18.211 Problem Set 6 (due Wednesday, November 27, 2019)

Problem 1. For positive integers $k < n$, consider the graph G on the vertex set $V = [n]$ such that vertices i and j are connected by an edge wheneven $|i-j| \leq k$. Calculate the chromatic polynomial $\chi_G(q)$ of the graph G, and the chromatic number of G (i.e., the smallest number of colors needed to properly color the graph).

Problem 2. For $n \geq 2$, find an explicit expression for the chromatic polynomial $\chi_n(q) := \chi_{C_n}(q)$ of the graph C_n that consists of a single cycle of length n. (For example, $C_3 = K_3$, so $\chi_3(q) = \chi_{K_3}(q) = q(q-1)(q-2)$.)

Problem 3. For $n \geq 1$, calculate the number of acyclic orientations of the $2 \times n$ grid graph. (The $2 \times n$ grid graph is the product of K_2 and the graph that consists of a single path with n vertices. See Problem 4 in the Problem Set 5 for the definition of product of graphs.)

Problem 4. Let P be a 3-dimensional polytope with V vertices, E edges, and F faces such that every face of P is a quadrilateral. Prove that the vector $(V-2, E, F)$ is a multiple of the vector $(1, 2, 1)$. (For example, for the cube, we have $(V-2, E, F) = 6(1, 2, 1)$.)

Problem 5. Let $G = (V, E)$ be a connected graph. Let us fix an orientation of all edges in G. For a positive integer k, a *nowhere-zero k-flow* on G is a map $f : E \to \{1, 2, \ldots, k-1\}$ such that, for any vertex $v \in V$, we have

$$\sum_{e \in E\,:\, e \text{ enters } v} f(e) - \sum_{e' \in E\,:\, e' \text{ exits from } v} f(e') \equiv 0 \pmod{k}.$$

(In other words, the total in-flow to vertex v minus the total out-flow from v should be divisible by k.) Let $C_G(k)$ be the number of nowhere-zero k-flows on G.

Notice that $C_G(k)$ does not depend on a choice of orientation of edges in G, because one can always reverse the orientation of any edge e and simultaneousely replace $f(e)$ by $-f(e)$. So the number $C_G(k)$ is an invariant of an *undirected* graph G.

Prove that $C_G(k)$ satisfies the deletion-contraction recurrence:

$$C_G(k) = C_{G/e}(k) - C_{G\setminus e}(k),$$

for any edge $e \in E$ that is not a loop nor a bridge. Deduce that $C_G(k)$ is given by a polynomial function in k.

(This polynomial $C_G(k)$ is called the *flow polynomial* of graph G.)

Bonus Problems

Problem 6. Let $G = (V, E)$ be an undirected graph on the vertex set $V = [n]$. A *score vector* for G is a vector $(d_1, \dots, d_n) \in \mathbb{Z}^n$ such that there exists an orientation $\mathcal{O}$ of all edges of G such that, for all $i \in V$, d_i equals the outdegree of the vertex i in the orientation $\mathcal{O}$.

Prove that the number of different score vectors for graph G equals the number of forests $F = (V, E')$, $E' \subset E$, in the graph G.

For example, for graph $G = K_3$ there are 7 different score vectors $(0, 1, 2)$, $(0, 2, 1)$, $(1, 0, 2)$, $(1, 2, 0)$, $(2, 0, 1)$, $(2, 1, 0)$, $(1, 1, 1)$. On the other hand, there are 7 forests in K_3.

Problem 7. Let us fix two positive integers m and n. Prove that the number of acylic orientations of the complete bipartite graph $K_{m,n}$ equals the number of permutations $w \in S_{m+n}$ such that such that $-m \le w(i) - i \le n$, for $i = 1, \dots, m + n$.

One can identify such permutations w with placements of $m + n$ pairwise non-attacking rooks of the chessboard $B_{m,n}$ with boxes (i, j) such that $-m \le i - j \le n$ and $i, j \in [m + n]$. (We label boxes of a chessboard by pairs (i, j) in the same way as one would label entries of a matrix.) In other words, the board $B_{m,n}$ is obtained from the $(m + n) \times (m + n)$ square chessboard by removing two triangular subsets of boxes of shapes $(n-1, n-2, \dots, 1)$ and $(m-1, m-2, \dots, 1)$ located in the North-East and South-West corners of the square.

For example, for $m = n = 1$, the graph $K_{1,1}$ has 2 acyclic orientations. On the other hand, $B_{1,1}$ is the 2×2 square. There are 2 rook placements on $B_{1,1}$. For $m = n = 2$, the graph $K_{2,2}$ is the 4-cycle that has $2^4 - 2 = 14$ acyclic orientations. On the other hand, $B_{2,2}$ is the 3×3 square with two boxes in the opposite corners removed. There are 14 rook placements on $B_{2,2}$.

Problem 8. Prove that a graph is G is chordal if and only if it has a perfect elimination ordering of vertices.

Problem 9. A graph G is called *outerplanar* if it can be drawn on the plane without crossing edges so that all vertices of G belong to the outer face (i.e., all vertices appear on the perimeter of the drawing). Prove that G is outerplanar if an only if G has no sugraph that is edge-equivalent to K_4 or $K_{2,3}$.

Problem 10. Fix a positive integer n. Let T be a binary tree with n vertices that have exactly 2 children (and $n+1$ leaves). Let us label all n non-leaf vertices of T by $1, \dots, n$. We say that such labelled binary tree is *increasing* if the label of a child is always greater than the label of its parent vertex. We also say that such labelled binary tree is *left-increasing* if the label of a *left* child is always greater than the label of its parent vertex (but the label of a right child may or may not be greater than the label of its parent).

(a) Find an expression of the number of inreasing labelled binary trees with n non-leaf vertices. Give a bijective proof for this expression.

(b) Find an expression of the number of left-inreasing labelled binary trees with n non-leaf vertices. Give a bijective proof for this expression.