18.212 Problem Set 2 (due Friday, April 5, 2019)

Problem 1. Show that the number of non-crossing partitions of the set $\{1, \ldots, n\}$ equals the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. (A bijective proof is preferable. For example, you can use the fact that C_n is equal to the number of Dyck paths with $2n$ steps.)

Problem 2. (a) Prove the recurrence relation for the signless Stirling numbers of the first kind

$$c(n+1, k) = n\, c(n, k) + c(n, k-1).$$

(b) Prove the recurrence relation for the Stirling numbers of the second kind:

$$S(n+1, k) = k\, S(n, k) + S(n, k-1).$$

Problem 3. The Bell number $B(n)$ is the total number of partitions of an n element set, i.e., $B(n) = S(n, 1) + S(n, 2) + \cdots + S(n, n)$.

Show that the Bell numbers can be calculated using the Bell triangle:

$$\begin{array}{ccccc} & & & & 1 \\ & & & 1 & 2 \\ & & 2 & 3 & 5 \\ & 5 & 7 & 10 & 15 \\ 15 & 20 & 27 & 37 & 52 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

In this triangle, the first number in each row (except the first row) equals the last number in the previous row; and any other number equals the sum of the two numbers to the left and above it. The Bell numbers $B(0) = 1, B(1) = 1, B(2) = 2, B(3) = 5, B(4) = 15, B(5) = 52, \ldots$ appear as the first entries (and also the last entries) in rows of this triangle.

Problem 4. Show that the Bell number $B(n)$ is given by

$$B(n) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}.$$

Problem 5. In class, we mentioned two ways to define a lattice.

(I) A set L with two binary operation called "meet" $\vee$ and "join" $\wedge$ that satisfy several axioms.

(II) A poset P such that, for any two elements $x, y \in P$, there is a unique minimal element u such that $u \geq x$ and $u \geq y$, and a unique maximal element v such that $v \leq x$ and $v \leq y$.

Show that these two defintions of lattices are equivalent.

Problem 6. Let L be a finite distributive lattice. Let P be the poset formed by all join-irreducible elements of L. Use axioms of distributive lattices to show that L is isomorphic to $J(P)$.

Problem 7. Let P be a finite poset. Prove Dilworth's theorem that claims that the maximal size $M(P)$ of an anti-chain in P equals the minimal number $m(P)$ of disjoint chains (not necessarily saturated) that cover all elements of P.

Problem 8. (a) Show that the Fibonacci number F_{n+1} equals the number of *compositions* of n with all parts equal to 1 or 2, that is, the number of ordered sequences $c_1 \dots c_l$ such that $c_1 + \cdots + c_l = n$ and all $c_i \in \{1, 2\}$. For example,

$$F_6 = \#\{11111, 1112, 1121, 1211, 2111, 122, 212, 221\} = 8.$$

(b) In class, we gave a recursive construction of the differential poset $\mathbb{F}$ called the *Fibonacci lattice.* Give a nonrecursive description of $\mathbb{F}$ as a certain order relation on compositions with parts equal to 1 or 2.

(c) Prove that $\mathbb{F}$ is indeed a lattice.

Problem 9. Let W_n be the number of walks with $2n$ steps on the Hasse diagram of the Young's lattice $\mathbb{Y}$ that start and end at the minimal element $\hat{0} = (0)$. (The walks can have up and down steps in any order.)

For example, $W_2 = 3$, because there are 3 walks with 4 steps:

$$(0) \to (1) \to (2) \to (1) \to (0)$$
$$(0) \to (1) \to (1,1) \to (1) \to (0)$$
$$(0) \to (1) \to (0) \to (1) \to (0)$$

Show that W_n equals the number of *perfect matchings* in the complete graph K_{2n}. Find a closed formula for W_n.

Problem 10. Let X and D be two operators that act on polynomials $f(x)$ as follows:

$$X : f(x) \mapsto x f(x) \qquad \text{and} \qquad D : f(x) \mapsto f'(x).$$

For $n \geq 0$, define the polynomials $f_n(x) := (X + D)^n(1)$. For example, $f_0 = 1$, $f_1 = x$, $f_2 = x^2 + 1$, $f_3 = x^3 + 3x$. Calculate the constant term $f_n(0)$ of the polynomial f_n.

Problem 11. Fix positive integers k and l. Define the weight function $w(x)$ on boxes $x = (i, j)$ of the $k \times l$ rectangular Young diagram by

$$w((i, j)) := (i - j + l)(j - i + k),$$

for $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, l\}$.

Show that, for any Young diagram λ that fits inside the $k \times l$ rectangle, we have

$$\sum_{x \in \mathrm{Add}(\lambda)} w(x) - \sum_{y \in \mathrm{Remove}(\lambda)} w(y) = k \cdot l - 2\,|\lambda|.$$

Here $\mathrm{Add}(\lambda)$ is the set of all boxes of the $k \times l$ rectangle that can be added to the Young diagram λ; and $\mathrm{Remove}(\lambda)$ is the set of all boxes that can be removed from λ.

Problem 12. Show that the poset $J(J([2] \times [n]))$ is unimodal. (This is the poset of all shifted Young diagrams that fit inside the shifted shape $(n, n-1, \ldots, 1)$ ordered by containement.)

Problem 13. Find a closed formula for the number of saturated chains from the minimal element $\hat{0}$ to the maximal element $\hat{1}$ in the partition lattice Π_n.

Problem 14. Let NC_n be the subposet of the partition lattice Π_n formed by all non-crossing partitions of the set $\{1, \ldots, n\}$. The poset NC_n is called the *lattice of non-crossing partitions.*

Find a closed formula for the number of saturated chains from the minimal element $\hat{0}$ to the maximal element $\hat{1}$ in the poset NC_n.

Problem 15. Find a bijection between partitions of n with all odd parts and partitions of n with all distinct parts.

Problem 16. Prove that the number of partitions of n with all distinct and odd parts equals the number of self-conjugate partitions of n, i.e., partitions λ such that $\lambda' = \lambda$.