cards marked by the numbers  $1, \dots, n$ . Initially, Alice has  $k$  cards and Ben has the remaining  $n - k$  cards. Each turn a random number generator produces a random integer  $i$  uniformly distributed on  $[n]$ , and the person who has the  $i$ -th card passes it to the opponent. The game ends when one person (the winner) collects all  $n$  cards. Find the probability that Alice wins.

**Problem 8.** Show that the following conditions on a function  $f : [n] \rightarrow [n]$  are equivalent to each other:

- $f$  is a parking function.
- $\#\{i \mid f(i) \geq n - k + 1\} \leq k$ , for  $k = 1, \dots, n$ .
- There exists a permutation  $w \in S_n$  such that  $f(i) \leq w(i)$ , for  $i = 1, \dots, n$ .

**Problem 9.** Find bijections between the following sets:

- The set of labelled trees on  $n + 1$  vertices.
- The set of plane binary trees on  $n$  vertices labelled by  $1, \dots, n$  such that the left child of a vertex always has a bigger label than its parent.
- The set of Dyck paths of length  $2n$  with up steps labelled by  $1, \dots, n$  (and unlabelled down steps) such that, for any two consecutive up steps, the label of the second step is greater than the label of the first step.
- The set of parking functions of size  $n$ .

**Problem 10.** The *tree inversion polynomial* is defined as the sum

$$I_n(x) := \sum_T x^{\text{inv}(T)},$$

where the sum is over all trees  $T$  on  $n + 1$  labelled vertices, and  $\text{inv}(T)$  is the number of inversions of  $T$ .

Show that

$$I_n(x) = \sum_f x^{\binom{n+1}{2} - (f(1) + \dots + f(n))},$$

where the sum is over all parking functions  $f : [n] \rightarrow [n]$ .

(Hint: One approach to this problem is based on showing that both sides satisfy the same recurrence relation.)

**Problem 11.** Show that the value  $I_n(-1)$  of the tree inversion polynomial equals the number  $A_n$  of *alternating permutations*  $w \in S_n$ , i.e., permutations such that  $w_1 < w_2 > w_3 < w_4 > \dots$ .

(Hint: You can use the result of the previous problem.)

**Problem 12.** Let  $A_n$  be the number of alternating permutations in  $S_n$ . (The numbers  $A_n$  are known under many different names: Euler numbers, zigzag numbers, up-down numbers, tangent numbers, secant numbers. The problem of calculating  $A_n$  is known as André's problem. The numbers  $A_n$  are also closely related Bernoulli numbers.)

Show that the exponential generating function for  $A_n$  is

$$\sum_{n \geq 0} A_n \frac{x^n}{n!} = \tan(x) + \sec(x).$$

**Problem 13.** The Bernoulli-Euler triangle is the triangular array of numbers

$$\begin{array}{cccccccc} & & & & & & & 1 \\ & & & & & & 0 & 1 \\ & & & & & 1 & 1 & 0 \\ & & & 0 & 1 & 2 & 2 \\ & 5 & 5 & 4 & 2 & 0 \\ 0 & 5 & 10 & 14 & 16 & 16 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

Each odd/even row of this triangle is obtained by adding the numbers in the row above it starting from the right/left.

Show that the numbers  $1, 1, 2, 5, 16, \dots$  that appear on the sides of this triangle are the numbers  $A_n$  of alternating permutations in  $S_n$ .

**Problem 14.** Calculate the determinant of the  $n \times n$  matrix  $A = (a_{ij})_{i,j \in [n]}$  such that  $a_{ij} = C_{i+j-1}$ , where  $C_k$  is the Catalan number  $\frac{1}{k+1} \binom{2k}{k}$ .

**Problem 15.** A *skew Young diagram*  $\lambda/\mu$  is the set-theoretic difference of two Young diagrams  $\lambda$  and  $\mu$  (such that  $\mu$  fits inside  $\lambda$ ). A skew Young diagram is *connected* if all its boxes are connected with each other by vertical and horizontal steps. (For example,  $(3,2,1)/(1)$  is connected, but  $(3,2,1)/(2)$  is not connected.)

We say that two skew Young diagrams are equivalent to each other if they can be obtained from each other by vertical and horizontal translations.

Find the number of equivalence classes of connected skew Young diagrams with perimeter  $2n$ , for any  $n \geq 2$ .

For example, for  $n = 2$  we have one diagram  $(1)$  (single box); for  $n = 3$ , we have 2 diagrams:  $(2)$  and  $(1, 1)$  (horizontal and vertical dominos); for  $n = 4$ , we have 5 diagrams:  $(3)$ ,  $(2, 1)$ ,  $(2, 2)/(1)$ ,  $(1, 1, 1)$ ,  $(2, 2)$ .

(Hint: Calculate the numbers of such diagrams for a few other small values of  $n$  and guess the answer. Then prove it.)