18.211 PROBLEM SET 5 (due Friday, November 15, 2019)

Problem 1. For a tree T on n labelled vertices $1, \ldots, n$, let $x^T := \prod_{i=1}^n x_i^{\deg_T(i)}$. Show that

$$\sum_T x^T = x_1 \cdots x_n \,(x_1 + \cdots + x_n)^{n-2},$$

where the sum is over all n^{n-2} labelled trees T on n vertices.

(Hint: Some properties of Prüfer's code might help you.)

In the following two problems, $G = (V, E)$ is a graph with positive weights $c(e) > 0$ assigned to edges $e \in E$. (The weight $c(e)$ is the *cost* of edge e.) The cost of a subgraph $H = (V, E')$, $E' \subset E$, of G is defined as $c(H) := \sum_{e \in E'} c(e)$. Let's try to use variations of the *greedy algorithm* to maximize the cost of several kinds of subgraphs of G.

Problem 2. We would like to find a matching M in G with maximal possible cost $c(M)$ among all matchings of G. Let us pick an edge e_1 with maximal cost $c(e_1)$, then pick an edge e_2 with maximal possible cost $c(e_2)$ among all edges e_2 such that $\{e_1, e_2\}$ is a matching, then pick an edge e_3 with maximal possible cost $c(e_3)$ among all edges such that $\{e_1, e_2, e_3\}$ is a matching, etc. Will this algorithm always produce a matching with maximal possible cost? Prove this or find a counterexample.

Problem 3. We would like to find a subgraph $H = (V, E')$ of G such that its complementary subgraph $G \setminus H = (V, E \setminus E')$ is connected and H has maximal possible cost among all subgraphs of G with this property. Let us pick an edge e_1 with maximal cost $c(e_1)$ among all edges of G such that $G \setminus \{e_1\}$ is connected, then pick an edge e_2 with maximal possible cost $c(e_2)$ among all edges e_2 such that $G \setminus \{e_1, e_2\}$ is still connected, then pick an edge e_3 with maximal possible cost $c(e_3)$ among all edges such that $G \setminus \{e_1, e_2, e_3\}$ is still connected, etc. Will this algorithm always produce a subgraph with the needed property? Prove this or find a counterexample.

Problem 4. Calculate the number of spanning trees of the product $K_3 \times K_3$ of two complete graphs K_3.

(The *product* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \times G_2$ on the vertex set $V_1 \times V_1 = \{(u, v) \mid u \in V_1,\, v \in V_2\}$ with edges of the form $\{(u, v), (u, v')\}$, for any $u \in V_1$ and $\{v, v'\} \in E_2$, and of the form $\{(u, v), (u', v)\}$, for any $\{u, u'\} \in E_1$ and $v \in V_2$.)

Problem 5. Calculate the number of spanning trees of the complete biparite graph $K_{m,n}$.

Problem 6. Find the number of perfect matchings of the bipartite graph $G \subset K_{n,n}$ on the vertices $1, \ldots, n, 1', \ldots, n'$ such that the vertices i and j' are connected by an edge in G if and only if $j \leq i + 3$.

Problem 7. Fix $n \geq 3$. Let C be the graph on n vertices that consists of a single cycle with n edges. Show that the number of all (not necessarily perfect) matchings of C equals the sum of two Fibonacci numbers $F_{n+1}+F_{n-1}$. (For example, the 3-cycle has $F_4+F_2 = 3+1 = 4$ matchings, namely, the empty matching and the 3 matchings with a single edge.)

Problem 8. Let $G \subset K_{n-1,\,n}$ be a biparite graph on the vertices $1, \ldots, n-1$ (the left part) and $1', 2', \ldots, n'$ (the right part). Let $N_i := \{j' \mid (i, j') \text{ is an edge of } G\}$, for $i \in [n-1]$. Show that the following two conditions are equivalent:

(A) For any $j \in [n]$, there exists a matching in G (with $n-1$ edges) that covers all vertices of G except the vertex j'.

(B) For any distinct $i_1, \ldots, i_k \in [n-1]$, we have

$$|N_{i_1} \cup N_{i_2} \cup \cdots \cup N_{i_k}| \geq k + 1.$$

Bonus Problems

Problem 9. Find the number of spanning trees of the *complete tripartite graph* $K_{m,n,k}$. (The graph $K_{m,n,k}$ is the graph with $m+n+k$ vertices $1, \ldots, m, 1', \ldots, n', 1'', \ldots, k''$, whose edges are (i, j'), (i, l''), and (j', l''), for any $i \in [n]$, $j \in [m]$, $l \in [k]$.)

Problem 10. Let D be a Dyck path with $2n$ steps. If the ith "up" step in D is the line segment $[(x, y), (x+1, y+1)]$, we define its *height* as $h_i = h_i(D) := y+1$. (For example, for the Dyck path represented by the sequence $(1, 1, 1, -1, 1, -1, -1, 1, -1, -1)$, the heights are $h_1 = 1$, $h_2 = 2$, $h_3 = 3$, $h_4 = 3$, and $h_5 = 2$.)

Find a closed expression for the sum

$$\sum_D \prod_{i=1}^{n} h_i(D),$$

over all Dyck paths D with $2n$ steps.